Midterm

Exam Location: Remote

PRINT your student ID: ____________________

PRINT AND SIGN your name: ____________________, ____________________ ____________________

(last) (first) (sign)

PRINT your discussion sections and (u)GSIs (the ones you attend): ____________________

Row Number: ____________________ Seat Number: ____________________

Name and SID of the person to your left: ____________________

Name and SID of the person to your right: ____________________

Name and SID of the person in front of you: ____________________

Name and SID of the person behind you: ____________________

1. **Honor Code (0 pts.)**

   Please copy the following statement in the space provided below and sign your name.

   As a member of the UC Berkeley community, I act with honesty, integrity, and respect for others. I will follow the rules and do this exam on my own.

   Note that if you do not copy the honor code and sign your name, you will get a 0 on the exam.

2. **What are you planning to do after the midterm? (2 pts.)**

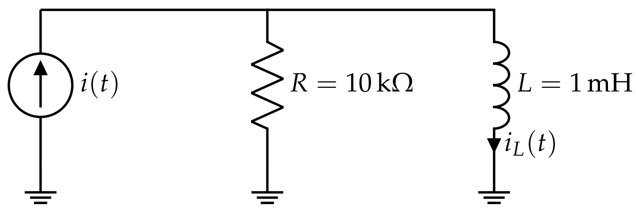
3. **What's your favorite thing to do in Berkeley? (2 pts.)**

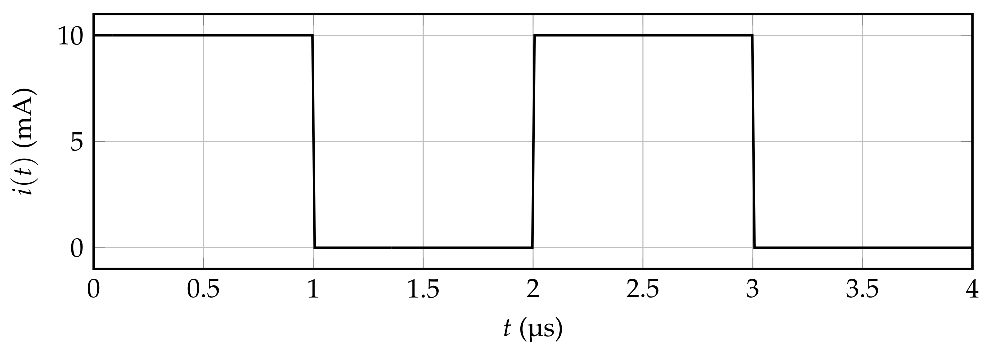
Do not turn this page until the proctor tells you to do so.
You can work on the above problems before time starts.

4. RL Square Wave (16 pts.)

In this problem, we will explore how an RL circuit behaves to a square wave input.

Suppose you have the following circuit, with the plot of input $i(t)$ shown below. Assume $i_L(0) = 0$.

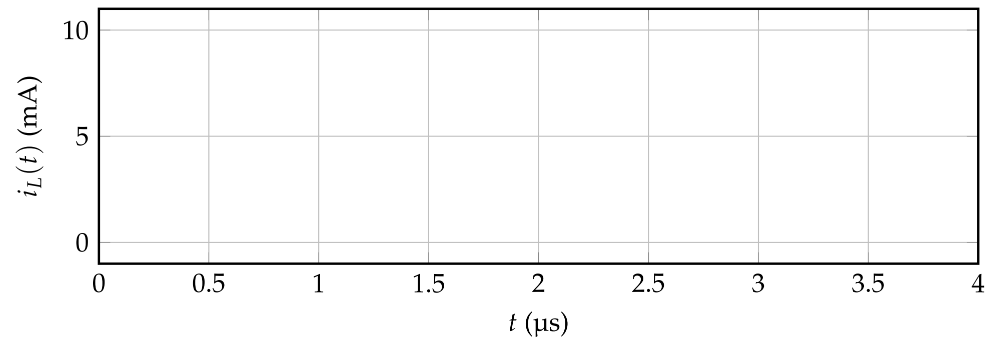




(a) (5 pts.) Find the **differential equation for** $i_L(t)$ in terms of R, L, and $i(t)$.

(b) (2 pts.) Find the **time constant** τ that describes this circuit both in terms of R and L, as well as numerically using the element values provided in the circuit diagram.

(c) (2 pts.) **Qualitatively draw** $i_L(t)$ **for** $0\,\mu s < t < 4\,\mu s$ on the provided plot (make sure to look at the units of the plot). Also, remember that $i_L(0) = 0$.



(HINT: Remember how the time constant relates to transitions between voltages.)

For the rest of the problem, we will focus on the time interval $0\,\mu s < t < 1\,\mu s$ (which represents the first transition for the circuit).

(d) (5 pts.) **Solve the differential equation for** $i_L(t)$ on the time interval $0\,\mu s < t < 1\,\mu s$.

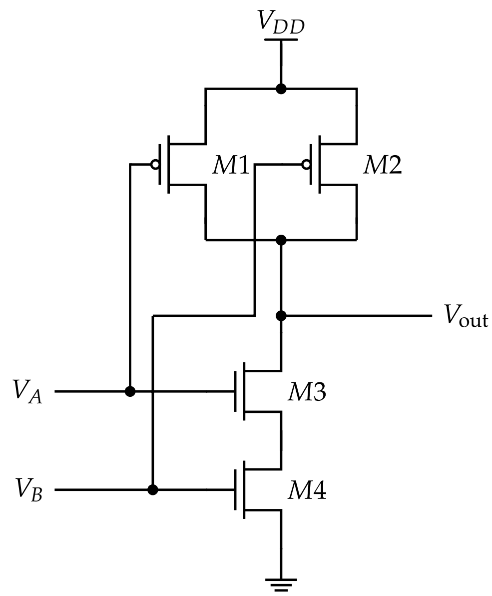
(e) (2 pts.) Suppose we want to shorten the time period of the input square wave $i(t)$ such that the first transition occurs over the time interval $0 < t < T$ (currently, $T = 1 \times 10^{-6}\,\text{s} = 1\,\mu s$). However, we want to still ensure that $i_L(t)$ reaches at least 90% of $i_{L,\text{desired}}$, its target/desired value (the value it approaches over the time interval). To find the minimum time T such that this condition is fulfilled, we can solve the following equation:

$$\frac{i_L(T)}{i_{L,\text{desired}}} = 0.9 \tag{1}$$

Using your answer from the previous part, **solve for the minimum time** T. Your answer should be numerical, but you do not need to simplify it.

5. Transistor Logic (10 pts.)

For this problem, we will analyze the following transistor logic gate:



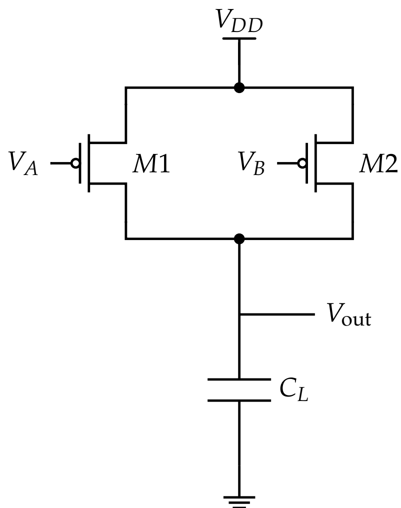
Assume that $V_{DD} > V_{Tn}, |V_{Tp}| > 0$ (where V_{Tn} is the threshold voltage for the NMOS transistors and V_{Tp} is the threshold voltage for the PMOS transistors) for all parts of the problem.

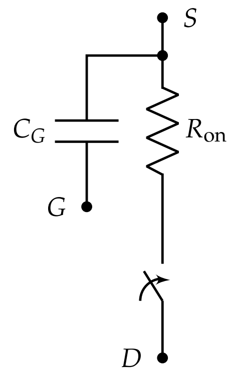
(a) (5 pts.) For each row in the following table, for the given input voltages V_A and V_B, fill in the bubble **for each transistor that is active (conducts current) for those input voltages** and fill out one of the bubbles for the **output** V_{out} ($V_{\text{out}} = V_{DD}$ or $V_{\text{out}} = 0$).

For this part, you may model the transistors as voltage-controlled switches (with no resistance or capacitance). Also, you do not have to choose whether to fill in the bubble or not for transistor M3 for $V_A = V_{DD}$ and $V_B = 0$ (the bubble is not present).

V_A	V_B	M1	M2	M3	M4	$V_{\text{out}} = V_{DD}$	$V_{\text{out}} = 0$
0	0	○	○	○	○	○	○
0	V_{DD}	○	○	○	○	○	○
V_{DD}	0	○	○		○	○	○
V_{DD}	V_{DD}	○	○	○	○	○	○

(b) (5 pts.) Now, consider the following circuit on the left, which can represent part of the transistor gate from the previous part when it drives some load capacitance C_L (which can represent a circuit connected to the output of the transistor gate). The circuit on the right is the model we will use for the two identical transistors (the state of the switch is determined by the relevant threshold voltage condition).



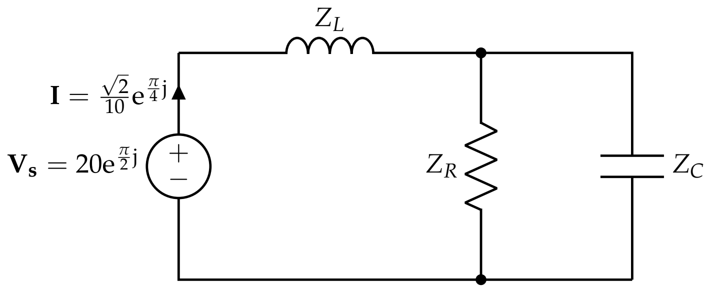


Suppose we have two scenarios:

- Scenario 1: $V_A = 0$, $V_B = V_{DD}$
- Scenario 2: $V_A = V_B = 0$

In which scenario will V_{out} transition between voltages the fastest? Explain your reasoning using your knowledge of time constants.

6. AC Power (5 pts.)



Find the ***average power*** **and** ***reactive power*** **taken from the source. Show your work.** Remember that complex power follows the equation

$$P_{\text{complex}} = \frac{1}{2}VI^* \tag{2}$$

and (average) power is $P_{\text{avg}} = \text{Re}\{P_{\text{complex}}\}$, while reactive power is $P_{\text{reactive}} = \text{Im}\{P_{\text{complex}}\}$.

7. System ID (10 pts.)

(a) (5 pts.) Consider the following *discrete-time* system:

$$x[i+1] = a_1 x[i] + a_2 x[i-1] + b u[i] \tag{3}$$

where $x[i], i \geq 0$ is the "state" of the system and $u[i], i \geq 0$ is the "input" into the system. The constants $a_1, a_2, b \in \mathbb{R}$ are unknown. Consider the following table of collected data points:

i	$x[i]$	$u[i]$
0	0	0
1	1	0
2	1	3
3	1.5	8
4	1.85	8
5	1.7	4
6	1.5	3
7	1.4	4

Set up a least squares problem, of the form $D\vec{p} = \vec{s}$ for some matrix D and vector $\vec{s}$ derived from the data above, where $\vec{p} := \begin{bmatrix} a_1 \\ a_2 \\ b \end{bmatrix}$.

(b) (5 pts.) Suppose the actual system was

$$x[i+1] = \underbrace{0.5}_{a_1} x[i] + \underbrace{0.2}_{a_2} x[i-1] + \underbrace{0.1}_{b} u[i] \tag{4}$$

Say you were still interested in "estimating" the values of a_1, a_2, b (even though you know what the values are). **Construct a sequence of control inputs** $u[0], u[1], u[2], u[3], u[4]$ **and initial states** $x[0]$ **and** $x[1]$ **such that:**

i. You can use least squares to estimate a_1, a_2, b**.**

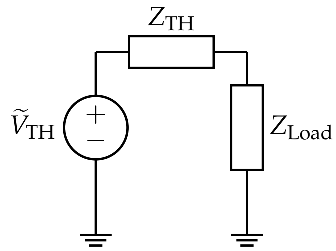
ii. You cannot use least squares to estimate a_1, a_2, b**.**

Assume there is no noise/disturbance term.

(HINT: What is the condition needed on the matrix D to apply least squares? Construct a sequence of control inputs so that you can/cannot satisfy this condition.)

8. Maximum Power Transfer Theorem (15 pts.)

Consider the following phasor domain circuit:



where Z_{TH} is written as its real and complex components as follows:

$$Z_{\text{TH}} = z_r + \text{j}z_i \tag{5}$$

Suppose Z_{Load} is constrained to be purely real (i.e., $\text{Im}\{Z_{\text{Load}}\} = 0$).

(a) (5 pts.) **Find the average power (or, equivalently, active power) delivered to the load in terms of** $z_r, z_i, \widetilde{V}_{\text{TH}}, Z_{\text{Load}}$**.** Remember that complex power follows the equation

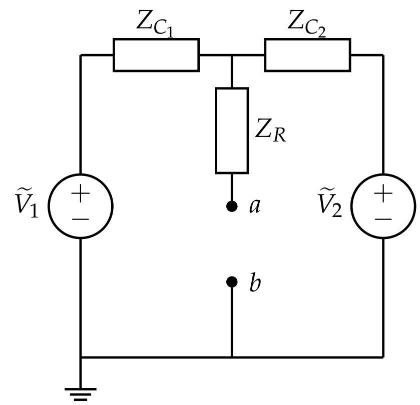
$$P_{\text{complex}} = \frac{1}{2} V I^* \tag{6}$$

and average power is $P_{\text{avg}} = \text{Re}\{P_{\text{complex}}\}$.

(b) (5 pts.) **Show that the value of Z_{Load} that maximizes P_{Load}, the average power delivered to Z_{Load}, is $Z_{\text{Load}} = |Z_{\text{TH}}| = \sqrt{z_r^2 + z_i^2}$.**
(HINT: Take a derivative of your average power expression from the previous part and set it to 0. Use this to solve for the maximizing value of Z_{Load}. Remember, impedance is nonnegative.)

[Extra page. If you want the work on this page to be graded, make sure you tell us on the problem's main page.]

(c) (4 pts.) Consider the following circuit:



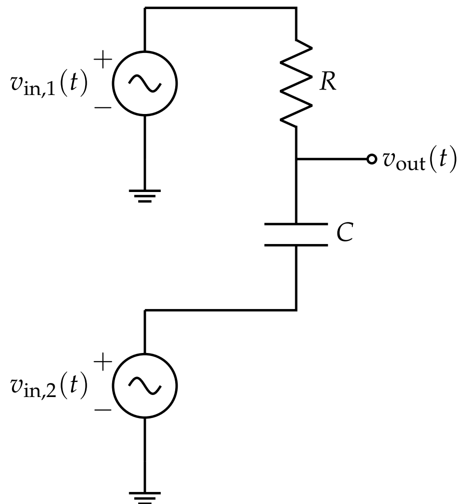
Here, $Z_R = R$, $Z_{C_1} = \frac{1}{j\omega C_1}$, and $Z_{C_2} = \frac{1}{j\omega C_2}$. You may assume that the value of ω is fixed (i.e., the time domain representations of $\widetilde{V}_1$ and $\widetilde{V}_2$ both have the same angular frequency). **Find the Thevenin equivalent impedance between ports** a **and** b**.**

(d) (1 pts.) Again, assume that your angular frequency ω is fixed as was the case in the previous part. Let's say you wanted to connect a lightbulb between ports a and b (a purely resistive/real Z_{Load}). **What should the resistance be, if you wanted to maximize brightness (i.e., maximize power delivered)?**

9. Transfer Function Superposition (20 pts.)

In this problem, we will explore how superposition relates to transfer functions.

Suppose you have the following circuit with two sinusoidal inputs.



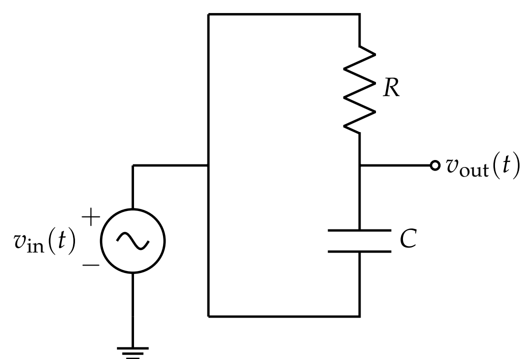
(a) (7 pts.) Find the **transfer function** $H_1(\mathrm{j}\omega_1) = \frac{\widetilde{V}_{\text{out},1}}{\widetilde{V}_{\text{in},1}}$ that describes the contribution of $v_{\text{in},1}(t)$ to the output voltage.

(HINT: What happens to the other voltage sources when superposition is used?)

(b) (7 pts.) Find the **transfer function** $H_2(\mathrm{j}\omega_2) = \frac{\widetilde{V}_{\text{out},2}}{\widetilde{V}_{\text{in},2}}$ that describes the contribution of $v_{\text{in},2}(t)$ to the output voltage.

(c) (3 pts.) Using your answers to the previous parts, **find an expression for** $\widetilde{V}_{\text{out}}$ in terms of $H_1(\mathrm{j}\omega_1)$, $H_2(\mathrm{j}\omega_2)$, $\widetilde{V}_{\text{in},1}$, and $\widetilde{V}_{\text{in},2}$.

(d) (3 pts.) Now, suppose we have the following circuit with one sinusoidal input.



Using your answers to the previous parts, **find** $H(\mathrm{j}\omega) = \frac{\widetilde{V}_{\text{out}}}{\widetilde{V}_{\text{in}}}$ **for this circuit**. Simplify your answer as much as possible.

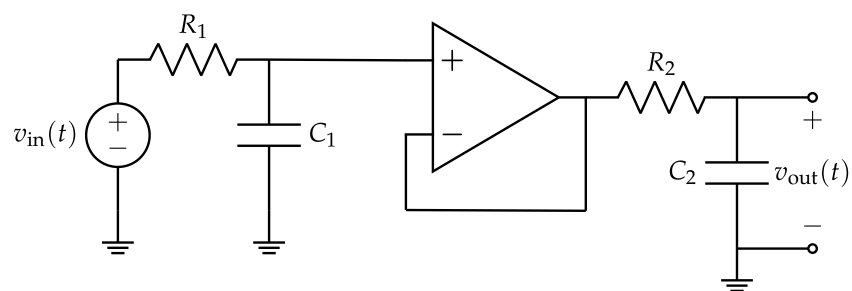
10. Loading Effect Demonstrated (30 pts.)

Consider the following two RC filters:

We know from lecture that these circuits represent low-pass filters. In this problem, we will demonstrate the loading effect and hence motivate the need for a unity gain buffer when cascading filters.

NOTE: For all Bode plotting questions, straight line approximations are fine.

(a) (5 pts.) Consider the following circuit:



Suppose $v_{\text{in}}(t)$ is some sinusoidal input, and the input and output voltage phasors are denoted $\widetilde{V}_{\text{in}}$ and $\widetilde{V}_{\text{out}}$ respectively. **Write the transfer function for this circuit.**

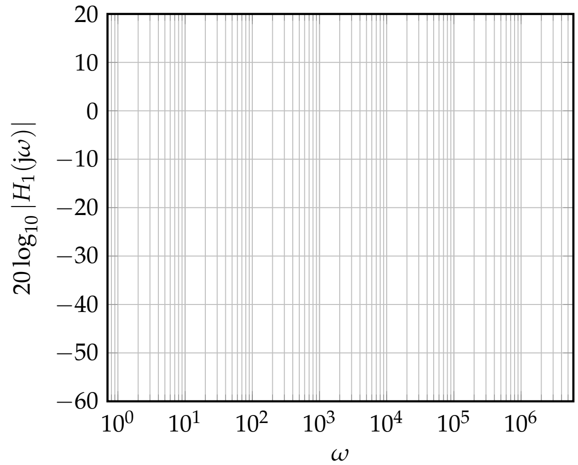
(b) (6 pts.) Regardless of your answer to the previous part, suppose that the transfer function is

$$H(\mathrm{j}\omega) = \frac{1}{\left(1+\frac{\mathrm{j}\omega}{10^3}\right)^2} = \frac{1}{\left(1+\frac{\mathrm{j}\omega}{10^3}\right)\left(1+\frac{\mathrm{j}\omega}{10^3}\right)} \tag{7}$$

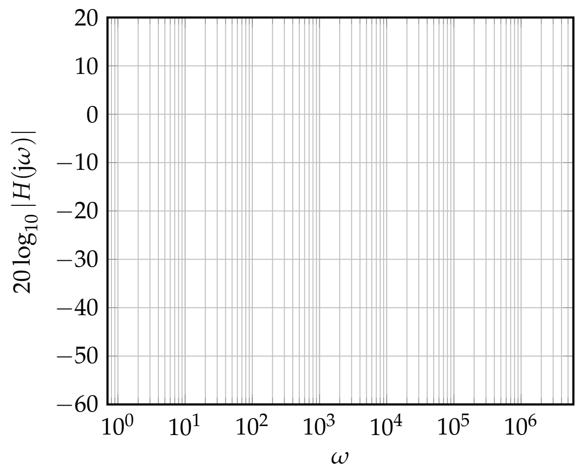
We notice that this transfer function is characterized by two poles at $\omega_c = 10^3$.

i. **First, draw the Bode magnitude plot for**

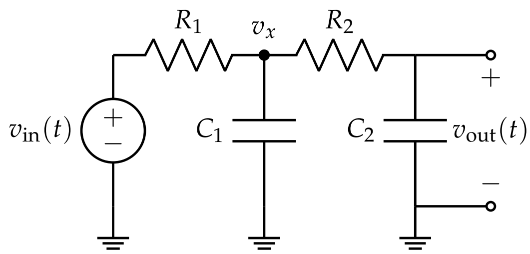
$$H_1(\mathrm{j}\omega) = \frac{1}{1 + \frac{\mathrm{j}\omega}{10^3}} \tag{8}$$



ii. **Next, draw the Bode magnitude plot for** $H(\mathrm{j}\omega)$**.**



(c) (6 pts.) Now, suppose we omitted the unity gain buffer above. This yields the following circuit:



Again, assume $v_{\text{in}}(t)$ is sinusoidal, and the phasor representations of the input and output voltages are $\widetilde{V}_{\text{in}}$ and $\widetilde{V}_{\text{out}}$ respectively. **Find $\widetilde{V}_x$, the phasor representation for v_x in the circuit above, in terms of $\widetilde{V}_{\text{in}}$.**

(d) (2 pts.) **Using your answer to the previous part, find the transfer function $H(\mathrm{j}\omega) = \frac{\widetilde{V}_{\text{out}}}{\widetilde{V}_{\text{in}}}$ that represents this circuit.**

(HINT: Find $\widetilde{V}_{\text{out}}$ in terms of $\widetilde{V}_x$ first.)

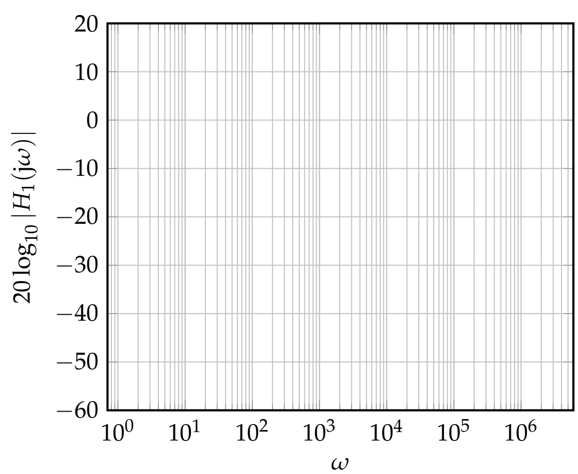
(e) (10 pts.) Regardless of your answer to the previous part, suppose that the transfer function can be written in factored form as

$$H(\mathrm{j}\omega) = \frac{1}{\left(1 + \frac{\mathrm{j}\omega}{10^2}\right)\left(1 + \frac{\mathrm{j}\omega}{10^4}\right)} \tag{9}$$

which has poles at $\omega_c = 10^2$ and $\omega_c = 10^4$.

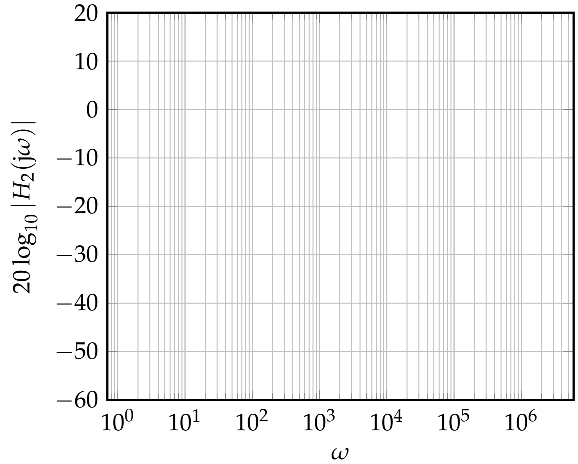
i. **Draw the Bode magnitude plot for the first part of the transfer function (the first pole), i.e.,**

$$H_1(\mathrm{j}\omega) = \frac{1}{1 + \frac{\mathrm{j}\omega}{10^2}} \tag{10}$$

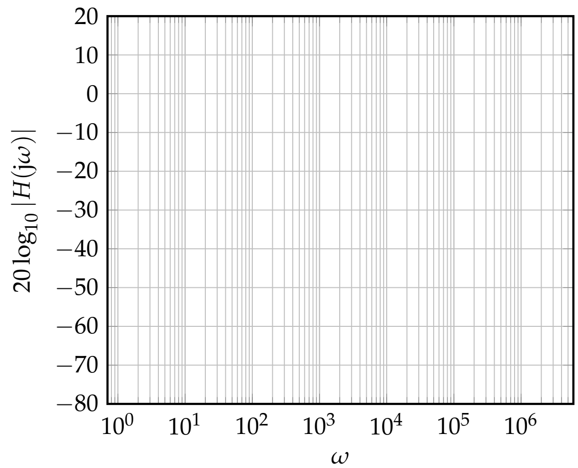


ii. **Draw the Bode magnitude plot for the second part of the transfer function (the second pole), i.e.,**

$$H_2(\mathrm{j}\omega) = \frac{1}{1 + \frac{\mathrm{j}\omega}{10^4}} \tag{11}$$

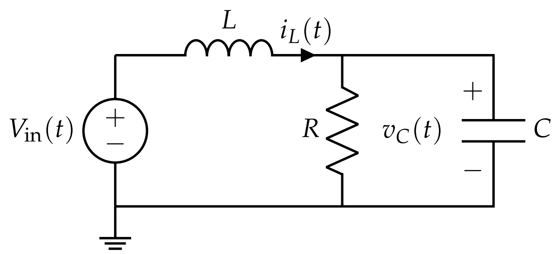


iii. **Draw the combined Bode magnitude plot for $H(j\omega) = H_1(j\omega) \cdot H_2(j\omega)$. Is the Bode plot the same as the one for the circuit with a unity gain buffer?**



11. RLC Vector Differential Equation (20 pts.)

In this problem, you will approach solving the following RLC circuit using *the vector differential equation method*.



(a) (6 pts.) **Derive a set of differential equations, one for $I_L(t)$ and another for $V_C(t)$.**

(b) (6 pts.) Suppose our vector differential equation was written as follows:

$$\frac{\mathrm{d}}{\mathrm{d}t}\vec{x}(t) = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \vec{x}(t) + \begin{bmatrix} e \\ f \end{bmatrix} V_{\text{in}}(t) \tag{12}$$

The state vector is defined as $\vec{x}(t) = \begin{bmatrix} V_C(t) \\ I_L(t) \end{bmatrix}$. **Find the values of a, b, c, d, e, and f in terms of R, L, C.**

(c) (6 pts.) Suppose you are now told that the system is given by:

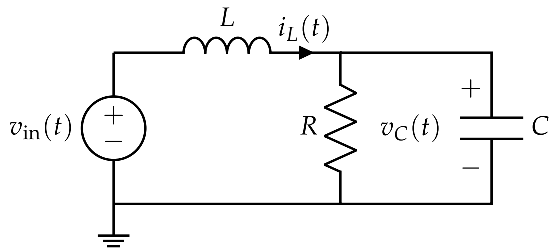
$$\frac{\mathrm{d}}{\mathrm{d}t}\vec{x}(t) = \begin{bmatrix} -6 & 8 \\ -1 & 0 \end{bmatrix} \vec{x}(t) + \begin{bmatrix} 0 \\ 1 \end{bmatrix} V_{\text{in}}(t) \tag{13}$$

Given that the eigenvalues of the above state matrix are -4 and -2 with eigenvectors $\begin{bmatrix} 4 \\ 1 \end{bmatrix}$ and $\begin{bmatrix} 2 \\ 1 \end{bmatrix}$ respectively, **rewrite the given system of differential equations in the diagonalized basis ($\vec{\widetilde{x}}(t) = \begin{bmatrix} \widetilde{V}_C(t) \\ \widetilde{I}_L(t) \end{bmatrix}$).**

(d) (2 pts.) Your initial conditions in the natural basis are given symbolically as $V_C(0)$ and $I_L(0)$. **Find the initial conditions in the diagonalized basis ($\widetilde{V}_C(0)$ and $\widetilde{I}_L(0)$).**

12. RLC 2nd Order Differential Equation (30 pts.)

In this problem, you will approach solving the same RLC circuit as the previous problem using your knowledge of 2nd Order Differential Equations.



(a) (8 pts.) **Find the differential equation for** $v_C(t)$ in the following form (where a, b, and c are constants that depend on R, L, and C):

$$\frac{\mathrm{d}^2 v_C(t)}{\mathrm{d}t^2} + a\frac{\mathrm{d}v_C(t)}{\mathrm{d}t} + bv_C(t) = cv_{\text{in}}(t) \tag{14}$$

(b) (6 pts.) Suppose that you found the differential equation

$$\frac{\mathrm{d}^2 v_C(t)}{\mathrm{d}t^2} + 6\frac{\mathrm{d}v_C(t)}{\mathrm{d}t} + 8v_C(t) = 8v_{\mathrm{in}}(t) \tag{15}$$

The general homogeneous solution $v_{C,h}(t)$ to this differential equation can be written in the following form:

$$v_{C,h}(t) = C_1 \mathrm{e}^{s_1 t} + C_2 \mathrm{e}^{s_2 t} \tag{16}$$

Find numerical values for s_1 **and** s_2 for this differential equation (the order of s_1 and s_2 does not matter).

(c) (6 pts.) Suppose $v_{\mathrm{in}}(t) = V_0 \cos\left(\frac{1}{\sqrt{LC}} t\right)$. Find the **particular solution** $v_{C,p}(t)$ for this differential equation (in terms of R, L, C, and V_0).

(HINT: Remember the relationship between the particular solution and steady state solution. What method can you use to find the steady state solution when the input is sinusoidal?)

(d) (4 pts.) To solve the 2nd Order Differential Equation, we need initial conditions for both $v_C(t)$ and $v_C'(t) = \frac{\mathrm{d}v_C(t)}{\mathrm{d}t}$. Suppose we know that $v_C(0) = 0$ and $i_L(0) = 0$. **Find $v_C'(0)$.**

(e) (6 pts.) Suppose that you found that the particular solution is $v_{C,p}(t) = \sin(\omega_p t)$. Then, you could write your total solution as

$$v_C(t) = v_{C,p}(t) + v_{C,h}(t) = \sin(\omega_p t) + C_1 e^{s_1 t} + C_2 e^{s_2 t} \tag{17}$$

Using the initial conditions $v_C(0)$ and $v_C'(0)$, **set up a system of equations to solve for C_1 and C_2** (your equations can contain the constants $v_C(0)$, $v_C'(0)$, ω_p, s_1, and s_2).

[Doodle page! Draw us something if you want or give us suggestions or complaints. You can also use this page to report anything suspicious that you might have noticed.

If needed, you can also use this space to work on problems. But if you want the work on this page to be graded, make sure you tell us on the problem's main page.]

[Extra page. If you want the work on this page to be graded, make sure you tell us on the problem's main page.]

