Exam Location: In Person

PRINT your student ID: _____

PRINT AND SIGN your name: _____ , _____ _____
                                            (last)   (first)   (sign)

PRINT your discussion sections and (u)GSIs (the ones you attend): _____

Row Number: _____   Seat Number: _____

Name and SID of the person to your left: _____

Name and SID of the person to your right: _____

Name and SID of the person in front of you: _____

Name and SID of the person behind you: _____

1. **Honor Code (0 pts.)**

   Please copy the following statement in the space provided below and sign your name.

   As a member of the UC Berkeley community, I act with honesty, integrity, and respect for others. I will follow the rules and do this exam on my own.

   Note that if you do not copy the honor code and sign your name, you will get a 0 on the exam.

   **Solution:** Any attempt to copy the honor code and sign should get full points.

2. **What is your favorite topic of the course so far? (2 pts.)**

   Solution: Any answer is sufficient.

Do not turn this page until the proctor tells you to do so.
You can work on the above problems before time starts.

3. **RC Circuit with a Dependent Source (30 pts.)**

Suppose you have the following circuit with constant input v_{in}, which features a dependent current source.
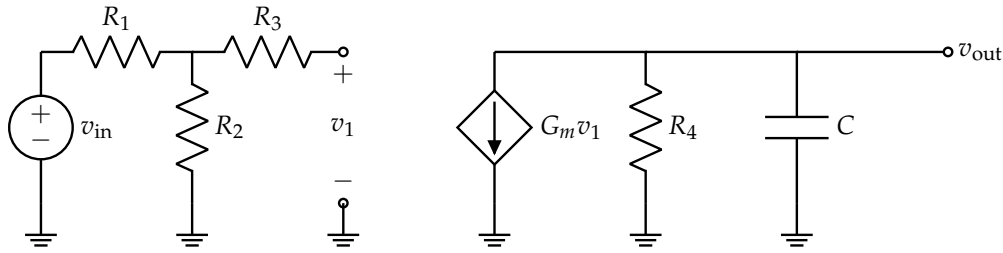
(a) (10 pts.) **Find the differential equation for $v_{\text{out}}(t)$ in terms of the component variables and the constant input v_{in}.** Note: Recall how dependent sources work; the current through the dependent current source in this case is $G_m v_1$, where v_1 is the voltage defined in the circuit diagram and G_m is some constant associated with the dependent current source.

Solution: The voltage v_1 is simply the output of a voltage divider between R_1 and R_2 (R_3 is irrelevant since no current can pass through it due to the open circuit):

$$v_1 = \frac{R_2}{R_1 + R_2} v_{\text{in}} \tag{1}$$

Then, we can perform KCL at the output node to find our differential equation (after some simplification) (note the direction of the dependent current source is important here):

$$\frac{v_{\text{out}}}{R_4} + C\frac{\mathrm{d}v_{\text{out}}(t)}{\mathrm{d}t} = -G_m v_1 \tag{2}$$

$$\frac{v_{\text{out}}}{R_4} + C\frac{\mathrm{d}v_{\text{out}}(t)}{\mathrm{d}t} = -G_m \frac{R_2}{R_1 + R_2} v_{\text{in}} \tag{3}$$

$$\frac{\mathrm{d}v_{\text{out}}(t)}{\mathrm{d}t} = -\frac{1}{R_4 C} v_{\text{out}} - \frac{G_m R_2}{C(R_1 + R_2)} v_{\text{in}} \tag{4}$$

(b) (2 pts.) **What is the time constant τ of this circuit?** Recall that for a first order differential equation, the time constant will be τ such that $e^{-\frac{t}{\tau}}$ is the relevant exponential present for your system.

Solution: We know that if we have a differential equation of the form

$$\frac{\mathrm{d}v(t)}{\mathrm{d}t} = \lambda v(t) + u(t) \tag{5}$$

the time constant τ is related to λ by $\lambda = -\frac{1}{\tau}$.

Thus, in our situation, we can observe based off our differential equation that $\tau = R_4 C$.

Note: You could also find the time constant without the differential equation by finding the Thevenin equivalent resistance seen by the capacitor; doing this, you would find that $R_{th} = R_4$ and thus the time constant would be $\tau = R_{th} C = R_4 C$.

(c) (5 pts.) **Find the steady state value of $v_{\text{out}}(t)$ in terms of the component variables and input v_{in}.**

Solution: We have a constant voltage source as the input, so we have a DC steady state scenario, where the capacitor will act like a short circuit.

The equivalent circuit for DC steady state is shown below:
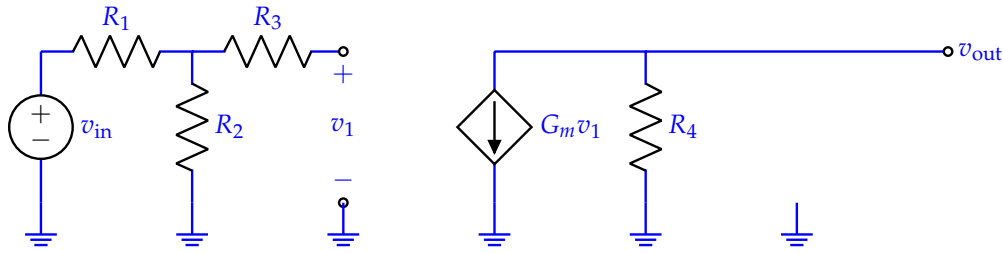
The steady state solution for $v_{\text{out}}(t)$, which we will call $v_{\text{out},p}(t)$ (since this is the same as the particular solution for the differential equation), can be found with some simple KCL and KVL:

$$v_{\text{out},p}(t) = -G_m R_4 v_1 = -G_m R_4 \left(\frac{R_2}{R_1 + R_2} v_{\text{in}} \right) = -\frac{G_m R_4 R_2}{R_1 + R_2} v_{\text{in}} \tag{6}$$

For the rest of the problem, suppose that when we use the component values, we get the following differential equation:

$$\frac{\mathrm{d}v_{\text{out}}(t)}{\mathrm{d}t} = -\frac{1}{2} v_{\text{out}}(t) - 5 \tag{7}$$

Additionally, suppose with component values, we find a steady state solution of -10V and we know that at time $t = 0$, the capacitor is discharged ($v_{\text{out}}(0) = 0$).

(d) (8 pts.) **Find the full solution for $v_{\text{out}}(t)$.**

Solution: We already have the particular solution from the steady state solution ($v_{\text{out},p}(t) = -10$).

We need to find the homogeneous solution, which we know for this relatively standard differential equation ($\lambda = -\frac{1}{2}$):

$$v_{\text{out},h}(t) = Ke^{-\frac{1}{2}t} \tag{8}$$

This means the full solution (with the arbitrary constant still present) is

$$v_{\text{out}}(t) = v_{\text{out},p}(t) + v_{\text{out},h}(t) = -10 + Ke^{-\frac{1}{2}t} \tag{9}$$

Now, we just need to use our initial condition to solve for the arbitrary constant K:

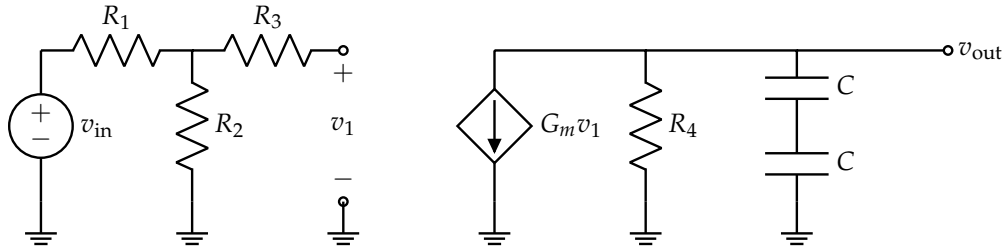$$v_{\text{out}}(0) = 0 \tag{10}$$

$$-10 + Ke^{-\frac{1}{2}(0)} = 0 \tag{11}$$

$$-10 + K = 0 \tag{12}$$

$$K = 10 \tag{13}$$

Thus, the full solution would be:

$$v_{\text{out}}(t) = -10 + 10e^{-\frac{1}{2}t} = -10(1 - e^{-\frac{1}{2}t}) \tag{14}$$

(e) (5 pts.) Suppose that we make a change to the circuit by adding an extra capacitor (with the same capacitance) in series with the current capacitor, as shown in the below circuit diagram:

Suppose the previous circuit (one capacitor) had time constant τ_0 and steady state solution V_{p0}.

Find the new circuit's time constant τ and steady state solution V_p in terms of τ_0 and V_{p0}. Briefly explain your answer.

Solution: The two capacitors in series can be combined with capacitor equivalence to be one capacitor with capacitance $\frac{C}{2}$.

Then, the circuit is effectively the same exact system as the previous one, which indicates that the effect we must consider is that of the capacitance being halved.

We know that the time constant is proportional to the capacitance, so $\tau = \frac{1}{2}\tau_0$.

The DC steady solution is that when the capacitors are left open, which does not change between the circuits, so $V_p = V_{p0}$.
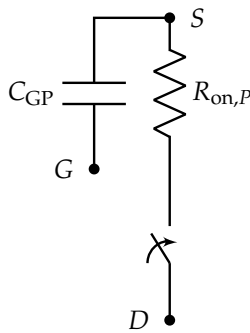
4. Transistor Inverter (20 pts.)

Suppose we have a transistor inverter, as you have likely seen before:

V_{DD}

$V_{\text{in}}(t)$ $V_{\text{out}}(t)$
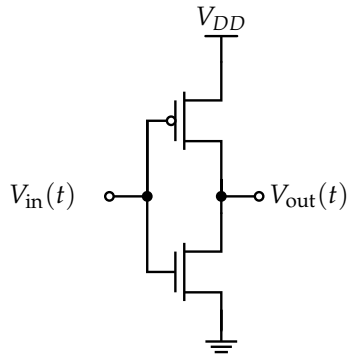
The RC models of NMOS and PMOS transistors we will use are these:
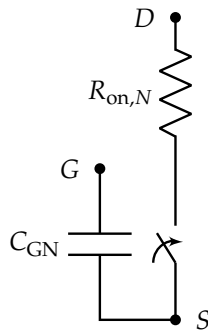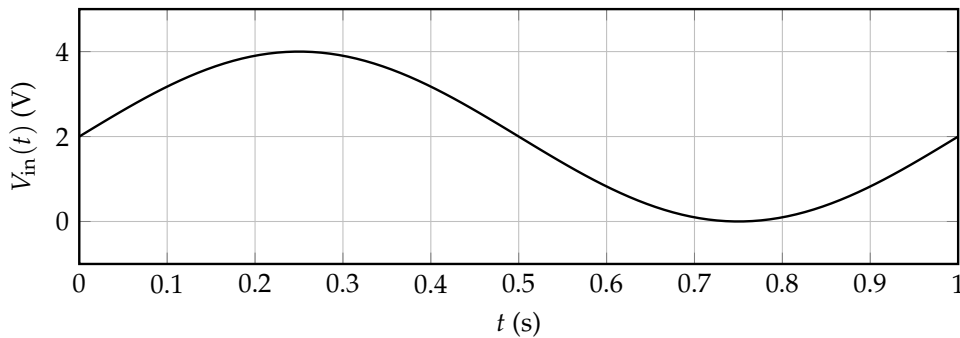
(a) NMOS Transistor RC Model

(b) PMOS Transistor RC Model

For the entire problem, we will use $V_{DD} = 4\,\text{V}$, threshold voltages $V_{THn} = 0.5\,\text{V}$ (for NMOS transistors) and $|V_{THp}| = 1.5\,\text{V}$ (for PMOS transistors), and resistances $R_{\text{on},N} = 1\,\text{k}\Omega$ (for NMOS transistors) and $R_{\text{on},P} = 3\,\text{k}\Omega$ (for PMOS transistors).

(a) (10 pts.) Suppose the plot of $V_{\text{in}}(t)$ is the sinusoidal function shown below:

Plot the output voltage $V_{\text{out}}(t)$. You do not need to be exactly precise with the time values.

(HINT: There are three possible situations depending on the input voltage at some specific time: both transistor are on, only the top transistor is on, or only the bottom transistor is on. Try to find what input and output voltages correspond to these three scenarios.)
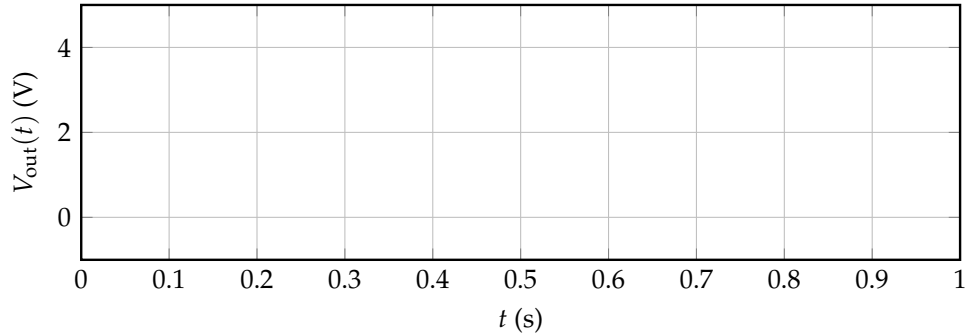
Solution: To determine what the output voltage is for some input voltage, we need to examine which transistors are on for that voltage (the top transistor is a PMOS transistor and the bottom transistor is an NMOS transistor).

For the NMOS transistor, $V_{GS} = V_{\text{in}}(t) - 0 = V_{\text{in}}(t)$. For the NMOS transistor to be on, we need $V_{GS} \geq V_{THn}$, which for our specific case means that the NMOS transistor is on when $V_{\text{in}}(t) \geq 0.5\,\text{V}$.

For the PMOS transistor, $V_{SG} = V_{DD} - V_{\text{in}}(t) = 4 - V_{\text{in}}(t)$. For the PMOS transistor to be on, we need $V_{SG} \geq |V_{THp}|$, which for our specific case means that the PMOS transistor is on when $4 - V_{\text{in}}(t) \geq 1.5$, or $V_{\text{in}}(t) \leq 2.5\,\text{V}$.

We also need to know what output voltage is set based on which transistors are on.

When only the NMOS transistor is on, the output voltage must be $0\,\text{V}$. Likewise, when only the PMOS transistor is on, the output voltage must be $V_{DD} = 4\,\text{V}$.

When both are on, using the provided transistor RC models, the output voltage must be a voltage divider between the two resistances $R_{\text{on},N}$ and $R_{\text{on},P}$:

$$V_{\text{out}} = \frac{R_{\text{on},N}}{R_{\text{on},N} + R_{\text{on},P}} V_{DD} = \frac{1}{1+3}(4) = 1\,\text{V} \tag{15}$$
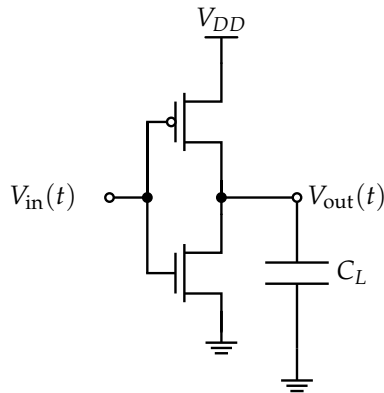
Thus, we can construct some ranges to set for our plot:

- When $0\,\text{V} < V_{\text{in}}(t) < 0.5\,\text{V}$, only the PMOS transistor is on so $V_{\text{out}}(t) = 4\,\text{V}$.
- When $0.5\,\text{V} < V_{\text{in}}(t) < 2.5\,\text{V}$, both transistors are on so $V_{\text{out}}(t) = 1\,\text{V}$.
- When $2.5\,\text{V} < V_{\text{in}}(t) < 4\,\text{V}$, only the NMOS transistor on so $V_{\text{out}}(t) = 0\,\text{V}$.

Thus, our output plot should be:
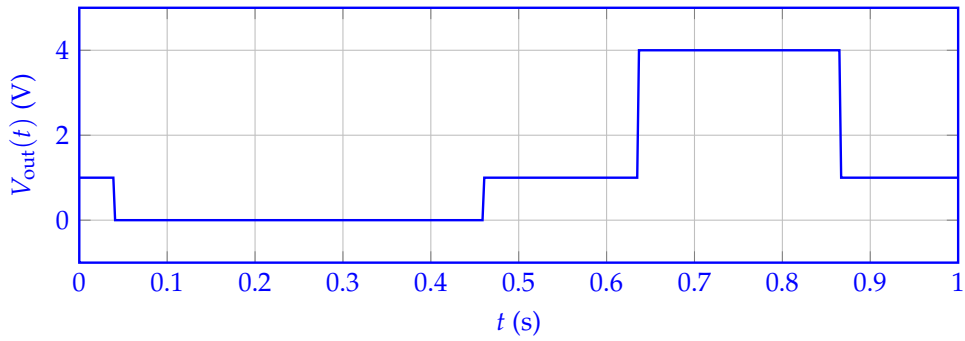
(b) (10 pts.) Suppose we connect this inverter to some other circuit, represented by a load capacitance C_L:


We are interested in the speed of our inverter for different output voltage transitions. Assume for this part that only one transistor is on at a time for each transition.

With the model values provided at the start of the problem, **will our inverter circuit be faster for pull up transitions (V_{out} from $0\,\text{V} \rightarrow 4\,\text{V}$) or pull down transitions (V_{out} from $4\,\text{V} \rightarrow 0\,\text{V}$)?** Please briefly explain using time constants.

(HINT: First, understand which transistor is on for each type of transition based on what the final voltage for each transition is. Then, think about how to apply the transistor RC model to the situation; which part of the RC model impacts the output time constant?)

Solution: For pull up transitions, the transition is done by the PMOS transistor, which has resistance $R_{\text{on},P}$. The equivalent circuit for this transition is:

V_{DD}
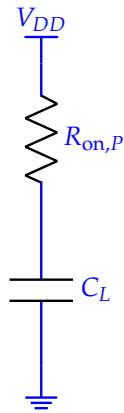
You could derive the differential equation for this circuit, but it is much easier to notice by inspection that since the capacitor sees a resistance of $R_{\text{on},P}$, the time constant for this circuit will be $\tau_{\text{up}} = R_{\text{on},P}C_L$.

For pull down transitions, the transition is done by the NMOS transistor, which has resistance $R_{\text{on},N}$. The equivalent circuit for this transition is:
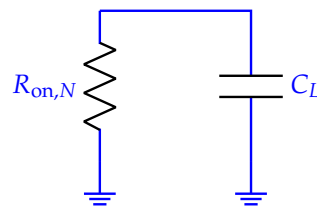

Similarly to the previous situation, the time constant for this circuit will be $\tau_{\text{down}} = R_{\text{on},N}C_L$.

When we compare the two time constants, we know that $R_{\text{on},N} < R_{\text{on},P}$, which implies that $\tau_{\text{down}} < \tau_{\text{up}}$ so this circuit will be faster for pull down transitions.

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5. **RLC 2nd Order Differential Equation (30 pts.)**

In this problem, you will approach solving this RLC circuit using your knowledge of 2nd Order Differential Equations.
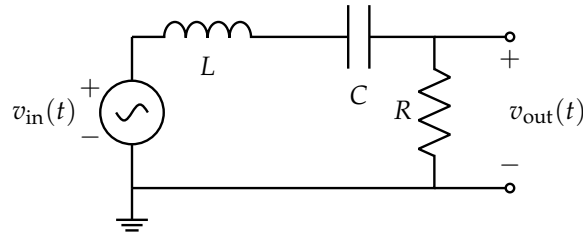
Figure 2: Circuit in "time domain"

(a) (8 pts.) **Find the differential equation for** $i_R(t)$ **in the following form** (where a, b, and c are constants that may depend on R, L, and C):

$$\frac{d^2 i_R(t)}{dt^2} + a\frac{di_R(t)}{dt} + bi_R(t) = c\frac{dv_{in}(t)}{dt} \tag{16}$$

(HINT: Note that $v_{in}(t)$ is some sinusoidal input and NOT a constant.)

Solution: We start by using KVL:

$$v_L(t) + v_C(t) + v_R(t) = v_{in}(t) \tag{17}$$

$$L\frac{di_L(t)}{dt} + v_C(t) + i_R(t)R = v_{in}(t) \tag{18}$$

$$L\frac{d^2 i_L(t)}{dt^2} + \frac{dv_C(t)}{dt} + R\frac{di_R(t)}{dt} = \frac{dv_{in}(t)}{dt} \tag{19}$$

$$\frac{d^2 i_R(t)}{dt^2} + \frac{R}{L}\frac{di_R(t)}{dt} + \frac{1}{LC}i_R(t) = \frac{1}{L}\frac{dv_{in}(t)}{dt} \tag{20}$$

$$\tag{21}$$

(b) (6 pts.) Suppose that you found the differential equation

$$\frac{d^2 i_R(t)}{dt^2} + 5\frac{di_R(t)}{dt} + 6i_R(t) = 2\frac{dv_{in}(t)}{dt} \tag{22}$$

The general *homogeneous* solution $i_{R,h}(t)$ to this differential equation can be written in the following form:

$$i_{R,h}(t) = C_1 e^{s_1 t} + C_2 e^{s_2 t} \tag{23}$$

Find numerical values for s_1 **and** s_2 for this differential equation (the order of s_1 and s_2 does not matter).

Solution: To start off, we will write the homogeneous differential equation, which is the differential equation without the input term:

$$\frac{d^2 i_{R,h}(t)}{dt^2} + 5\frac{di_{R,h}(t)}{dt} + 6i_{R,h}(t) = 0 \tag{24}$$

We will plug in $i_{R,h}(t) = e^{st}$ to the homogeneous differential equation. This will create a quadratic equation that will provide two solutions for s, which will be the s_1 and s_2 exponent coefficients for the general homogeneous solution.

$$s^2 e^{st} + 5s e^{st} + 6e^{st} = 0 \tag{25}$$

$$s^2 + 5s + 6 = 0 \tag{26}$$

$$(s+3)(s+2) = 0 \tag{27}$$

Thus, $s = -3, -2$ are the two solutions.

(c) (6 pts.) Suppose $v_{\text{in}}(t) = V_0 \cos\left(\frac{1}{\sqrt{LC}} t + \frac{\pi}{2}\right)$. Find the **particular solution** $i_{R,p}(t)$ for this differential equation (in terms of R, L, C, and V_0).

(HINT: Remember the relationship between the particular solution and steady state solution. What method can you use to find the steady state solution when the input is sinusoidal?)

Solution: We can convert the circuit into the frequency domain to more easily find the steady state solution. Following this, one option is to simplify the circuit using Ohm's law. But another method is to recognize that the circuit is in resonance. At resonance, the imaginary impedances cancel out, giving:
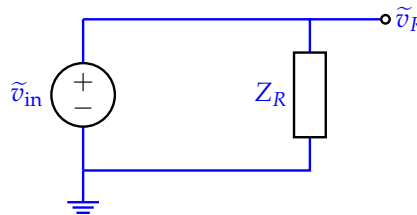
And thus, our current phasor is given by $\widetilde{i}_R = \frac{\widetilde{v}_R}{Z_R}$. In the time domain, this makes our answer $i_{R,p} = \frac{V_0}{R} \cos\left(\frac{1}{\sqrt{LC}} t + \frac{\pi}{2}\right)$.

(d) (4 pts.) To solve the 2nd Order Differential Equation, we need initial conditions for both $i_R(t)$ and $i'_R(t) = \frac{di_R(t)}{dt}$. Suppose we know that $i_R(0) = 0$ and $v_C(0) = 0$. **Find $i'_R(0)$. Show all your work.**

(HINT: Consider using an intermediate step from part (a) and information from part (c) to help you.)

Solution: The following intermediate step from part (a) is most relevant to this question:

$$L \frac{di_L(t)}{dt} + v_C(t) + i_R(t)R = v_{\text{in}}(t) \tag{28}$$

Additionally, using our input from part (c), tells us that $v_{\text{in}}(0) = V_0 \cos\left(\frac{\pi}{2}\right) = 0$. Thus, we find simply that

$$L \frac{di_L(t)}{d0} + v_C(0) + i_R(0)R = v_{\text{in}}(0) \tag{29}$$

$$L i'_R(0) + 0 + 0 = 0 \tag{30}$$

$$i'_R(0) = 0 \tag{31}$$

(e) (6 pts.) Suppose that you found that the particular solution is $i_{R,p}(t) = \sin(\omega_p t)$. Then, you could write your total solution as

$$i_R(t) = i_{R,p}(t) + i_{R,h}(t) = \sin(\omega_p t) + C_1 e^{s_1 t} + C_2 e^{s_2 t} \tag{32}$$

Using the initial conditions $i_R(0)$ and $i'_R(0)$, **set up a system of equations to solve for** C_1 **and** C_2 (your equations can contain the constants $i_R(0)$, $i'_R(0)$, ω_p, s_1, and s_2).

Solution: We can simply use the equation for $i_R(t)$ with constants C_1 and C_2 to set up the equations. For $i_R(0)$:

$$i_R(0) = i_R(0) \tag{33}$$
$$\sin(\omega_p(0)) + C_1 e^{s_1(0)} + C_2 e^{s_2(0)} = i_R(0) \tag{34}$$
$$C_1 + C_2 = i_R(0) \tag{35}$$

By differentiating $i_R(t)$, we get $i'_R(t) = \omega_p \cos(\omega_p t) + s_1 C_1 e^{s_1 t} + s_2 C_2 e^{s_2 t}$. For $i'_R(0)$:

$$i'_R(0) = i'_R(0) \tag{36}$$
$$\omega_p \cos(\omega_p(0)) + s_1 C_1 e^{s_1(0)} + s_2 C_2 e^{s_2(0)} = i'_R(0) \tag{37}$$
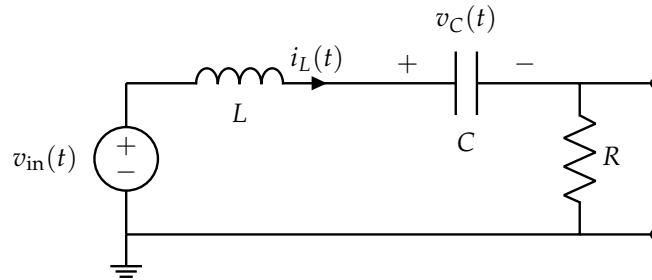$$\omega_p + s_1 C_1 + s_2 C_2 = i'_R(0) \tag{38}$$

Thus, the system of equations for C_1 and C_2 is:

$$C_1 + C_2 = i_R(0) \tag{39}$$
$$\omega_p + s_1 C_1 + s_2 C_2 = i'_R(0) \tag{40}$$

6. RLC Vector Differential Equation (30 pts.)

In this problem, you will approach solving the following RLC circuit using *the vector differential equation method*.

Figure 3: Circuit in "time domain"

(a) (4 pts.) **Derive a pair of differential equations, one for $\frac{di_L(t)}{dt}$ and another for $\frac{dv_C(t)}{dt}$ in terms of $i_L(t), v_C(t)$, and any necessary constants..** **Solution:** We start with KVL:

$$v_L(t) + v_C(t) + v_R(t) = v_{in}(t) \tag{41}$$

$$L\frac{di_L(t)}{dt} + v_C(t) + i_L(t)R = v_{in}(t) \tag{42}$$

$$\frac{di_L(t)}{dt} = -\frac{1}{L}v_C(t) - \frac{R}{L}i_L(t) + \frac{1}{L}v_{in}(t) \tag{43}$$

KCL on the node containing the inductor and capacitor should give us a second valid equation.

$$i_L(t) = i_C(t) \tag{44}$$

$$\frac{dv_C(t)}{dt} = \frac{1}{C}i_L(t) \tag{45}$$

And thus, we find our two differential equations.

(b) (6 pts.) Suppose our vector differential equation was written as follows:

$$\frac{d}{dt}\vec{x}(t) = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \vec{x}(t) + \begin{bmatrix} e \\ f \end{bmatrix} V_{\text{in}}(t) \tag{46}$$

The state vector is defined as $\vec{x}(t) = \begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix} = \begin{bmatrix} i_L(t) \\ v_C(t) \end{bmatrix}$. **Find the values of a, b, c, d, e, and f in terms of R, L, C.** **Solution:** Our first step is to obtain a vectorized version of the two differential equations we found in part (a). We do this by stacking complementary terms.

As suggested, we will take $\vec{x}(t) = \begin{bmatrix} i_L(t) \\ v_C(t) \end{bmatrix}$. With this, we find that the system can be written as:

$$\frac{d}{dt}\begin{bmatrix} i_L(t) \\ v_C(t) \end{bmatrix} = \begin{bmatrix} -\frac{R}{L} & -\frac{1}{L} \\ \frac{1}{C} & 0 \end{bmatrix}\begin{bmatrix} i_L(t) \\ v_C(t) \end{bmatrix} + \begin{bmatrix} \frac{1}{L} \\ 0 \end{bmatrix} V_{\text{in}} \tag{47}$$

Thus, $a = -\frac{R}{L}$, $b = -\frac{1}{L}$, $c = \frac{1}{C}$, $d = 0$, $e = \frac{1}{L}$, $f = 0$.

(c) (6 pts.) Suppose you are now told that the system is given by:

$$\frac{d}{dt}\vec{x}(t) = \begin{bmatrix} -3 & -1 \\ 2 & 0 \end{bmatrix} \vec{x}(t) \tag{48}$$

where $V_{\text{in}}(t)$ is simply zero. **Find the eigenvalues ($\lambda_1 \leq \lambda_2$) and eigenvectors of the given state matrix. Ensure your eigenvectors are in the form $\begin{bmatrix} c \\ 1 \end{bmatrix}$ where c is some constant. Show all your work.** **Solution:**

To find our eigenvalues, we use the method used in EECS 16A:

$$A\vec{v} = \lambda\vec{v} \tag{49}$$

$$A\vec{v} - \lambda\vec{v} = 0 \tag{50}$$

$$(A - \lambda I_2)\vec{v} = 0 \tag{51}$$

$$\det(A - \lambda I_2) = 0 \tag{52}$$

$$\det\left(\begin{bmatrix} -3 & -1 \\ 2 & 0 \end{bmatrix} - \begin{bmatrix} \lambda & 0 \\ 0 & \lambda \end{bmatrix}\right) = 0 \tag{53}$$

$$\det\left(\begin{bmatrix} -3-\lambda & -1 \\ 2 & -\lambda \end{bmatrix}\right) = 0 \tag{54}$$

$$(\lambda + 2)(\lambda + 1) = 0 \tag{55}$$

Therefore our eigenvalues are $\lambda_1 = -2$ and $\lambda_2 = -1$.

We find the corresponding eigenvectors by plugging in our eigenvalues:

For $\lambda_1 = -2$:

$$\left(\begin{bmatrix} -3 & -1 \\ 2 & 0 \end{bmatrix} - \begin{bmatrix} -2 & 0 \\ 0 & -2 \end{bmatrix}\right)\vec{v_1} = 0 \tag{56}$$

$$\left(\begin{bmatrix} -1 & -1 \\ 2 & 2 \end{bmatrix}\right)\vec{v_1} = 0 \tag{57}$$

From inspection or Gaussian Elimination, we find that $\vec{v}_1 = \begin{bmatrix} -1 \\ 1 \end{bmatrix}$.

For $\lambda_2 = -1$:

$$\left(\begin{bmatrix} -3 & -1 \\ 2 & 0 \end{bmatrix} - \begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix} \right)\vec{v}_1 = 0 \tag{58}$$

$$\left(\begin{bmatrix} -2 & -1 \\ 2 & 1 \end{bmatrix} \right)\vec{v}_2 = 0 \tag{59}$$

From inspection or Gaussian Elimination, we find that $\vec{v}_2 = \begin{bmatrix} -0.5 \\ 1 \end{bmatrix}$.

(d) (4 pts.) Now, we would like to diagonalize our system to more easily solve for the solution of the vector differential equation. We can write our system in the standard basis as $\frac{d\vec{x}(t)}{dt} = A\vec{x}(t)$.

For clarity in the upcoming parts, suppose $\Lambda = \begin{bmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{bmatrix}$ and V are defined as they have been in class.

Show what change of basis you would use to diagonalize the system by writing $\vec{x}(t)$ in terms of $\vec{y}(t)$, AND then symbolically write your system in the diagonalized basis in terms of $\frac{d\vec{y}(t)}{dt}$ and $\vec{y}(t)$. You may use any variables defined in this problem part.   **Solution:** Our change of basis would be given by $\vec{x}(t) = V\vec{y}(t)$. Now we show the full diagonalization process given this basis:

$$\frac{d}{dt}\vec{x}(t) = A\vec{x}(t) \tag{60}$$

$$V^{-1}\left(\frac{d}{dt}\vec{x}(t) \right) = \underbrace{V^{-1}AV}_{\Lambda}\vec{y}(t) \tag{61}$$

$$\frac{d}{dt}\vec{y}(t) = \Lambda\vec{y}(t) \tag{62}$$

(e) (4 pts.) Your initial conditions in the standard basis are given symbolically as $x_1(0) = 2$ and $x_2(0) = 1$. **Find the initial conditions in the diagonalized basis ($y_1(0)$ and $y_2(0)$).**   **Solution:** To perform our change of basis on our original state vector $\vec{x}$, we had to apply the following transformation:

$$\vec{y}(t) = V^{-1}\vec{x}(t) \tag{63}$$

There are a few ways to find V^{-1}. One is to use the determinant as follows:

$$V^{-1} = \frac{1}{\det(V)} \begin{bmatrix} 1 & 0.5 \\ -1 & -1 \end{bmatrix} \tag{64}$$

$$V^{-1} = \frac{1}{-0.5} \begin{bmatrix} 1 & 0.5 \\ -1 & -1 \end{bmatrix} \tag{65}$$

$$V^{-1} = \begin{bmatrix} -2 & -1 \\ 2 & 2 \end{bmatrix} \tag{66}$$

Therfore, to get our initial conditions in the diagonalized basis, we would perform the same transformation:

$$\vec{y}(0) = V^{-1}\vec{x}(t) \tag{67}$$

$$\vec{y}(0) = \begin{bmatrix} -2 & -1 \\ 2 & 2 \end{bmatrix} \begin{bmatrix} x_1(0) \\ x_2(0) \end{bmatrix} \tag{68}$$

$$\vec{y}(0) = \begin{bmatrix} -2x_1(0) - x_2(0) \\ 2x_1(0) + 2x_2(0) \end{bmatrix} \tag{69}$$

Thus, $\widetilde{i}_R(0) = -2x_1(0) - x_2(0) = -5$ and $\widetilde{v}_C(0) = 2x_1(0) + 2x_2(0) = 6$.

(f) (6 pts.) **Using your answer to the previous question, solve for $i_L(t)$ and $v_C(t)$**   **Solution:**
With our system being diagonalized now, we can solve our differential equations separately (i.e.
$y(t) = \begin{bmatrix} y_1(t) \\ y_2(t) \end{bmatrix}$)
For $y_1(t)$:

$$\frac{dy_1(t)}{dt} = -2y_1(t) \tag{70}$$

$$y_1(t) = y_1(0)e^{-2t} \tag{71}$$

$$= -5e^{-2t} \tag{72}$$

Similarly for $y_2(t)$:

$$\frac{dy_2(t)}{dt} = -1y_2(t) \tag{73}$$

$$y_2(t) = y_2(0)e^{-t} \tag{74}$$

$$= 6e^{-t} \tag{75}$$

Finally, we get back our solution in the standard basis by doing our change of basis in reverse:

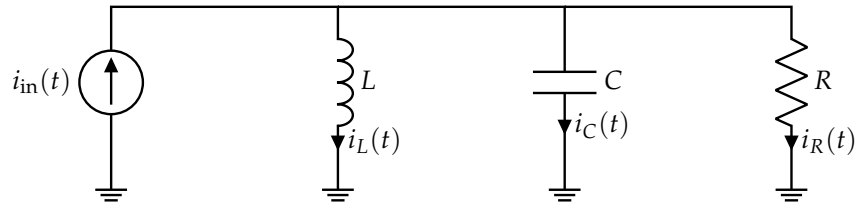$$\vec{x}(t) = V\vec{y}(t) \tag{76}$$

$$= \begin{bmatrix} -1 & -0.5 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} -5e^{-2t} \\ 6e^{-t} \end{bmatrix} \tag{77}$$

$$= \begin{bmatrix} 5e^{-2t} - 3e^{-t} \\ -5e^{-2t} + 6e^{-t} \end{bmatrix} \tag{78}$$

And since $\vec{x}(t) = \begin{bmatrix} i_L(t) \\ v_C(t) \end{bmatrix} = \begin{bmatrix} 5e^{-2t} - 3e^{-t} \\ -5e^{-2t} + 6e^{-t} \end{bmatrix}$, this give $i_L(t) = 5e^{-2t} - 3e^{-t}$ and $v_C(t) = -5e^{-2t} + 6e^{-t}$.

7. Parallel Resonance (20 pts.)

For this problem, we will examine a parallel resonance circuit, a common resonance circuit just like the series resonance circuit.

(a) (5 pts.) **Show that Z_{eq}, the equivalent impedance of the inductor, capacitor, and resistor as seen by the current source is:**

$$Z_{eq} = \frac{j\omega L}{1 + j\omega \frac{L}{R} + (j\omega)^2 LC} \tag{79}$$

Solution: The inductor, capacitor, and resistor are in parallel, so we can apply the parallel operator to the three component impedances.

$$Z_{eq} = Z_L || Z_C || Z_R \tag{80}$$

$$= j\omega L || \frac{1}{j\omega C} || R \tag{81}$$

$$= \frac{(j\omega L)(\frac{1}{j\omega C})}{j\omega L + \frac{1}{j\omega C}} || R \tag{82}$$

$$= \frac{j\omega L}{1 + (j\omega)^2 LC} || R \tag{83}$$

$$= \frac{(\frac{j\omega L}{1+(j\omega)^2 LC})R}{\frac{j\omega L}{1+(j\omega)^2 LC} + R} \tag{84}$$

$$= \frac{j\omega LR}{j\omega L + R + (j\omega)^2 LCR} \tag{85}$$

$$= \frac{j\omega L}{1 + j\omega \frac{L}{R} + (j\omega)^2 LC} \tag{86}$$

(b) (5 pts.) **Find the transfer function $H(j\omega) = \frac{\tilde{I}_R}{\tilde{I}_{in}}$ for the current through the resistor** in terms of the component variables and frequency ω.

(HINT: The provided expression for Z_{eq} may be helpful here.)

Solution: We can either use our expression for Z_{eq} to find the voltage across the resistor $\tilde{V}_R$ first and then the current $\tilde{I}_R$, or directly apply a current divider equation to the system.

Method 1 (Voltage then Current): We can find the voltage across the resistor (and also the inductor and capacitor), by using Ohm's law with the equivalent impedance Z_{eq} from the previous problem part:

$$\tilde{V}_R = \tilde{I}_{in} Z_{eq} = \frac{j\omega L}{1 + j\omega \frac{L}{R} + (j\omega)^2 LC} \tilde{I}_{in} \tag{87}$$

Then, we can use Ohm's law again with the resistor itself to find the current through it in terms of the input current, which leads to $H(j\omega)$:

$$\widetilde{I}_R = \frac{\widetilde{V}_R}{R} \tag{88}$$

$$\widetilde{I}_R = \frac{j\omega \frac{L}{R}}{1 + j\omega \frac{L}{R} + (j\omega)^2 LC} \widetilde{I}_{in} \tag{89}$$

$$H(j\omega) = \frac{\widetilde{I}_R}{\widetilde{I}_{in}} = \frac{j\omega \frac{L}{R}}{1 + j\omega \frac{L}{R} + (j\omega)^2 LC} \tag{90}$$

Method 2 (Current Divider): We can apply a current divider between $Z_L || Z_C$ and Z_R:

$$H(j\omega) = \frac{Z_L || Z_C}{Z_L || Z_C + Z_R} \tag{91}$$

We can expand this out with the impedance expressions to obtain the same answer as with the previous method.

(c) (5 pts.) **Find the frequency ω_{max} that maximizes the value of the transfer function.** Additionally, **draw an equivalent circuit at this frequency**.

(HINT: You can do this problem even if you did not obtain the transfer function in the previous part. What frequency is crucial for RLC circuits?)

Solution: The key idea here is resonance; at resonance, the inductor and capacitor will have the same magnitude impedance with opposite phase, which causes specific circuit behavior. In this case, the inductor and capacitor in parallel will resonate to have infinite equivalent impedance (open circuit).

To be more thorough, the resonant frequency for this circuit (and many other common resonance circuits) is $\omega_0 = \frac{1}{\sqrt{LC}}$ (you can verify that the equivalent impedance is real at this frequency, which is the condition for resonance). At this frequency, $Z_C || Z_L = \frac{j\omega_0 L}{1 - \omega_0^2 LC} = \infty$, as stated earlier. Thus, the equivalent circuit in this case would be:

Thus, the input current directly translates to the resistor current, implying that the transfer function at the resonant frequency is $H(j\omega_0) = 1$.

(d) (3 pts.) The quality factor for this circuit can be defined as the magnitude of the ratio of current across the capacitor to current across the resistor ($Q_p = \left| \frac{\widetilde{I}_C}{\widetilde{I}_R} \right|$) at the resonant frequency ω_0. **Show that the quality factor for this circuit is $Q_p = \omega_0 RC$.**

Solution: The current through the capacitor and resistor can be found in terms of the voltage across them, which is the same ($\widetilde{V} = \widetilde{V}_C = \widetilde{V}_R$):

$$\widetilde{I}_R = \frac{\widetilde{V}}{Z_R} \tag{92}$$

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$$\widetilde{I}_C = \frac{\widetilde{V}}{Z_C} \tag{93}$$

We can notice that the ratio of these two expressions will cancel out $\widetilde{V}$, leaving only constant values (since we know we are at the resonant frequency) for our quality factor:

$$Q_p = \left| \frac{\widetilde{I}_C}{\widetilde{I}_R} \right| = \left| \frac{Z_R}{Z_C} \right| = \left| \frac{R}{\frac{1}{j\omega_0 C}} \right| = \omega_0 R C \tag{94}$$

(e) (2 pts.) Suppose we want to use this circuit as a filter, but with the current component values we have used, the filter is not sharp enough (too low quality factor).

Suppose we have the ability to double or halve exactly one of the component values, with the goal of maximizing our quality factor.

Which component value should we change and should we double it or halve it? Please briefly explain why.

(HINT: The resonant frequency ω_0 will depend on the component values as well so you should consider this factor.)

Solution: To start off, we should write our quality factor in terms of the three component variables, R, L, and C:

$$Q_p = \omega_0 R C = \frac{1}{\sqrt{LC}} R C = R \sqrt{\frac{L}{C}} \tag{95}$$

From this expression, we can clearly see that factor changes in the resistance will have the most impact on our quality factor, so we should change component value R.

Since the quality factor is proportional to R, we need to increase it to increase the quality factor so we should double it.

8. **Bode Plot Design (20 pts.)**

   For this question, we will design a filter based on the following provided Bode plot.
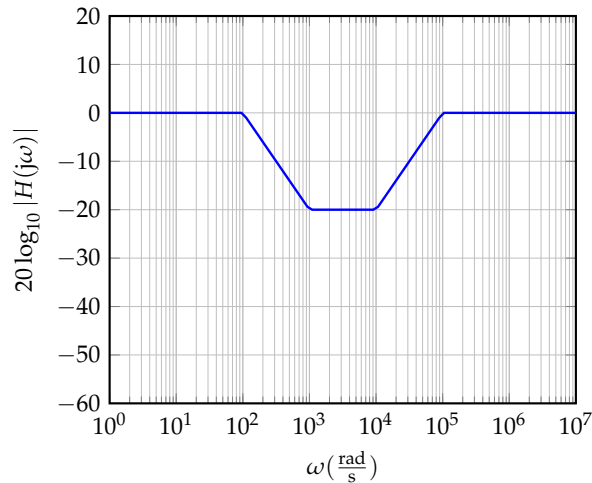
   (a) (5 pts.) **What are the pole and zero frequencies of this Bode plot?**

   Solution: For the magnitude Bode plot, the pole frequencies are where the slope decreases by $20\frac{\text{dB}}{\text{dec}}$, while the zero frequencies are where the slope increases by $20\frac{\text{dB}}{\text{dec}}$.

   Based on the plot, the slope starts off at $0\frac{\text{dB}}{\text{dec}}$, decreases to $-20\frac{\text{dB}}{\text{dec}}$ at $\omega = 10^2\frac{\text{rad}}{\text{s}}$, increases to $0\frac{\text{dB}}{\text{dec}}$ at $\omega = 10^3\frac{\text{rad}}{\text{s}}$, increases to $+20\frac{\text{dB}}{\text{dec}}$ at $\omega = 10^4\frac{\text{rad}}{\text{s}}$, and then finally decreases to $0\frac{\text{dB}}{\text{dec}}$ at $\omega = 10^5\frac{\text{rad}}{\text{s}}$.

   Thus, the pole and zero frequencies are:

   - Poles: $\omega_{p1} = 10^2\frac{\text{rad}}{\text{s}}$, $\omega_{p2} = 10^5\frac{\text{rad}}{\text{s}}$
   - Zeros: $\omega_{z1} = 10^3\frac{\text{rad}}{\text{s}}$, $\omega_{z2} = 10^4\frac{\text{rad}}{\text{s}}$

   (b) (5 pts.) **Find a transfer function $H(\text{j}\omega)$ that would produce the provided magnitude Bode plot.**

   Solution: We can use our knowledge of poles and zeros to directly correspond each of them to factors of the transfer function in the numerator (for zeros) and denominator (for poles), where each factor takes the form of $1 \pm \text{j}\frac{\omega}{\omega_{zp}}$ with ω_{zp} as the zero or pole frequency (you can choose either plus or minus for each factor as this will not impact the magnitude Bode plot).

   We also know we do not need a factor in front because the Bode plot starts at 0dB or 1 and there are no poles or zeros at $\omega = 0$ based on the flat slope the plot starts with.
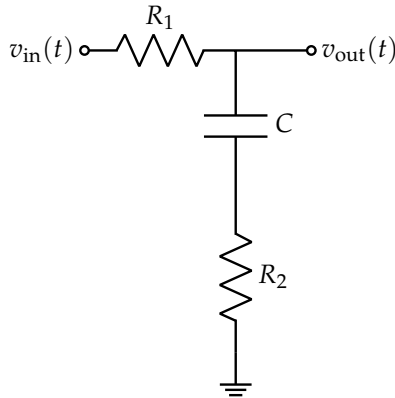
   Thus, the transfer function will be:

   $$H(\text{j}\omega) = \frac{(1 \pm \text{j}\frac{\omega}{10^3})(1 \pm \text{j}\frac{\omega}{10^4})}{(1 \pm \text{j}\frac{\omega}{10^2})(1 \pm \text{j}\frac{\omega}{10^5})} \tag{96}$$

   (c) (10 pts.) Suppose we had a different Bode plot and found our transfer function to be:

   $$H(\text{j}\omega) = \frac{(1 + \text{j}\frac{\omega}{10^4})(1 + \text{j}\frac{\omega}{10^6})}{(1 + \text{j}\frac{\omega}{10^3})(1 + \text{j}\frac{\omega}{10^5})} \tag{97}$$

We want to design a filter that will implement this transfer function. We know the transfer function for the following circuit:

The transfer function for this is:

$$H(j\omega) = \frac{1 + j\omega R_2 C}{1 + j\omega(R_1 + R_2)C} \tag{98}$$

Suppose we only have $1\,\mathrm{k\Omega}$ and $9\,\mathrm{k\Omega}$ resistors, but we have tunable capacitors (so we can choose any capacitance values we want). We also have op-amps present.

Draw a filter that implements the transfer function. Please label all component values on your diagram.

Solution: Since the pole and zero frequencies seem to be separated by factors of 10, it makes sense to choose $R_2 = 1\,\mathrm{k\Omega}$ and $R_1 = 9\,\mathrm{k\Omega}$ (since the zeros will have the factor of $R_2 = 1\,\mathrm{k\Omega}$ and the poles will have the factor of $R_1 + R_2 = 10\,\mathrm{k\Omega}$).

We can split our transfer function into two logical parts (based on the resistance values we chose):

$$H_1(j\omega) = \frac{1 + j\frac{\omega}{10^4}}{1 + j\frac{\omega}{10^3}} \tag{99}$$

$$H_2(j\omega) = \frac{1 + j\frac{\omega}{10^6}}{1 + j\frac{\omega}{10^5}} \tag{100}$$

This is done because the pole frequencies are now $\frac{1}{10}$ of the zero frequencies, which matches the resistance value ratios.

The overall transfer function is $H(j\omega) = H_1(j\omega)H_2(j\omega)$, so we will need to cascade two filters together, with a buffer in between to prevent loading between the stages.

For the first stage, we need $R_2 C = 10^{-4}$, so we choose $C = 10^{-7} = 0.1\,\mathrm{\mu F}$. We can verify that $(R_1 + R_2)C = 10^{-3}$, as desired.
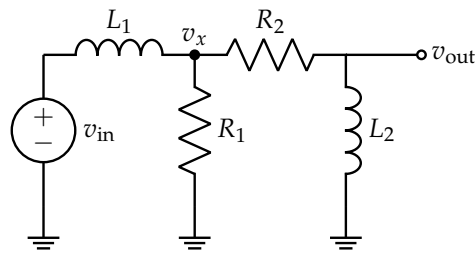
For the second stage, we need $R_2 C = 10^{-6}$, so we choose $C = 10^{-9} = 1\,\mathrm{nF}$. We can verify that $(R_1 + R_2)C = 10^{-5}$, as desired.

With these component values, we can draw the following circuit diagram for our filter:
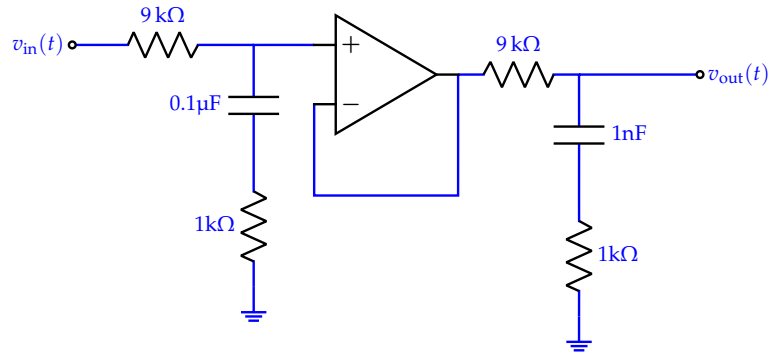
9. Transfer Functions (40 pts.)

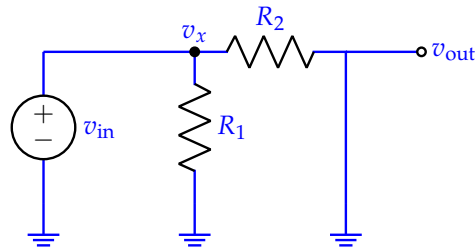For this problem, we will analyze the following circuit.

We want to find the overall transfer function $H(j\omega) = \frac{\widetilde{V}_{out}}{\widetilde{V}_{in}}$.

(a) (5 pts.) **What are $H(j0)$ and $H(j\infty)$?**

(HINT: You can (and are intended to) do this without finding the whole transfer function.)

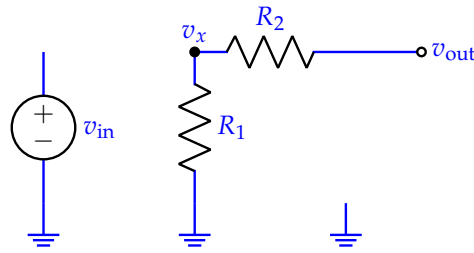**Solution:** For $\omega = 0$, we know the inductors act like short circuits ($Z_L(0) = 0$).

Thus, the equivalent circuit for $\omega = 0$ will be:

Since $\widetilde{V}_{out} = 0$ here (connected directly to ground), $H(j0) = 0$.

For $\omega = \infty$, we know the inductors act like open circuits ($Z_L(\infty) = \infty$).

Thus, the equivalent circuit for $\omega = \infty$ will be:

Since $\widetilde{V}_{\text{out}} = 0$ here (the input is disconnected from the output so there is no current through the resistors and thus, the ground connection translates to the output), $H(j\infty) = 0$.

(b) (5 pts.) **Find the equivalent impedance of R_1, R_2, and L_2 seen by node v_x.**

Please simplify to the following form:

$$Z_{eq} = Z_0 \frac{1 + j\omega\tau_1}{1 + j\omega\tau_2} \tag{101}$$

where Z_0, τ_1, and τ_2 should be in terms of R_1, R_2, and L_2 (this equivalent impedance will not include L_1).

Solution: The resistor R_1 is in parallel with the series connection of R_2 and L_2, so we can perform these operations with the impedance formulas for each component.

$$Z_{eq} = Z_{R_1} || (Z_{R_2} + Z_{L_2}) \tag{102}$$

$$= R_1 || (R_2 + j\omega L_2) \tag{103}$$

$$= \frac{R_1(R_2 + j\omega L_2)}{R_1 + R_2 + j\omega L_2} \tag{104}$$

$$= \frac{R_1 R_2 + j\omega R_1 L_2}{R_1 + R_2 + j\omega L_2} \tag{105}$$

$$= \frac{R_1 R_2}{R_1 + R_2} \frac{1 + j\omega \frac{L_2}{R_2}}{1 + j\omega \frac{L_2}{R_1 + R_2}} \tag{106}$$

$$= (R_1 || R_2) \frac{1 + j\omega \frac{L_2}{R_2}}{1 + j\omega \frac{L_2}{R_1 + R_2}} \tag{107}$$

To compare to the desired form, $Z_0 = R_1 || R_2$, $\tau_1 = \frac{L_2}{R_2}$, $\tau_2 = \frac{L_2}{R_1 + R_2}$.

(c) (8 pts.) **Find the transfer function from input to node v_x, $H_1(j\omega) = \frac{\widetilde{V}_x}{\widetilde{V}_{\text{in}}}$.**

You may use Z_0, τ_1 and τ_2 (along with the component variables) in your answer. Additionally, please simplify your answer to the following form:

$$H_1(j\omega) = A_0 \frac{1 + j\omega\tau_3}{1 + j\omega\tau_4 + (j\omega)^2 \tau_5} \tag{108}$$

where A_0, τ_3, τ_4, and τ_5 should be in terms of L_1, Z_0, τ_1, and τ_2.

Solution: Just as the hint advises, we can use Z_{eq} to simplify our calculations; in this case, it allows us to view the problem as a voltage divider between Z_{L_1} and Z_{eq}.

$$H_1(j\omega) = \frac{Z_{eq}}{Z_{eq} + Z_{L_1}} \tag{109}$$

$$= \frac{Z_0 \frac{1+j\omega\tau_1}{1+j\omega\tau_2}}{Z_0 \frac{1+j\omega\tau_1}{1+j\omega\tau_2} + j\omega L_1} \tag{110}$$

$$= \frac{Z_0(1+j\omega\tau_1)}{Z_0(1+j\omega\tau_1) + j\omega L_1 + (j\omega)^2 L_1 \tau_2} \tag{111}$$

$$= \frac{1+j\omega\tau_1}{1+j\omega(\tau_1 + \frac{L_1}{Z_0}) + (j\omega)^2 \frac{L_1 \tau_2}{Z_0}} \tag{112}$$

To compare to the desired form, $A_0 = 1$, $\tau_3 = \tau_1$, $\tau_4 = \tau_1 + \frac{L_1}{Z_0}$, $\tau_5 = \frac{L_1 \tau_2}{Z_0}$.

(d) (5 pts.) **Find the transfer function from node v_x to output, $H_2(j\omega) = \frac{\widetilde{V}_{\text{out}}}{\widetilde{V}_x}$.**

Solution: This problem is just a voltage divider between Z_{R_2} and Z_{L_2}.

$$H_2(j\omega) = \frac{Z_{L_2}}{Z_{R_2} + Z_{L_2}} = \frac{j\omega L_2}{R_2 + j\omega L_2} = \frac{j\omega \frac{L_2}{R_2}}{1 + j\omega \frac{L_2}{R_2}} \tag{113}$$

(e) (2 pts.) **What is $H(j\omega)$ in terms of $H_1(j\omega)$ and $H_2(j\omega)$?**

Solution: The overall transfer function will just be the multiplication of the partial transfer functions as the variables indicate:
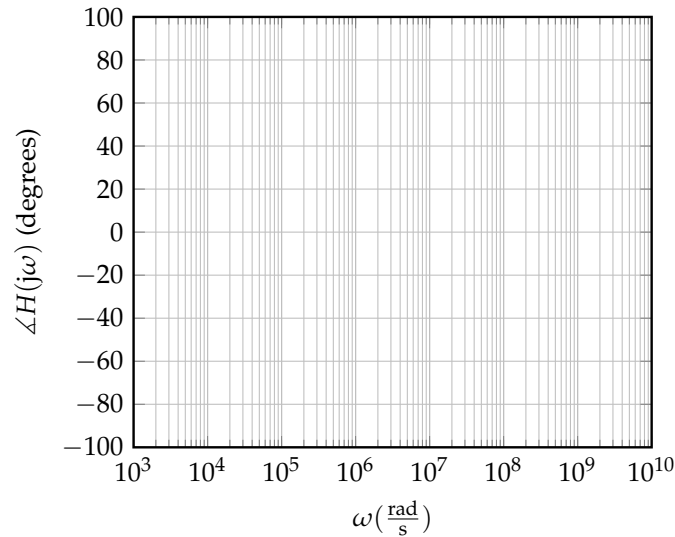
$$H(j\omega) = \frac{\widetilde{V}_{\text{out}}}{\widetilde{V}_{\text{in}}} = \frac{\widetilde{V}_x}{\widetilde{V}_{\text{in}}} \cdot \frac{\widetilde{V}_{\text{out}}}{\widetilde{V}_x} = H_1(j\omega) H_2(j\omega) \tag{114}$$

(f) (15 pts.) Suppose that with component values, we find that

$$H(j\omega) = \frac{j\omega 10^{-7}}{1 + j\omega(10^{-8} + 10^{-5}) + (j\omega)^2 10^{-13}} \tag{115}$$

What are the pole and zero frequencies of this transfer function? Use these to plot the Bode plots for $H(j\omega)$ on the provided plots below. *(HINT: It may be useful to know that $H(j10^7) \approx \frac{1}{100}$.)*
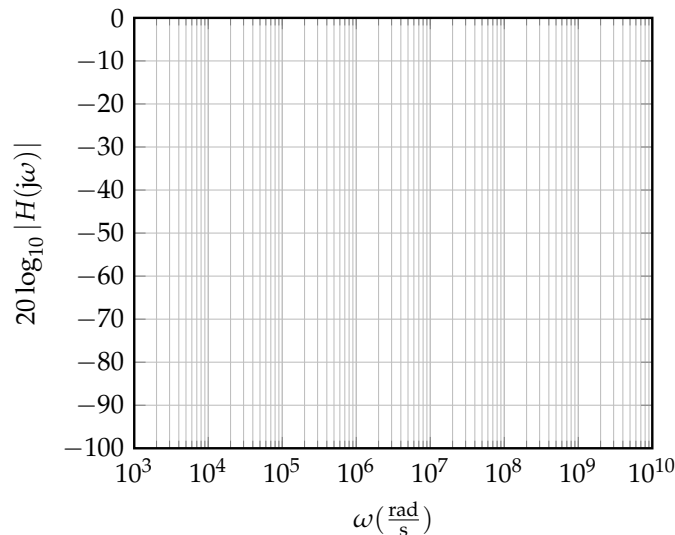
Solution: To find the poles and zeros, we should first try to factor the transfer function into the desirable rational function form:

$$H(j\omega) = \frac{j\omega 10^{-7}}{(1 + j\omega 10^{-5})(1 + j\omega 10^{-8})} = \frac{j\frac{\omega}{10^7}}{(1 + j\frac{\omega}{10^5})(1 + j\frac{\omega}{10^8})} \tag{116}$$

From this form, we can clearly identify the transfer function's poles and zeros:

- Poles: $\omega_{p1} = 10^5 \frac{\text{rad}}{\text{s}}$, $\omega_{p2} = 10^8 \frac{\text{rad}}{\text{s}}$
- Zeros: $\omega_z = 0$

For the magnitude plot, we know that we will start with a slope of $+20\frac{\text{dB}}{\text{dec}}$ (due to $\omega_z = 0$), decrease to a slope of $0\frac{\text{dB}}{\text{dec}}$ due to $\omega_{p1} = 10^5 \frac{\text{rad}}{\text{s}}$, and then decrease to a slope of $-20\frac{\text{dB}}{\text{dec}}$ due to $\omega_{p2} = 10^8 \frac{\text{rad}}{\text{s}}$.

We also need to know what value the transfer function will have when the slope is $0\frac{\text{dB}}{\text{dec}}$ to properly construct the plot, which is why the hint tells us that $H(j10^7) = \frac{1}{100}$ (a frequency value within this range) and thus indicates that the value for this flat part will be $20\log\left(\frac{1}{100}\right) = -40\text{dB}$.

For the phase plot, we will start at $90°$ due to $\omega_z = 0$. The phase will drop from $90°$ to $0°$ from $\omega = 10^4$ to $\omega = 10^6$ due to $\omega_{p1} = 10^5 \frac{\text{rad}}{\text{s}}$. Similarly, the phase will drop from $0°$ to $-90°$ from $\omega = 10^7$ to $\omega = 10^9$ due to $\omega_{p1} = 10^8 \frac{\text{rad}}{\text{s}}$.

The Bode plots are thus shown below:
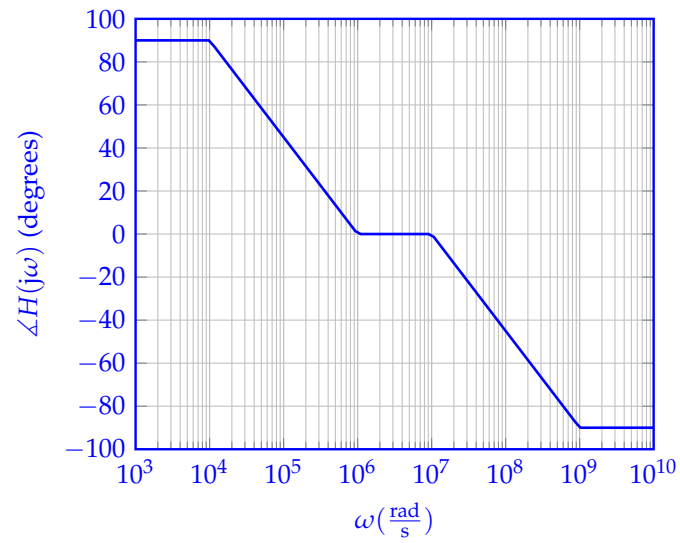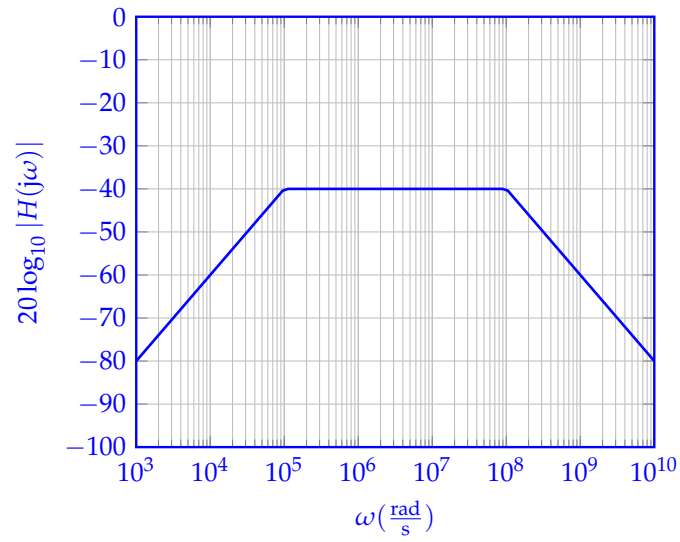
10. **All-Pass Reverb (20 pts.)**

    You've been hired as a sound engineer for a music production company.

    You are given a request to create a filter that processes audio signals. The filter has the following requirements.

    (a) The filter should attenuate all frequencies of audio signals by the same amount (in contrast to something like a low-pass filter which only attenuates low frequency signals)

    (b) The filter should allow for precise changes to the inputted audio signal's phase

    After consulting your helpful TAs, Chancharik and Nikhil, you receive the suggestion to look into the **all-pass filter**. A circuit diagram of an all-pass filter is shown where it is build using an *ideal* op-amp.
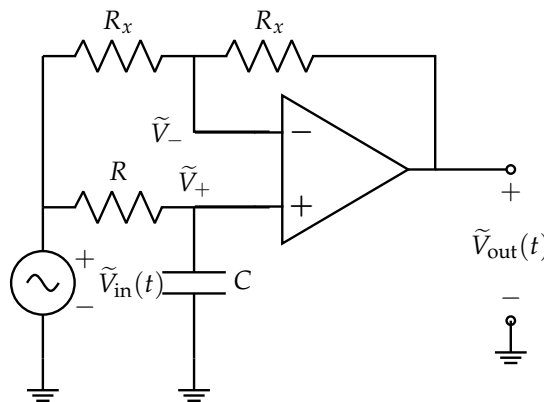
    Intrigued, you decide to probe the nature of the transfer function of this filter further using phasors and frequency domain analysis.

    (a) (3 pts.) **First, solve for $\widetilde{V}_+$ in terms of $\widetilde{V}_{in}$, $j\omega$, and any other necessary components.**

    Solution: $\widetilde{V}_+$ is clearly the output voltage of a low-pass RC circuit. Thus,

    $$\widetilde{V}_+ = \frac{Z_C}{Z_C + Z_R}\widetilde{V}_{in} \tag{117}$$

    $$\widetilde{V}_+ = \frac{1}{1 + j\omega RC}\widetilde{V}_{in} \tag{118}$$

    (b) (4 pts.) **Write $\widetilde{V}_{out}$ in terms of $\widetilde{V}_{in}$, $\widetilde{V}_+$, and any other necessary components.**

    (HINT: Properties of ideal op-amps might be helpful here.)

    Solution: We start with KCL on the $\widetilde{V}_-$ node, knowing that (1) $\widetilde{I}_- = 0$ and (2) $\widetilde{V}_+ = \widetilde{V}_-$ by properties of ideal op-amps:

    $$\widetilde{I}_{2R_x} = \widetilde{I}_- + \widetilde{I}_{R_x} \tag{119}$$

    $$\frac{\widetilde{V}_{in} - \widetilde{V}_-}{R_x} = \frac{\widetilde{V}_- - \widetilde{v}_{out}}{R_x} \tag{120}$$

    $$\widetilde{V}_{out} = 2\widetilde{V}_- - \widetilde{V}_{in} \tag{121}$$

$$\widetilde{V}_{\text{out}} = 2\widetilde{V}_{+} - \widetilde{V}_{\text{in}} \tag{122}$$

(c) (5 pts.) **Now, find the transfer function, $H(j\omega)$ for this filter.**

Solution: Let's expand our answers from the previous part.

$$v_{out}(t) = 2\widetilde{V}_{+} - \widetilde{V}_{\text{in}} \tag{123}$$

$$= \frac{2}{1 + j\omega RC}\widetilde{V}_{\text{in}} - \frac{1 + j\omega RC}{1 + j\omega RC}\widetilde{V}_{\text{in}} \tag{124}$$

$$= \frac{1 - j\omega RC}{1 + j\omega RC}\widetilde{V}_{\text{in}} \tag{125}$$

Thus, our transfer function is given by $H(j\omega) = \frac{1-j\omega RC}{1+j\omega RC}$.

(d) (4 pts.) Regardless of your answers to the previous problem part, suppose that the transfer function of the all-pass filter is given by:

$$H(j\omega) = \frac{1 - j\omega RC}{2(1 + j\omega RC)} \tag{126}$$

Find $|H(j\omega)|$ and $\angle H(j\omega)$. Qualitatively describe what the filter is doing and how it aligns with your goals mentioned previously (one or two sentences maximum).

Solution: From inspection, we can see that $|H(j\omega)| = \frac{1}{2}$.

For phase:

$$\angle H(j\omega) = \angle(1 - j\omega RC) - \angle(1 + j\omega RC) \tag{127}$$

$$= -2\arctan(\omega RC) \tag{128}$$

(e) (4 pts.) Using your expression for $\angle H(j\omega)$, **Propose a design for your filter (i.e. suggest values for R and C) to shift the phase of a signal with $\omega = 10^3$ by $-\frac{\pi}{2}$.**

Solution:

$$-2\arctan(\omega RC) = -\frac{\pi}{2} \tag{129}$$

$$\arctan(\omega RC) = \frac{\pi}{4} \tag{130}$$

$$RC = 10^{-3} \tag{131}$$

Therefore any combination of R and C values that have a product of 10^{-3} is a valid answer here.