

**The midterm redo is due on Monday, March 28, 2022, at 11:59PM.**

## 1. Complex Numbers

You are given the graph in Figure 1.

In this problem, you may use the  $\text{atan2}(b, a)$  function to compute the angle (phase) for the complex number  $a + jb$  as necessary.

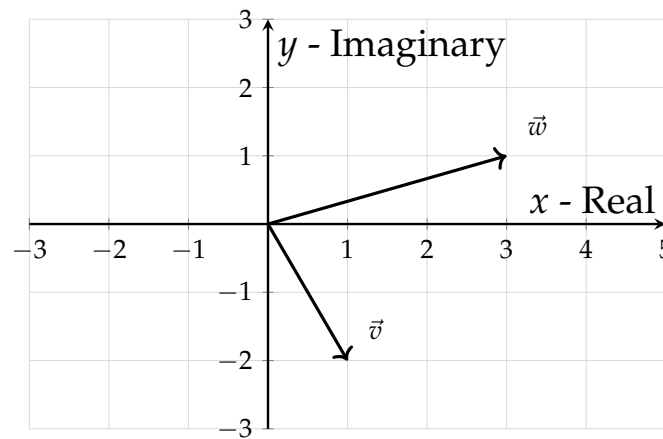


Figure 1: Vectors in the  $x - y$  plane

- (a) What are the Cartesian  $((x, y))$  and Polar  $(re^{j\theta})$  coordinates of  $\vec{v}$ ?

**Solution:** Vector  $\vec{v}$  Cartesian =  $(1, -2)$

Vector  $\vec{v}$  Polar =  $\sqrt{5}e^{j\text{atan2}(-2,1)}$

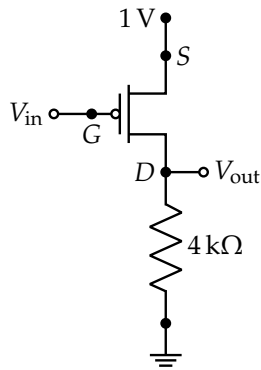
- (b) What are the Cartesian  $((x, y))$  and Polar  $(re^{j\theta})$  coordinates of  $\vec{w}$ ?

**Solution:** Vector  $\vec{w}$  Cartesian =  $(3, 1)$

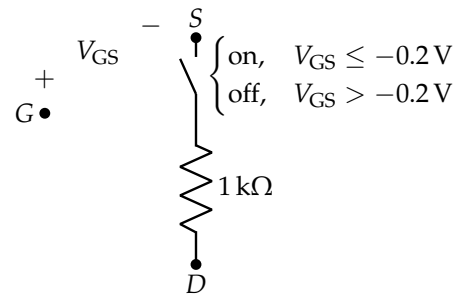
Vector  $\vec{w}$  Polar =  $\sqrt{10}e^{j\text{atan2}(1,3)}$

## 2. PMOS Transistor Inverter

Consider the following schematic and PMOS model.



(a) A PMOS transistor circuit

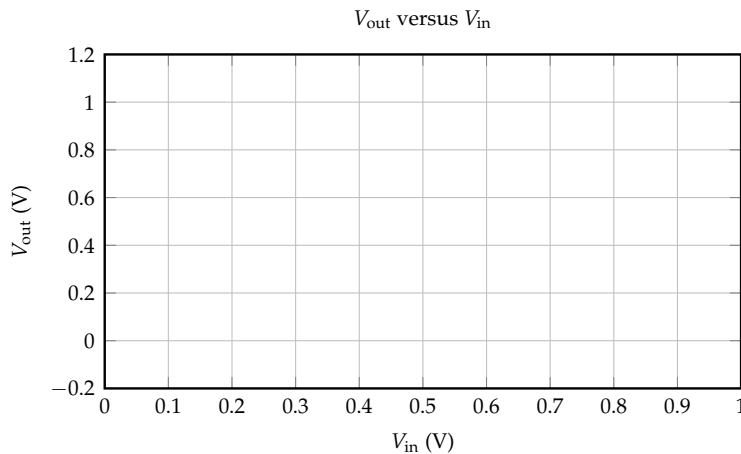


(b) Resistor and switch model for PMOS transistor.

Figure 2: PMOS figures.

Please plot the output  $V_{out}$  for the input  $V_{in}$  ranging from 0 V to 1 V. Justify your answer.

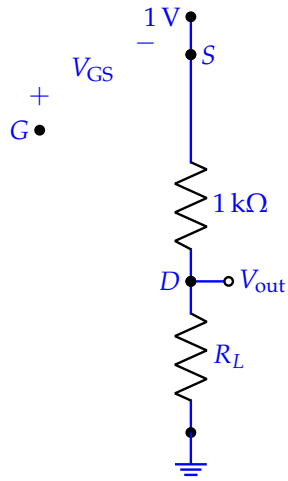
NOTE: The  $y$ -axis ticks starts from  $-0.2$  V.



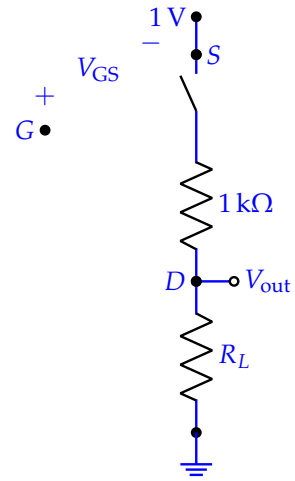
(I)  $V_{out}$  versus  $V_{in}$  Graph

**Solution:** First substitute in the transistor model, and examine  $V_{GS}$  to see if we expect the PMOS to be on or off. When  $V_{in} \leq 0.8$  V, since  $V_{GS} \leq -0.2$  V, the switch is on and the PMOS circuit becomes figure 4a. We have a voltage divider.  $V_{out} = \frac{R_L}{1k\Omega + R_L} 1 \text{ V} = \frac{4k\Omega}{1k\Omega + 4k\Omega} 1 \text{ V} = 0.8 \text{ V}$ .

When  $V_{in} > 0.8$  V, as shown in figure 4b, the switch is off, and  $V_{out} = 0$  V.



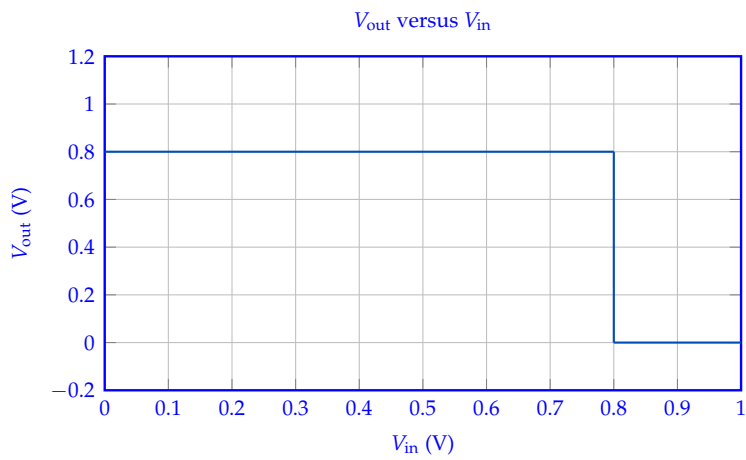
(a) PMOS circuit with transistor model substituted in.



(b) PMOS circuit with transistor model substituted in.

Figure 4: PMOS figures.

Therefore, the correct plot will be like:

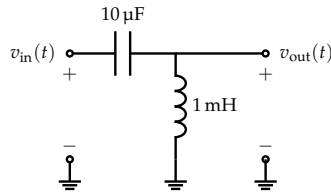


(I)  $V_{out}$  versus  $V_{in}$  Graph

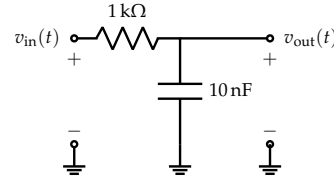
3. Filter Circuits

(a) Below, you have filter circuits A, B, C, D, each with specific component values. **Fill in the bubbles to match each filter to its corresponding magnitude transfer function plot out of choices I, II, III, IV.** Note that each plot may be assigned to filters **once, more than once, or not at all.** Each filter has **exactly one** corresponding plot.

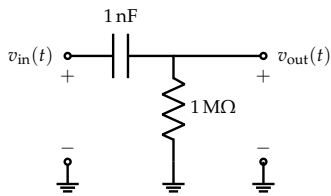
**SI Prefixes and Exponent definitions:** nano (n):  $10^{-9}$ ; micro ( $\mu$ ):  $10^{-6}$ ; milli (m):  $10^{-3}$ ; kilo (k):  $10^3$ ; mega (M):  $10^6$ ; giga (G):  $10^9$



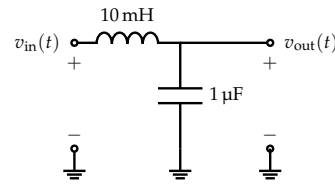
(A) Filter A.



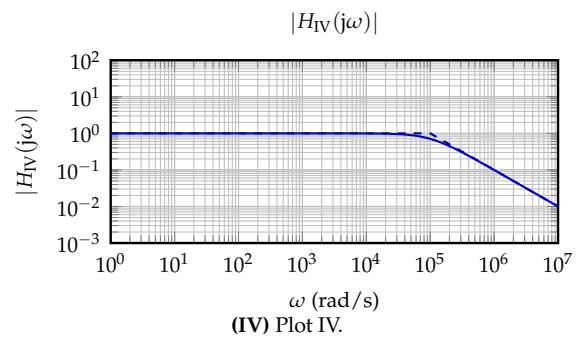
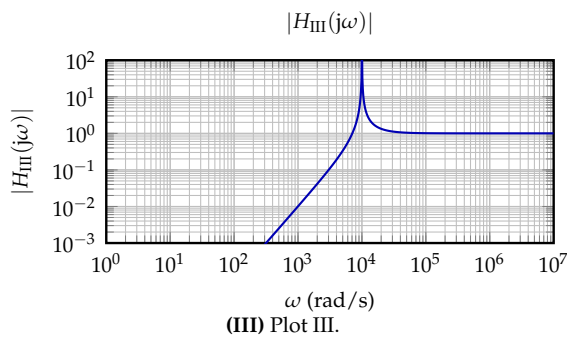
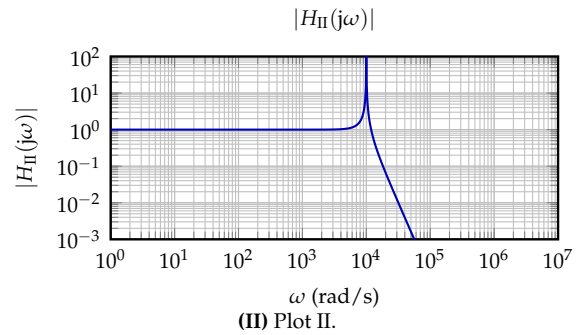
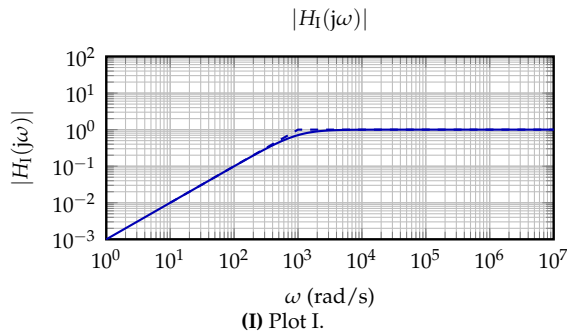
(B) Filter B.



(C) Filter C.



(D) Filter D.



| Filter Letter | Plot I                | Plot II               | Plot III              | Plot IV               |
|---------------|-----------------------|-----------------------|-----------------------|-----------------------|
| A             | <input type="radio"/> | <input type="radio"/> | <input type="radio"/> | <input type="radio"/> |
| B             | <input type="radio"/> | <input type="radio"/> | <input type="radio"/> | <input type="radio"/> |
| C             | <input type="radio"/> | <input type="radio"/> | <input type="radio"/> | <input type="radio"/> |
| D             | <input type="radio"/> | <input type="radio"/> | <input type="radio"/> | <input type="radio"/> |

**Solution:**

| Filter Letter | Plot I                           | Plot II                          | Plot III                         | Plot IV                          |
|---------------|----------------------------------|----------------------------------|----------------------------------|----------------------------------|
| A             | <input type="radio"/>            | <input type="radio"/>            | <input checked="" type="radio"/> | <input type="radio"/>            |
| B             | <input type="radio"/>            | <input type="radio"/>            | <input type="radio"/>            | <input checked="" type="radio"/> |
| C             | <input checked="" type="radio"/> | <input type="radio"/>            | <input type="radio"/>            | <input type="radio"/>            |
| D             | <input type="radio"/>            | <input checked="" type="radio"/> | <input type="radio"/>            | <input type="radio"/>            |

First of all, (A) and (D) are both LC circuits, where a sharp peak can be observed at  $\omega = \frac{1}{\sqrt{LC}}$ , therefore, they map to Plot II and III. For filter (A) and (D), the capacitor becomes short when  $\omega$  goes to  $\infty$ , then we have  $v_{\text{out}} = v_{\text{in}}$  for (A) and  $v_{\text{out}} = 0$  for (D). Therefore, filter (A)  $\rightarrow$  Plot III, filter (D)  $\rightarrow$  Plot II.

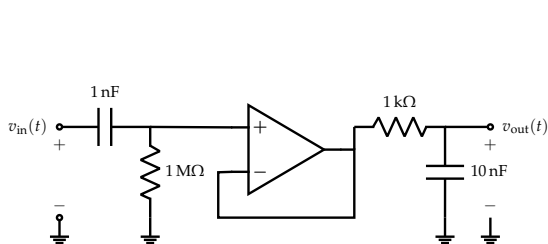
(B) and (C) are RC circuits, where (B) is a first-order low-pass filter and (C) is a first-order high-pass filter, therefore, filter (B)  $\rightarrow$  Plot IV, filter (C)  $\rightarrow$  Plot I.

For reference, we provide the following transfer functions:

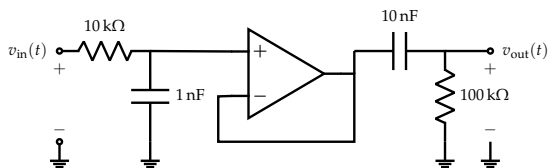
| Filter Letter | $H(j\omega)$   |
|---------------|--|
| A             | $H(j\omega) = \frac{j\omega L}{\frac{1}{j\omega C} + j\omega L}$           |
| B             | $H(j\omega) = \frac{\frac{1}{j\omega C}}{\frac{1}{j\omega C} + R}$         |
| C             | $H(j\omega) = \frac{R}{\frac{1}{j\omega C} + R}$                           |
| D             | $H(j\omega) = \frac{\frac{1}{j\omega C}}{\frac{1}{j\omega C} + j\omega L}$ |

- (b) Now, in order to design a band-pass filter, one possible way is to cascade two filters above. Below, you have filter circuits A, B, C, each with specific component values. **Fill in the bubbles to match each filter to its corresponding magnitude transfer function plot out of choices I, II, III.**

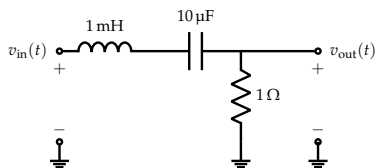
Note that each plot may be assigned to filters **once, more than once, or not at all**. Each filter has **exactly one** corresponding plot.



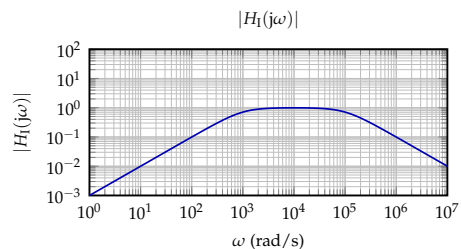
(A) Filter A.



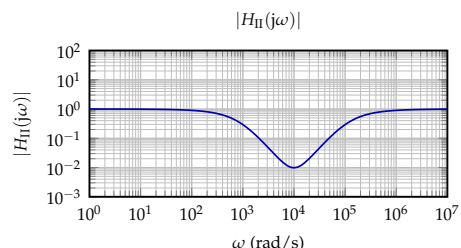
(B) Filter B.



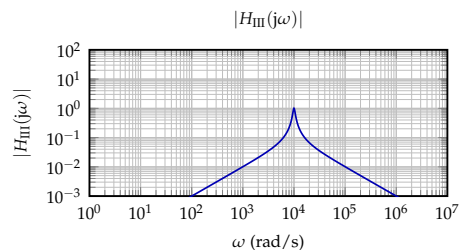
(C) Filter C.



(I) Plot I.



(II) Plot II.



III Plot III.

| Filter Letter | Plot I                | Plot II               | Plot III              |
|---------------|-----------------------|-----------------------|-----------------------|
| A             | <input type="radio"/> | <input type="radio"/> | <input type="radio"/> |
| B             | <input type="radio"/> | <input type="radio"/> | <input type="radio"/> |
| C             | <input type="radio"/> | <input type="radio"/> | <input type="radio"/> |

**Solution:**

| Filter Letter | Plot I                           | Plot II               | Plot III                         |
|---------------|----------------------------------|-----------------------|----------------------------------|
| A             | <input checked="" type="radio"/> | <input type="radio"/> | <input type="radio"/>            |
| B             | <input checked="" type="radio"/> | <input type="radio"/> | <input type="radio"/>            |
| C             | <input type="radio"/>            | <input type="radio"/> | <input checked="" type="radio"/> |

First of all, filter (A) and (B) have the same transfer functions, which equals to the product of two transfer functions (left and right). Swapping the order doesn't change the overall transfer function.

Filter (C) is a RLC band-pass filter which has a much sharper peak, which corresponds to Plot III.

For reference, we provide the following transfer functions:

| Filter Letter | $H(j\omega)$   |
|---------------|--|
| A             | $H(j\omega) = \frac{1}{j\omega C} \cdot \frac{R}{\frac{1}{j\omega C} + R}$ |
| B             | $H(j\omega) = \frac{1}{j\omega C} \cdot \frac{R}{\frac{1}{j\omega C} + R}$ |
| C             | $H(j\omega) = \frac{R}{\frac{1}{j\omega C} + R + j\omega L}$               |

#### 4. Magnitude, Phase, and Cascades

Suppose you have the transfer function  $H(j\omega)$  for a system as given in eq. (1) below:

$$H(j\omega) = \frac{20}{1 + j\frac{\omega}{\omega_0}} \quad (1)$$

where  $\omega_0 := 1 \times 10^7 \frac{\text{rad}}{\text{s}}$ .

Answer the following questions.

- (a) What is the transfer function's magnitude  $|H(j\omega)|$  at  $\omega = 0 \frac{\text{rad}}{\text{s}}$ ?

**Solution:**  $|H(j \cdot 0)| = \left| \frac{20}{1 + j\frac{0}{10^7}} \right| = 20.$

- (b) What is the transfer function's magnitude  $|H(j\omega)|$  at  $\omega = \infty \frac{\text{rad}}{\text{s}}$ ?

**Solution:**  $|H(j \cdot \infty)| = \left| \frac{20}{1 + j\frac{\infty}{10^7}} \right| = 0.$

- (c) What is the transfer function's phase  $\angle H(j\omega)$  at  $\omega = 1 \times 10^7 \frac{\text{rad}}{\text{s}}$ ?

**Solution:**  $\angle H(j \cdot 1 \times 10^7 \frac{\text{rad}}{\text{s}}) = \angle \left( \frac{20}{1 + j\frac{10^7}{10^7}} \right) = \angle \left( \frac{20}{1 + j} \right) = -\frac{\pi}{4} \text{ rad or } -45^\circ$

- (d) You cascade the systems  $S_1$  as defined by the transfer function:

$$H_1(j\omega) = \frac{20}{1 + j\frac{\omega}{\omega_1}} \quad (2)$$

with another system  $S_2$  as defined by the transfer function:

$$H_2(j\omega) = \frac{100}{1 + j\frac{\omega}{\omega_2}} \quad (3)$$

where  $\omega_1 := 1 \times 10^7 \frac{\text{rad}}{\text{s}}$  and  $\omega_2 := 1 \times 10^4 \frac{\text{rad}}{\text{s}}$ .

You place  $S_2$  after  $S_1$ , with unity-gain buffers in between. **Write the overall transfer function  $H_{\text{cascade}}(j\omega)$  in terms of  $j\omega$ . You do not need to simplify your answer for this subpart.** **Solution:**

Cascading filters leads to the product of their transfer functions as the output of one filter will become the input of the next. Thus we have:  $H_{\text{cascade}}(j\omega) = H_1(j\omega) \times H_2(j\omega) = \left( \frac{20}{1 + j\frac{\omega}{10^7}} \right) \times \left( \frac{100}{1 + j\frac{\omega}{10^4}} \right).$



### 5. Stability of Discrete-Time System

Suppose we are working with a linear model which has the form:

$$\underbrace{\begin{bmatrix} x_1[i+1] \\ x_2[i+1] \end{bmatrix}}_{\vec{x}[i+1]} = \underbrace{\begin{bmatrix} 0 & 1 \\ k & 1 \end{bmatrix}}_{A_d} \underbrace{\begin{bmatrix} x_1[i] \\ x_2[i] \end{bmatrix}}_{\vec{x}[i]} + \begin{bmatrix} 0 \\ 1 \end{bmatrix} w[i], \quad (4)$$

where  $k \in \mathbb{R}$  is an unknown variable.

- (a) **Give the range of  $k$ , such that the matrix  $A_d$  has only real eigenvalues.** *Justify your answer.*

**Solution:** To find the eigenvalues of  $A_d$ , we will solve  $\det(A_d - \lambda I) = 0$ . In other words, we aim to find  $\lambda$ , such that

$$\det(A_d - \lambda I) = \det\left(\begin{bmatrix} 0 - \lambda & 1 \\ k & 1 - \lambda \end{bmatrix}\right) \quad (5)$$

$$= \lambda(\lambda - 1) - k \quad (6)$$

$$= \lambda^2 - \lambda - k \quad (7)$$

$$= 0. \quad (8)$$

The quadratic formula gives

$$\lambda = \frac{1 \pm \sqrt{1 + 4k}}{2} \quad (9)$$

Now, in order to find the range of  $k$  to ensure that  $\lambda$ s are real,  $1 + 4k$  has to be greater than or equal to 0. Therefore, the answer is  $k \geq -\frac{1}{4}$ .

- (b) **Choose the possible  $k$  value(s) from the following options such that the above model is stable.** *Select all choices that apply.*

i.  $k = -\frac{3}{4}$ .

ii.  $k = 1$ .

iii.  $k = -\frac{1}{4}$ .

iv.  $k = \frac{1}{4}$ .

**Solution:** The condition of a stable system is that the magnitudes of all the eigenvalues are less than 1.

As given by equation 9 and the previous sub-problem, when  $k \geq -\frac{1}{4}$ , the eigenvalues are real. Therefore, to make sure the system is stable, we need to have

$$-1 < \frac{1 - \sqrt{1 + 4k}}{2} \leq \frac{1 + \sqrt{1 + 4k}}{2} < 1. \quad (10)$$

Solving it we have  $-\frac{1}{4} \leq k < 0$ .

On the other hand, when  $k < -\frac{1}{4}$ , the eigenvalues become not purely real, and we have

$$\lambda = \frac{1 \pm j\sqrt{-1 - 4k}}{2}, \quad (11)$$

Now, for both eigenvalues,

$$|\lambda| = \frac{\sqrt{1 + (-1 - 4k)}}{2} = \frac{\sqrt{-4k}}{2} < 1. \quad (12)$$

Solving it we have  $-\frac{1}{4} > k > -1$ .

In summary, to ensure that the system is stable, the range of  $k$  is  $-1 < k < 0$ .

Therefore, (i) and (iii) are the correct answers.

## 6. Controllability and Eigenvalue Placement

Suppose we are working with a linear model with two-dimensional state  $\vec{x}: \mathbb{N} \rightarrow \mathbb{R}^2$  but one-dimensional input  $u: \mathbb{N} \rightarrow \mathbb{R}$ :

$$\underbrace{\begin{bmatrix} x_1[i+1] \\ x_2[i+1] \end{bmatrix}}_{=\vec{x}[i+1]} = \underbrace{\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}}_{=A} \underbrace{\begin{bmatrix} x_1[i] \\ x_2[i] \end{bmatrix}}_{=\vec{x}[i]} + \underbrace{\begin{bmatrix} b_1 \\ b_2 \end{bmatrix}}_{=\vec{b}} u[i] \quad (13)$$

where  $b_1 \neq 0$  and  $b_2 \neq 0$ .

- (a) Show that the model in Equation (13) is not controllable.

**Solution:** The controllability matrix of Equation (13) is

$$C_2 := \begin{bmatrix} A\vec{b} & \vec{b} \end{bmatrix} = \begin{bmatrix} \vec{b} & \vec{b} \end{bmatrix} \quad (14)$$

which is (column) rank 1 and thus not controllable.

- (b) Suppose we add feedback control of the form

$$u[i] := \underbrace{\begin{bmatrix} f_1 & f_2 \end{bmatrix}}_{=\vec{f}^\top} \underbrace{\begin{bmatrix} x_1[i] \\ x_2[i] \end{bmatrix}}_{=\vec{x}[i]}. \quad (15)$$

Show that one of the eigenvalues of  $A_{CL} := A + \vec{b}\vec{f}^\top$  is 1, regardless of the values of  $f_1$  and  $f_2$ .

**Solution:**

First Solution:

Since  $A = I$ , we can write

$$A_{CL} - \lambda I = A + \vec{b}\vec{f}^\top - \lambda I \quad (16)$$

$$= I + \vec{b}\vec{f}^\top - \lambda I \quad (17)$$

$$= (1 - \lambda)I + \vec{b}\vec{f}^\top \quad (18)$$

and if  $\lambda = 1$  then  $A_{CL} - \lambda I$  is not full rank (i.e., is rank 1). Hence  $\lambda$  is an eigenvalue of  $A_{CL}$ .

Alternative Solution:

We calculate

$$A_{CL} = A + \vec{b}\vec{f}^\top \quad (19)$$

$$= \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} + \begin{bmatrix} b_1 \\ b_2 \end{bmatrix} \begin{bmatrix} f_1 & f_2 \end{bmatrix} \quad (20)$$

$$= \begin{bmatrix} 1 + b_1 f_1 & b_1 f_2 \\ b_2 f_1 & 1 + b_2 f_2 \end{bmatrix}. \quad (21)$$

Then we have

$$A_{CL} - \lambda I = \begin{bmatrix} 1 + b_1 f_1 - \lambda & b_1 f_2 \\ b_2 f_1 & 1 + b_2 f_2 - \lambda \end{bmatrix}. \quad (22)$$

At this point one can plug in  $\lambda = 1$  and conclude  $A_{CL} - I$  has rank 1, so 1 is an eigenvalue of  $A_{CL}$ . This is another alternate solution.

Continuing on, the characteristic polynomial is

$$p_{A_{CL}}(\lambda) := \det(A_{CL} - \lambda I) \quad (23)$$

$$= \det\left(\begin{bmatrix} 1 + b_1 f_1 - \lambda & b_1 f_2 \\ b_2 f_1 & 1 + b_2 f_2 - \lambda \end{bmatrix}\right) \quad (24)$$

$$= (1 + b_1 f_1 - \lambda)(1 + b_2 f_2 - \lambda) - b_1 b_2 f_1 f_2 \quad (25)$$

$$= \lambda^2 - (b_1 f_1 + b_2 f_2 + 2)\lambda + (b_1 f_1 + b_2 f_2 + 1). \quad (26)$$

Plugging in  $\lambda = 1$  gives

$$p_{A_{CL}}(1) = 1^2 - (b_1 f_1 + b_2 f_2 + 2) \cdot 1 + (b_1 f_1 + b_2 f_2 + 1) \quad (27)$$

$$= (1 - b_1 f_1 - b_2 f_2 - 2) + (b_1 f_1 + b_2 f_2 + 1) \quad (28)$$

$$= -(b_1 f_1 + b_2 f_2 + 1) + (b_1 f_1 + b_2 f_2 + 1) \quad (29)$$

$$= 0 \quad (30)$$

so 1 is an eigenvalue of  $A_{CL}$  as desired.

How to solve this if you didn't know an eigenvalue:

Suppose  $A_{CL}$  has eigenvalues  $\lambda_1$  and  $\lambda_2$ . Since the leading coefficient of  $\lambda$  in  $p_{A_{CL}}(\lambda)$  is 1, we can expand

$$p_{A_{CL}}(\lambda) = (\lambda - \lambda_1)(\lambda - \lambda_2) \quad (31)$$

$$= \lambda^2 - (\lambda_1 + \lambda_2)\lambda + \lambda_1 \lambda_2. \quad (32)$$

Thus by matching coefficients of powers of  $\lambda$ ,

$$\lambda_1 + \lambda_2 = b_1 f_1 + b_2 f_2 + 2 \quad (33)$$

$$\lambda_1 \lambda_2 = b_1 f_1 + b_2 f_2 + 1. \quad (34)$$

This system is solved by

$$\lambda_1 = b_1 f_1 + b_2 f_2 + 1 \quad \lambda_2 = 1 \quad (35)$$

or the opposite assignment. Thus one of the eigenvalues is 1, no matter what the feedback  $f_1, f_2$  are, though the other eigenvalue can be set to whatever we want.

### 7. Brain Stimulation

For his neuron-modeling project Krishna thought of consulting his close friend Radhika, who is a neuroscientist. According to Radhika’s suggestions, Krishna came up with the following model of the cell-membrane of a neuron:

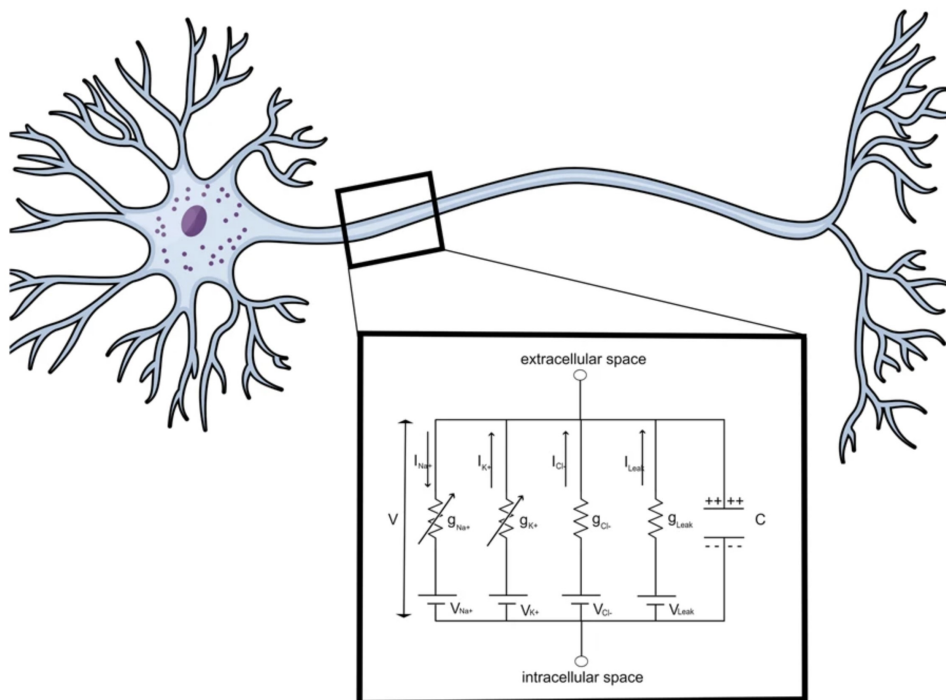


Figure 9: Electrical model of the neuron membrane

- (a) Now Krishna wants to see how the neuron behaves to an external current stimulus. As he found the complete model very difficult to analyze, he starts his analysis with the following simple model:

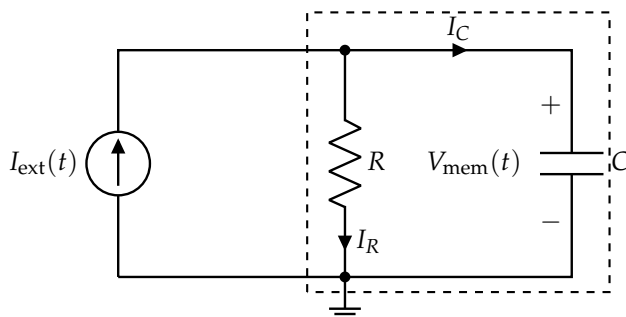
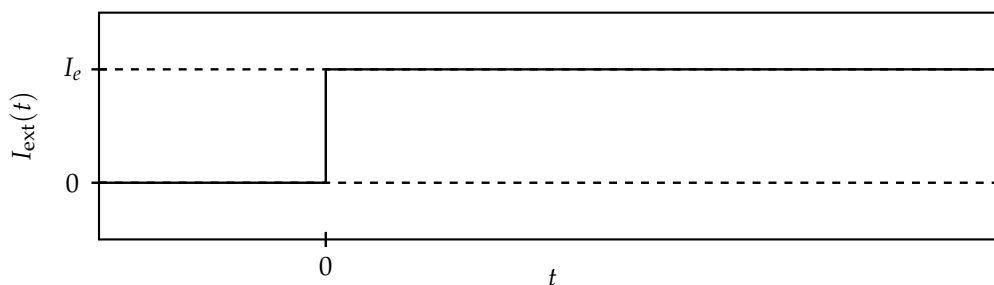


Figure 10: Simplified circuit model of a neuron membrane with an external current stimulus.

For all parts of this problem the external stimulus  $I_{ext}(t)$  is a piece-wise constant function as shown below:



- i. Find the value of  $V_{\text{mem}}(0)$  assuming the system reached steady-state for  $t < 0$ .

**Solution:** As  $I_{\text{ext}}(t) = 0$  for  $t < 0$ , we can conclude  $V_{\text{mem}}(0) = 0$ .

- ii. Solve for  $V_{\text{mem}}(t)$  where  $t \geq 0$ . Show your work.

**Solution:** KCL for the given circuit model can be written as

$$I_{\text{ext}}(t) = \frac{V_{\text{mem}}(t)}{R} + C \frac{d}{dt} V_{\text{mem}}(t) \quad (36)$$

$$\frac{d}{dt} V_{\text{mem}}(t) = -\frac{V_{\text{mem}}(t)}{RC} + \frac{I_{\text{ext}}(t)}{C} \quad (37)$$

Note that for  $t \geq 0$ ,  $I_{\text{ext}}(t) = I_e$ . Hence, substituting  $(V_{\text{mem}}(t) - RI_e)$  with  $\tilde{V}_{\text{mem}}(t)$  we get

$$\frac{d}{dt} \tilde{V}_{\text{mem}}(t) = -\frac{\tilde{V}_{\text{mem}}(t)}{RC} \quad (38)$$

Solving eq. 38 we get

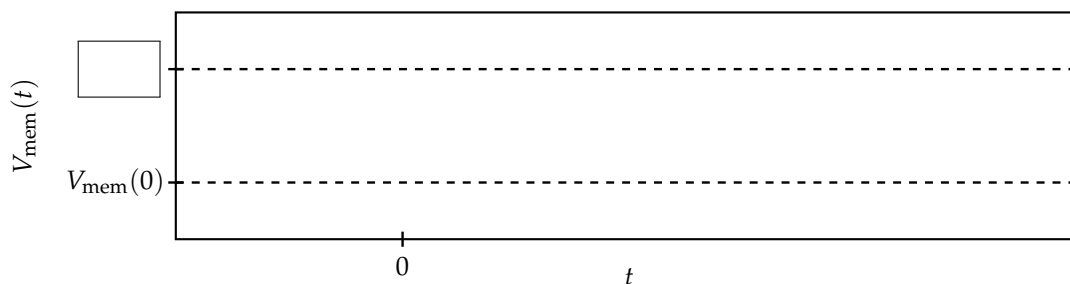
$$\tilde{V}_{\text{mem}}(t) = K_0 e^{-\frac{t}{RC}} \quad (39)$$

As  $V_{\text{mem}}(0) = 0$ , we have  $\tilde{V}_{\text{mem}}(0) = -RI_e$ . Applying this initial condition in eq. 39, we get

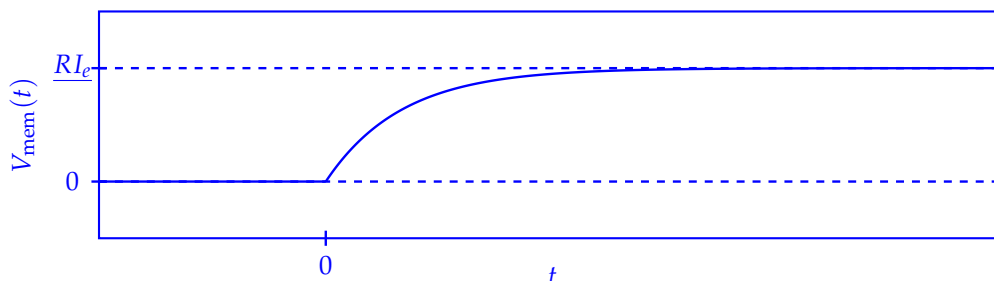
$$\tilde{V}_{\text{mem}}(t) = -RI_e e^{-\frac{t}{RC}} \quad (40)$$

$$V_{\text{mem}}(t) = RI_e \left(1 - e^{-\frac{t}{RC}}\right) \quad (41)$$

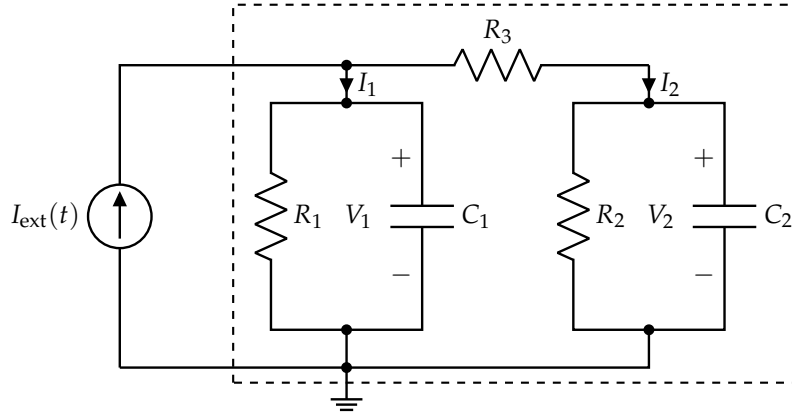
- iii. Qualitatively sketch  $V_{\text{mem}}(t)$  on the below plot, and label the steady-state value by filling in the un-filled  $y$ -axis label.



**Solution:** A sketch of  $V_{\text{mem}}(t)$  is shown below:



- (b) As a part of his project, Krishna needs to measure the neural potentials. However, in the measurement process multiple neurons can come into contact. He came up with the following circuit modeling two neuron membranes in contact with each other.



**Figure 11:** Simplified circuit model for two neuron membranes in contact.

By doing nodal analysis of the circuit, he found that the membrane voltages  $V_1(t)$  and  $V_2(t)$  are related to the external current stimulus ( $I_{\text{ext}}(t)$ ) through the following vector differential equation:

$$\frac{d}{dt} \begin{bmatrix} V_1(t) \\ V_2(t) \end{bmatrix} = \begin{bmatrix} a & -\frac{1}{R_3 C_1} \\ \frac{1}{R_3 C_2} & -\frac{1}{(R_2 || R_3) C_2} \end{bmatrix} \begin{bmatrix} V_1(t) \\ V_2(t) \end{bmatrix} + \begin{bmatrix} b \\ 0 \end{bmatrix} I_{\text{ext}}(t) \quad (42)$$

where  $R_i || R_j = \frac{R_i R_j}{R_i + R_j}$ . **Find expressions for  $a$  and  $b$  in terms of  $R_1, R_2, R_3, C_1$  and  $C_2$ .**

**Solution:** Applying KCL and element equations on the circuit gives the following set of equations

$$I_1(t) = \frac{V_1(t)}{R_1} + C_1 \frac{d}{dt} V_1(t) \quad (43)$$

$$I_2(t) = \frac{V_2(t)}{R_2} + C_2 \frac{d}{dt} V_2(t) \quad (44)$$

$$I_1(t) + I_2(t) = I_{\text{ext}}(t) \quad (45)$$

$$I_2(t) = \frac{V_1(t) - V_2(t)}{R_3} \quad (46)$$

substituting eq. 46 in eq. 45 we get

$$I_1(t) = I_{\text{ext}}(t) - \frac{V_1(t) - V_2(t)}{R_3} \quad (47)$$

substituting the results from eq. 47 and eq. 46 in eq. 43 and eq. 44 respectively gives us the following differential equations of  $V_1$  and  $V_2$ ,

$$\frac{d}{dt} V_1(t) = -\frac{V_1(t)}{R_1 C_1} + \frac{I_{\text{ext}}(t)}{C_1} - \frac{V_1 - V_2}{R_3 C_1} \quad (48)$$

$$\frac{d}{dt} V_2(t) = -\frac{V_2(t)}{R_2 C_2} + \frac{V_1(t) - V_2(t)}{R_3 C_2} \quad (49)$$

Hence, the resulting vector differential equation becomes,

$$\frac{d}{dt} \begin{bmatrix} V_1(t) \\ V_2(t) \end{bmatrix} = \begin{bmatrix} -\frac{1}{(R_1 || R_3) C_1} & \frac{1}{R_3 C_1} \\ \frac{1}{R_3 C_2} & -\frac{1}{(R_2 || R_3) C_2} \end{bmatrix} \begin{bmatrix} V_1(t) \\ V_2(t) \end{bmatrix} + \begin{bmatrix} \frac{1}{C_1} \\ 0 \end{bmatrix} I_{\text{ext}}(t) \quad (50)$$

Which means,

$$a = -\frac{1}{(R_1 || R_3)C_1} \quad (51)$$

$$b = \frac{1}{C_1} \quad (52)$$

- (c) Suppose for some appropriate component values, the vector differential equation 42 can be written in the following form

$$\frac{d}{dt}\vec{x}(t) = \begin{bmatrix} -30 & 10 \\ 10 & -30 \end{bmatrix} \vec{x}(t) + \begin{bmatrix} 10^3 \\ 0 \end{bmatrix} u(t) \quad (53)$$

where  $\vec{x}(t) = \begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix} = \begin{bmatrix} V_1(t) \\ V_2(t) \end{bmatrix}$  and  $u(t) = I_{\text{ext}}(t)$ . The external current source,  $I_{\text{ext}}(t)$  is same as in part (a) with  $I_e = 10$  mA. Plugging the value of  $I_{\text{ext}}(t)$ , for  $t > 0$  eq. 53 becomes

$$\frac{d}{dt}\vec{x}(t) = \begin{bmatrix} -30 & 10 \\ 10 & -30 \end{bmatrix} \vec{x}(t) + \begin{bmatrix} 10 \\ 0 \end{bmatrix} \quad (54)$$

Let's say the two eigenvalues of  $\begin{bmatrix} -30 & 10 \\ 10 & -30 \end{bmatrix}$  are  $\lambda_1, \lambda_2$  and the corresponding eigenvectors are  $\vec{v}_1, \vec{v}_2$  respectively. Let's also define  $V = [\vec{v}_1 \quad \vec{v}_2]$ . It's given to you that  $\lambda_1 = -40, \vec{v}_1 = \begin{bmatrix} 1 \\ -1 \end{bmatrix}$  and  $\vec{v}_2 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$ . You can also consider  $V^{-1} = \frac{1}{2} \begin{bmatrix} 1 & -1 \\ 1 & 1 \end{bmatrix}$ .

- i. Find the value of  $\lambda_2$ .

**Solution:** Plugging in the component values,  $A$  can be found as,

$$A = \begin{bmatrix} -30 & 10 \\ 10 & -30 \end{bmatrix} \quad (55)$$

Hence, the eigenvalues of  $A$  can be found as follows,

$$\det(A - \lambda I) = (\lambda + 30)^2 - 10^2 = 0 \quad (56)$$

which gives the eigenvalues to be,

$$\lambda_1 = -40 \quad (57)$$

$$\lambda_2 = -20. \quad (58)$$

**Alternative approach:**

As eigenvector  $\vec{v}_2$  is given, one can write,

$$\begin{bmatrix} -30 & 10 \\ 10 & -30 \end{bmatrix} \begin{bmatrix} 1 \\ 1 \end{bmatrix} = \lambda_2 \begin{bmatrix} 1 \\ 1 \end{bmatrix} \quad (59)$$

which gives  $\lambda_2 = -20$ .

- ii. Let's define  $\vec{\tilde{x}}(t)$  such that  $\vec{x}(t) = V\vec{\tilde{x}}(t)$ . **Determine**  $\vec{\tilde{x}}(0)$ . You can assume the initial condition of the circuit to be  $\vec{x}(0) = \vec{0}$ .

**Solution:**

$$\vec{\tilde{x}}(0) = V^{-1}\vec{x}(0) = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \quad (60)$$

iii. Now diagonalize the system given by eq. 54 and solve for  $\vec{x}(t)$  for  $t \geq 0$ .

**Solution:** The diagonalized system can be written as

$$\frac{d}{dt} \vec{\tilde{x}}(t) = \frac{d}{dt} \begin{bmatrix} \tilde{x}_1(t) \\ \tilde{x}_2(t) \end{bmatrix} = \begin{bmatrix} -40 & 0 \\ 0 & -20 \end{bmatrix} \begin{bmatrix} \tilde{x}_1(t) \\ \tilde{x}_2(t) \end{bmatrix} + \begin{bmatrix} \frac{1}{2} & -\frac{1}{2} \\ \frac{1}{2} & \frac{1}{2} \end{bmatrix} \begin{bmatrix} 10^3 \\ 0 \end{bmatrix} I_{\text{ext}}(t) \quad (61)$$

for  $t > 0$ , the individual scalar differential equations can be written as follows

$$\frac{d}{dt} \tilde{x}_1(t) = -40\tilde{x}_1(t) + 500I_e \quad (62)$$

$$= -40\left(\tilde{x}_1(t) - \frac{1}{8}\right) \quad (63)$$

$$\frac{d}{dt} \tilde{x}_2(t) = -20\tilde{x}_2(t) + 500I_e \quad (64)$$

$$= -20\left(\tilde{x}_2(t) - \frac{1}{4}\right) \quad (65)$$

To solve eq. 63 and 65, we can substitute  $(\tilde{x}_1(t) - \frac{1}{8})$  with  $y_1(t)$  and  $(\tilde{x}_2(t) - \frac{1}{4})$  with  $y_2(t)$ . Solving for  $y_1(t)$ ,  $y_2(t)$  and substituting back with  $\tilde{x}_1(t)$ ,  $\tilde{x}_2(t)$  we get

$$\tilde{x}_1(t) = K_1 e^{-40t} + \frac{1}{8} \quad (66)$$

$$\tilde{x}_2(t) = K_2 e^{-20t} + \frac{1}{4} \quad (67)$$

Plugging in the initial condition  $\vec{\tilde{x}}(0) = 0$  in eq. 66 and 67 we obtain,

$$\vec{\tilde{x}}(t) = \begin{bmatrix} \tilde{x}_1(t) \\ \tilde{x}_2(t) \end{bmatrix} = \begin{bmatrix} \frac{1}{8} (1 - e^{-40t}) \\ \frac{1}{4} (1 - e^{-20t}) \end{bmatrix} \quad (68)$$

iv. Use the result in the previous part to find  $\vec{x}(t)$  for  $t \geq 0$ .

**Solution:**

$$\vec{x}(t) = V \vec{\tilde{x}}(t) = \begin{bmatrix} 1 & 1 \\ -1 & 1 \end{bmatrix} \begin{bmatrix} \frac{1}{8} (1 - e^{-40t}) \\ \frac{1}{4} (1 - e^{-20t}) \end{bmatrix} \quad (69)$$

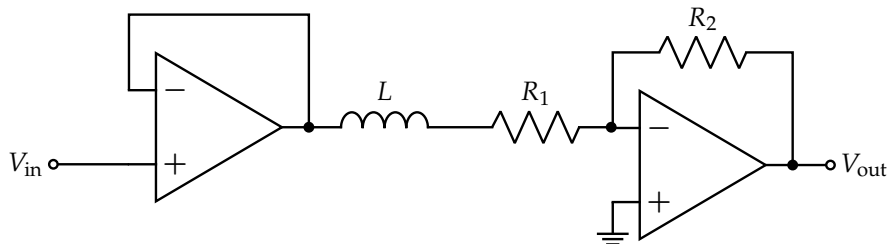
$$= \begin{bmatrix} \frac{3}{8} - \frac{1}{8}e^{-40t} - \frac{1}{4}e^{-20t} \\ \frac{1}{8} + \frac{1}{8}e^{-40t} - \frac{1}{4}e^{-20t} \end{bmatrix} \quad (70)$$



## 8. Active Filter

**NOTE:** This problem doesn't use any result from the previous problem on brain stimulation.

Krishna wants to measure the membrane potential of the neuron to characterize the neuron behavior as a part of his neural-modeling project. However, he knows that to accurately measure the neuron membrane potential, he needs to cancel out any external interference which may corrupt the neural signals. To do that he designed the following active filter:



**Figure 12:** Schematic of the active filter used to cancel unwanted interference to the neuron membrane potential

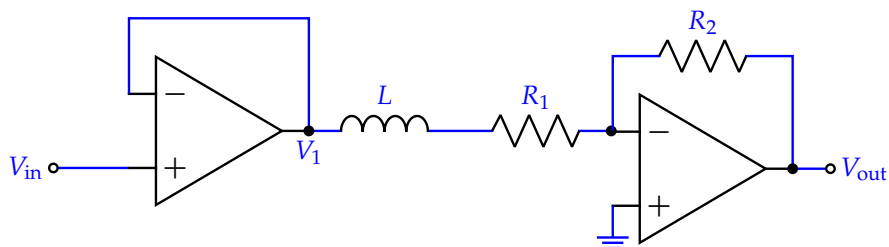
- (a) Assume the op-amps used in the filter are ideal. Which of the following best describes the type of this filter?

| Filter type                            | Select one            |
|--|-----------------------|
| 2 <sup>nd</sup> -order low-pass filter | <input type="radio"/> |
| 1 <sup>st</sup> -order low-pass filter | <input type="radio"/> |
| High-pass filter                       | <input type="radio"/> |
| Band-pass filter                       | <input type="radio"/> |

**Solution:** 1<sup>st</sup>-order low-pass filter.

- (b) Derive the transfer function of the filter.

**Solution:**



$$H(j\omega) = \frac{V_{\text{out}}}{V_{\text{in}}} = \frac{V_1}{V_{\text{in}}} \times \frac{V_{\text{out}}}{V_1} \quad (71)$$

$$= 1 \times \left( -\frac{R_2}{R_1 + j\omega L} \right) \quad (72)$$

$$= -\frac{R_2}{R_1 + j\omega L} \quad (73)$$

- (c) Assuming  $R_2 = 10 \times R_1$ , find the magnitude of the transfer function of the filter at  $\omega = 0$  (i.e.  $|H(j \cdot 0)|$ ).

**Solution:**

$$|H(j \cdot 0)| = \frac{R_2}{R_1} = 10 \quad (74)$$

- (d) Now assume the frequency of the neural signal,  $\omega_s$  can be in the range of 0 Hz to 100 Hz and the interference signal frequency,  $\omega_{\text{int}}$  is 60 kHz. The filter cut-off frequency,  $\omega_c$  needs to be positioned so that the interference is attenuated by at least a factor of 100 compared to  $|H(j \cdot 0)|$  (i.e.  $|H(j\omega_{\text{int}})| \leq \frac{|H(j \cdot 0)|}{100}$ ) and the neural signal doesn't see any attenuation compared to  $|H(j \cdot 0)|$  (i.e.  $|H(j\omega_s)| \approx |H(j \cdot 0)|$ ). **Which of the following is an acceptable range of cut-off frequencies ( $\omega_c$ ) for the active filter that Krishna designed? Justify your answer.**

| Frequency range | Select one            |
|-----------------|-----------------------|
| 50 Hz - 60 Hz   | <input type="radio"/> |
| 500 Hz - 600 Hz | <input type="radio"/> |
| 5 kHz - 6 kHz   | <input type="radio"/> |
| 50 kHz - 60 kHz | <input type="radio"/> |

**Solution:** The magnitude response of the filter can be given as,

$$|H(j\omega)| = \frac{R_2}{R_1} \times \frac{1}{\sqrt{1 + \left(\frac{\omega L}{R_1}\right)^2}} \quad (75)$$

Assuming the interference frequency,  $\omega_{\text{int}}$  to be much larger than the filter cut-off frequency  $\omega_c (= \frac{R_1}{L})$ , the magnitude of the transfer functions at  $\omega_{\text{int}}$  can be approximated as,

$$|H(j\omega_{\text{int}})| \approx \frac{R_2}{R_1} \times \frac{R_1}{\omega_{\text{int}} L} \quad (76)$$

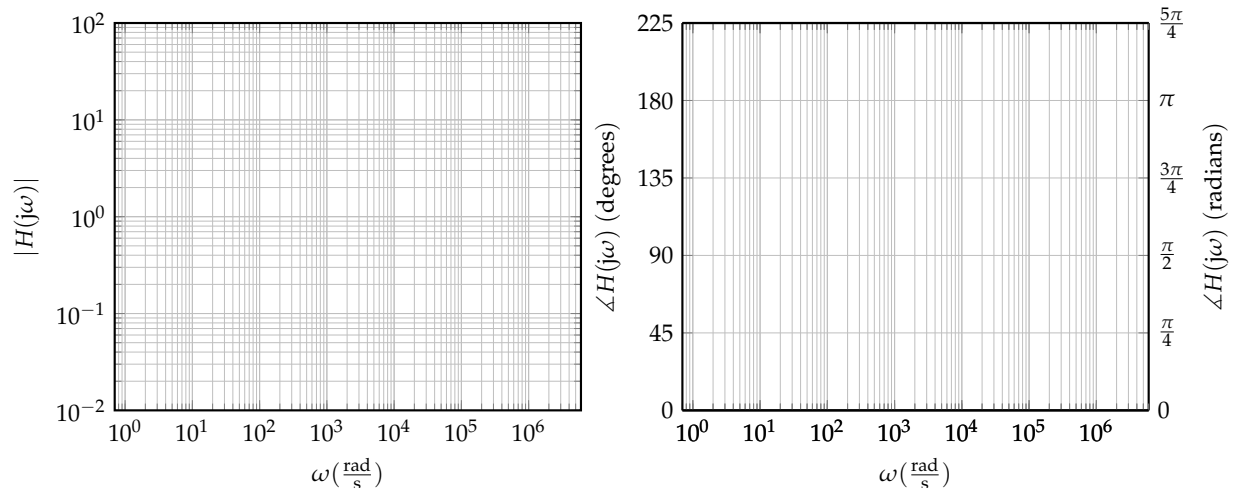
on applying the given condition  $|H(j\omega_{\text{int}})| \leq \frac{1}{100} \times |H(j \cdot 0)|$  we get,

$$\omega_c = \frac{R_1}{L} \leq \frac{\omega_{\text{int}}}{100} = 2\pi \times \frac{60 \text{ kHz}}{100} \quad (77)$$

$$\omega_c \leq 2\pi \times 600 \text{ Hz} \quad (78)$$

Also,  $\omega_c$  should be significantly greater than 100 Hz to make sure that  $|H(j\omega_s)| \approx |H(j \cdot 0)|$ . Hence, the correct range of cut-off frequencies is 500 Hz-600 Hz.

- (e) Suppose you have  $R_1 = 10 \Omega$ ,  $R_2 = 100 \Omega$  and  $L = 10 \text{ mH}$ . **Draw the Bode plot (straight-line approximations to the transfer function) for the magnitude and phase of the active filter.**



**Solution:** Plugging in the component values in eq. 73 the transfer function can be found as

$$H(j\omega) = -\frac{10}{1 + j\frac{\omega}{10^3}} \quad (79)$$

The magnitude response is given by

$$|H(j\omega)| = \frac{10}{\sqrt{1 + \left(\frac{\omega}{10^3}\right)^2}} \quad (80)$$

Hence we have the following properties for the magnitude Bode plot approximation:

- i. At  $\omega < 10^3$ , the magnitude is 10.
- ii. At  $\omega > 10^3$ , the magnitude drops by 20 dB per decade of increase in  $\omega$ .

The phase is given by

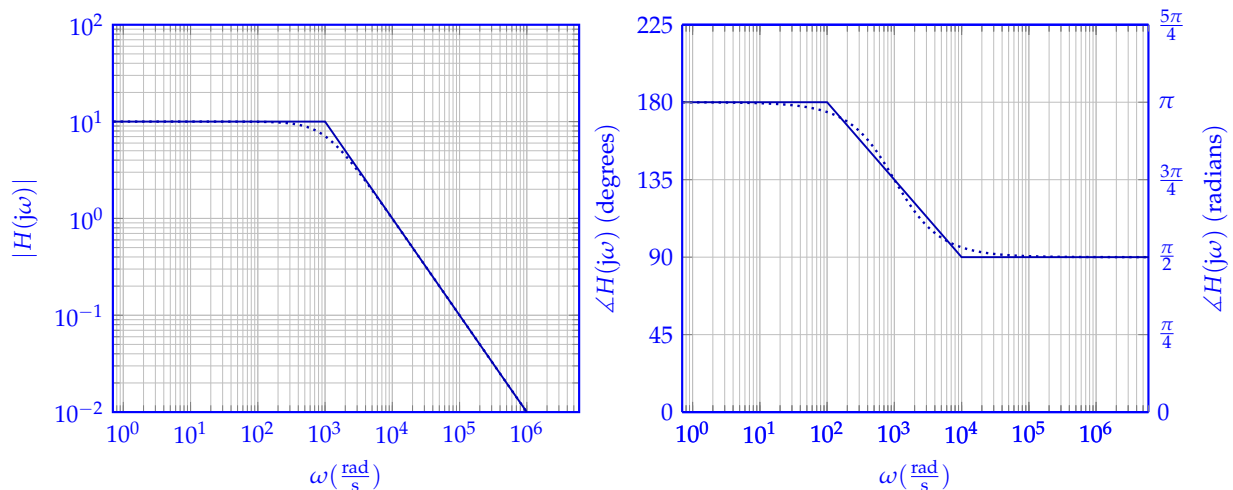
$$\angle(H(j\omega)) = \angle(-1) - \angle\left(1 + \frac{j \cdot \omega}{10^3}\right) \quad (81)$$

$$= 180^\circ - \text{atan2}\left(\frac{\omega}{10^3}, 1\right) \quad (82)$$

Hence we have the following properties for the phase Bode plot:

- i. At  $\omega < 10^2$ , the phase is  $180 - \text{atan2}(0, 1) = 180^\circ$ .
- ii. At  $\omega > 10^4$ , the phase is  $180 - \text{atan2}(\infty, 1) = 90^\circ$ .
- iii. At  $\omega = 10^3$ , the phase is  $180 - \text{atan2}(1, 1) = 135^\circ$ .

Hence the magnitude and phase Bode plots look as follows:



## 9. Affine Control

In this problem, we will analyze a *affine* model of the form

$$x[i + 1] = \alpha x[i] + \beta u[i] + \gamma \quad (83)$$

where  $\alpha, \beta, \gamma \in \mathbb{R}$ ,  $x: \mathbb{N} \rightarrow \mathbb{R}$  is the state, and  $u: \mathbb{N} \rightarrow \mathbb{R}$  is the input. Affine models are ubiquitous in control theory – in fact, our robot car from lab obeys a two-state-variable affine model.

(a) Suppose (for this part only) that:

- $\alpha = 1$ ,
- $\beta = 0$ ,
- $\gamma \neq 0$ ,
- $x[0]$  is anything.

so the model is of the form

$$x[i + 1] = x[i] + \gamma. \quad (84)$$

**Is the state  $x$  bounded?** *Justify your answer.*

**Solution:** No. In particular we have

$$x[i] = x[0] + \gamma \cdot i. \quad (85)$$

Since  $\gamma \neq 0$ , we have that

$$\lim_{i \rightarrow \infty} |x[i]| = \lim_{i \rightarrow \infty} |x[0] + \gamma \cdot i| = \infty. \quad (86)$$

(b) Suppose (for this part only) that the state evolves according to Equation (83), i.e.,

$$x[i + 1] = \alpha x[i] + \beta u[i] + \gamma \quad (87)$$

and

- $\alpha \neq 0$ ,
- $\beta > 0$ ,
- $\gamma \neq 0$ ,
- $x[0] = 0$ .

Suppose that we supply feedback control of the form

$$u[i] = f \cdot x[i] \quad (88)$$

for  $f \in \mathbb{R}$ .

i. **For the specific case of  $f = \frac{-1-\alpha}{\beta}$ , show that the state  $x$  is bounded.**

**Solution:** After plugging in feedback control, our model becomes

$$x[i + 1] = (\alpha + \beta f)x[i] + \gamma. \quad (89)$$

If  $f = \frac{-1-\alpha}{\beta}$ , then  $\alpha + \beta f = -1$ , so we have

$$x[i + 1] = -x[i] + \gamma \quad (90)$$

which has state trajectory

$$x[i] = \begin{cases} \gamma & i \text{ is odd} \\ 0 & i \text{ is even} \end{cases} \quad (91)$$

which is bounded by  $|\gamma|$ . Thus if  $f = \frac{-1-\alpha}{\beta}$  then  $x$  is bounded.

ii. In terms of  $\alpha$  and  $\beta$ , give a range of  $f$  that keeps the state  $x$  bounded.

**Solution:** The answer is

$$f \in \left[ \frac{-1-\alpha}{\beta}, \frac{1-\alpha}{\beta} \right). \quad (92)$$

The endpoint (non)inclusions are important.

Quick and dirty reasoning: With feedback control, our model becomes

$$x[i+1] = (\alpha + \beta f)x[i] + \gamma \quad (93)$$

We know that for a regular linear system of the form

$$x[i+1] = \kappa x[i] + u[i] \quad (94)$$

if the input  $u$  is bounded and  $|\kappa| < 1$ , then the state  $x$  is bounded; this corresponds to  $\alpha + \beta f \in (-1, 1)$ , which is if and only if  $f \in \left( \frac{-1-\alpha}{\beta}, \frac{1-\alpha}{\beta} \right)$ . It turns out that, if we restrict ourselves to *constant* inputs  $\gamma = u[i]$  (which is a sub-class of bounded inputs), then  $\kappa = -1$  (corresponding to  $f = \frac{-1-\alpha}{\beta}$ ) also keeps the state bounded – and we can see this by part i.

(Not Required) Formal proof

We use the same notation, letting  $\kappa := \alpha + \beta f$ , so that Equation (89) becomes

$$x[i+1] = \kappa x[i] + \gamma. \quad (95)$$

Indeed, the general trajectory of Equation (95) becomes

$$x[i] = \begin{cases} \gamma \cdot i & \kappa = 1 \\ \gamma \cdot \frac{\kappa^i - 1}{\kappa - 1} & \kappa \neq 1 \end{cases}. \quad (96)$$

Thus

$$|x[i]| = \begin{cases} |\gamma| i & \kappa = 1 \\ |\gamma| \frac{|\kappa^i - 1|}{|\kappa - 1|} & \kappa \neq 1 \end{cases}. \quad (97)$$

From here there are some cases:

- $|\kappa| = 1$ . Here there are two more cases:

- If  $\kappa = 1$  then

$$|x[i]| = |\gamma| i \rightarrow \infty \quad (98)$$

and  $x$  is unbounded.

- If  $\kappa = -1$  then

$$|x[i]| = |\gamma| \frac{|(-1)^i - 1|}{2} = \begin{cases} |\gamma| & i \text{ is odd} \\ 0 & i \text{ is even} \end{cases} \quad (99)$$

which is bounded by  $|\gamma|$ .

- $|\kappa| > 1$ . Then

$$|x[i]| = |\gamma| \frac{|\kappa^i - 1|}{|\kappa - 1|} \rightarrow \infty \quad (100)$$

and  $x$  is unbounded.

- $|\kappa| < 1$ . Then

$$|x[i]| = |\gamma| \frac{|\kappa^i - 1|}{|\kappa - 1|} \rightarrow \frac{|\gamma|}{|\kappa - 1|} \quad (101)$$

and  $x$  is bounded.

Thus  $x$  is bounded if and only if  $\kappa \in [-1, 1)$ . Since  $\kappa = \alpha + \beta f$ , this is if and only if

$$f \in \left[ \frac{-1 - \alpha}{\beta}, \frac{1 - \alpha}{\beta} \right). \quad (102)$$

(c) Suppose (for this part only) that the state evolves according to Equation (83), i.e.,

$$x[i + 1] = \alpha x[i] + \beta u[i] + \gamma \quad (103)$$

and

- $\alpha$  is anything,
- $\beta$  is anything,
- $\gamma$  is anything,
- $x[0]$  is anything.

Suppose that we are setting up a least-squares system identification procedure to learn  $\alpha$ ,  $\beta$ , and  $\gamma$ , and that we have data of the form  $(x[i], u[i], x[i + 1])$ , for  $i \in \{0, 1, \dots, \ell - 1\}$ . **Set up a least-squares problem  $D\vec{p} \approx \vec{s}$  to learn estimates for  $\alpha, \beta, \gamma$ . What are  $D, \vec{p}$ , and  $\vec{s}$ ?**

NOTE: Your answer for  $D$  should be as compact as possible.

NOTE: You do not need to solve the least squares problem; just set it up.

**Solution:** We follow the procedure on the hint and the lab. Namely, we have

$$D := \begin{bmatrix} x[0] & u[0] & 1 \\ \vdots & \vdots & \vdots \\ x[\ell - 1] & u[\ell - 1] & 1 \end{bmatrix} \quad \vec{p} := \begin{bmatrix} \alpha \\ \beta \\ \gamma \end{bmatrix} \quad \vec{s} := \begin{bmatrix} x[1] \\ \vdots \\ x[\ell] \end{bmatrix}. \quad (104)$$

(d) Suppose (for this part only) that the state evolves according to Equation (83), i.e.,

$$x[i + 1] = \alpha x[i] + \beta u[i] + \gamma \quad (105)$$

and

- $\alpha > 1$ ,
- $\beta > 0$ ,
- $\gamma > 0$ ,
- $x[0]$  is anything.

Suppose that we actually got our discrete-time model

$$x[i + 1] = \alpha x[i] + \beta u[i] + \gamma \quad (106)$$

by discretizing a continuous-time model

$$\frac{d}{dt}x(t) = ax(t) + bu(t) + c \quad (107)$$

where the sampling interval length is  $\Delta = 1$ , i.e.,  $x[i] = x(i\Delta)$ , and  $u(t)$  is piecewise constant over intervals of length  $\Delta$ , i.e.,  $u(t) = u(i\Delta) = u[i]$  for  $t \in [i\Delta, (i + 1)\Delta)$ . **In terms of  $\alpha, \beta, \gamma$ , what are  $a, b$ , and  $c$ ?**

(HINT: You can use any discretization formulas we derived in class, as long as they apply. Alternatively, you may use the following formula in your derivation.

For a constant input  $v$ , and a time  $t_0$  for which  $x(t_0)$  is known, the solution to the differential equation

$$\frac{d}{dt}x(t) = ax(t) + v \quad t \geq t_0 \quad (108)$$

is given by

$$x(t) = e^{a(t-t_0)}x(t_0) + \frac{e^{a(t-t_0)} - 1}{a} \cdot v, \quad t \geq t_0. \quad (109)$$

when  $a \neq 0$ . Also, recall from the problem statement above that the sampling interval length  $\Delta = 1$ .)

**Solution:** First Solution Using Integral Formula:

Starting from time  $t_0 = i\Delta$  and the value  $x(t_0) = x[i]$ , we use the piecewise constant assumption to simplify the continuous model.

$$\frac{d}{dt}x(t) = ax(t) + bu[i] + c, \quad t \in [i\Delta, (i+1)\Delta). \quad (110)$$

We use Equation (109) to find  $x(t) = x((i+1)\Delta) = x[i+1]$ . Using the substitution  $v = bu[i] + c$ , we have

$$x(t) = e^{a(t-i\Delta)}x[i] + \frac{e^{a(t-i\Delta)} - 1}{a}(bu[i] + c). \quad (111)$$

Plugging in  $t = i\Delta$  gets

$$x[i+1] = x((i+1)\Delta) \quad (112)$$

$$= e^{a((i+1)\Delta - i\Delta)}x[i] + \frac{e^{a((i+1)\Delta - i\Delta)} - 1}{a}(bu[i] + c) \quad (113)$$

$$= e^{a\Delta}x[i] + \frac{e^{a\Delta} - 1}{a}(bu[i] + c) \quad (114)$$

$$= e^{a\Delta}x[i] + \frac{e^{a\Delta} - 1}{a} \cdot bu[i] + \frac{e^{a\Delta} - 1}{a} \cdot c \quad (115)$$

$$= e^a x[i] + \frac{e^a - 1}{a} \cdot bu[i] + \frac{e^a - 1}{a} \cdot c \quad (116)$$

Thus we get

$$\alpha = e^a \quad (117)$$

$$\beta = \frac{e^a - 1}{a} \cdot b \quad (118)$$

$$\gamma = \frac{e^a - 1}{a} \cdot c. \quad (119)$$

Thus

$$a = \log(\alpha) \quad (120)$$

$$b = \frac{a}{e^a - 1} \beta = \frac{\log(\alpha)}{\alpha - 1} \beta \quad (121)$$

$$c = \frac{a}{e^a - 1} \gamma = \frac{\log(\alpha)}{\alpha - 1} \gamma. \quad (122)$$

Alternative Solution Using Known Discretization Coefficients:

Letting  $v(t) = bu(t) + c$ , we have that  $v(t)$  is piecewise constant over intervals of length  $\Delta$ . We have the continuous-time model

$$\frac{d}{dt}x(t) = ax(t) + v(t) \quad (123)$$

from which we can read off the discretization

$$x[i+1] = e^a x[i] + \frac{e^a - 1}{a} v[i] \quad (124)$$

$$= e^a x[i] + \frac{e^a - 1}{a} (bu[i] + c) \quad (125)$$

$$= e^a x[i] + \frac{e^a - 1}{a} bu[i] + \frac{e^a - 1}{a} c \quad (126)$$

from which the solution proceeds the same as before.