$$\boxed{\text{Exam Location: In Person}}$$

PRINT your student ID: _____

PRINT AND SIGN your name: _____ , _____ _____
                                          (last)   (first)   (sign)

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Row Number: _____   Seat Number: _____

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1. **Honor Code (0 pts.)**

   Please copy the following statement in the space provided below and sign your name.

   As a member of the UC Berkeley community, I act with honesty, integrity, and respect for others. I will follow the rules and do this exam on my own.

   Note that if you do not copy the honor code and sign your name, you will get a 0 on the exam.

   Solution: Any attempt to copy the honor code and sign should get full points.

2. **(EXAM EXTRA CREDIT) What is your favorite topic of the course? (2 pts.)**

   Solution: Any answer is sufficient.

| Do not turn this page until the proctor tells you to do so. |
| You can work on the above problems before time starts. |

3. **Diode System ID (10 pts.)**

Note: This problem does not require the use of your knowledge of circuits or diodes. The problem is about system ID.

Suppose that we want to characterize a diode D_1 with parasitic resistance R_p, represented by the following model:
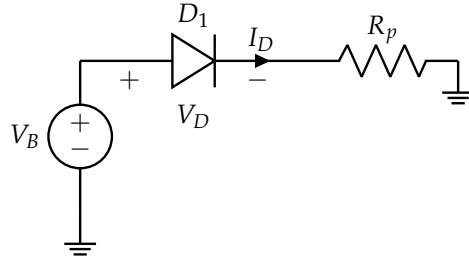
The current through the diode (forward-biased) is modeled as

$$I_D = I_0 e^{\frac{V_D}{V_T}} \tag{1}$$

where V_D is the voltage across the diode (as shown in the diagram), I_0 is a parameter associated with the diode, and V_T is the thermal voltage, a temperature dependent variable.

With some analysis, the system equation can be simplified to:

$$I_D = I_0 e^{\frac{V_B - I_D R_p}{V_T}} \tag{2}$$

To collect data, we will vary the temperature to vary V_T and measure the current I_D through the diode.

(a) (2 pts.) **Use the provided system equation to show that**

$$I_D R_p - V_T \ln(I_0) = V_B - V_T \ln(I_D) \tag{3}$$

is a valid equation to represent the system. *(HINT: Use the logarithm identity $\ln\left(\frac{a}{b}\right) = \ln(a) - \ln(b)$.)*

Solution:

$$I_D = I_0 e^{\frac{V_B - I_D R_p}{V_T}} \tag{4}$$

$$V_T \ln\left(\frac{I_D}{I_0}\right) = V_B - I_D R_p \tag{5}$$

$$V_T \ln(I_D) - V_T \ln(I_0) = V_B - I_D R_p \tag{6}$$

$$I_D R_p - V_T \ln(I_0) = V_B - V_T \ln(I_D) \tag{7}$$

(b) (4 pts.) Suppose we use $V_B = v_B$V and collect the following data points:

- For $V_T = v_{T1}$V, $I_D = i_{D1}$A.
- For $V_T = v_{T2}$V, $I_D = i_{D2}$A.

- For $V_T = v_{T3}$V, $I_D = i_{D3}$A.

Find the matrix D and vector $\vec{s}$ to set up a least-squares problem $D\vec{p} \approx \vec{s}$ where $\vec{p} = \begin{bmatrix} \ln(I_0) \\ R_p \end{bmatrix}$.

Solution: Each data point can be used with the provided system equation from the first part to create a system of linear equations with the unknowns being $\ln(I_0)$ and R_p (remember that the problem provides that $V_B = v_B$V):

- $-v_{T1}\ln(I_0) + i_{D1}R_p = v_B - v_{T1}\ln(i_{D1})$
- $-v_{T2}\ln(I_0) + i_{D2}R_p = v_B - v_{T2}\ln(i_{D2})$
- $-v_{T3}\ln(I_0) + i_{D3}R_p = v_B - v_{T3}\ln(i_{D3})$

The matrix form of this system is:

$$\begin{bmatrix} -v_{T1} & i_{D1} \\ -v_{T2} & i_{D2} \\ -v_{T3} & i_{D3} \end{bmatrix} \begin{bmatrix} \ln(I_0) \\ R_p \end{bmatrix} = \begin{bmatrix} v_B - v_{T1}\ln(i_{D1}) \\ v_B - v_{T2}\ln(i_{D2}) \\ v_B - v_{T3}\ln(i_{D3}) \end{bmatrix} \tag{8}$$

Thus, for the least-squares problem, $D = \begin{bmatrix} -v_{T1} & i_{D1} \\ -v_{T2} & i_{D2} \\ -v_{T3} & i_{D3} \end{bmatrix}$ and $\vec{s} = \begin{bmatrix} v_B - v_{T1}\ln(i_{D1}) \\ v_B - v_{T2}\ln(i_{D2}) \\ v_B - v_{T3}\ln(i_{D3}) \end{bmatrix}$

(c) (4 pts.) **Explain how you would use the least-squares system from the previous part to find model parameters I_0 and R_p.** You do not need to use actual values for this part; you explanation can just use D and $\vec{s}$ as known variables to show what calculations you would do to find I_0 and R_p.

Solution: We would first find the optimal parameter $\hat{p}$ with the least squares formula:

$$\hat{p} = (D^\top D)^{-1}D^\top \vec{s} \tag{9}$$

The elements of $\hat{p}$ represent optimal values for $\ln(I_0)$ and R_p so we set the elements of $\hat{p}$ equal to those parameters for our model:

$$\hat{p}_{v_B} = \ln(I_0) \rightarrow I_0 = e^{\hat{p}_1} \tag{10}$$

$$\hat{p}_2 = R_p \rightarrow R_p = \hat{p}_2 \tag{11}$$

Thus, we have found parameter values for I_0 and R_p from the least-squares system.

4. **Feedback Control of an Aircraft (8 pts.)**

Let the current state of an aircraft be described by its height above ground, vertical speed, and pitch (in order). Let the inputs to the system be the aircraft's engine thrust and the angle of the elevators (which control pitch). Moreover, assume that the aircraft's instruments are perfect so that there is no noise. The system is, in reality, quite complex, so you approximate it by linearization such that, the state system looks like:

$$\vec{x}[i+1] = A\vec{x}[i] + B\vec{u}[i]$$

where $A \in \mathbb{R}^{3\times3}, B \in \mathbb{R}^{3\times2}$, and

$$\vec{x}[i] = \begin{bmatrix} h[i] \\ s[i] \\ p[i] \end{bmatrix} \in \mathbb{R}^3$$

$$\vec{u}[i] = \begin{bmatrix} u_1[i] \\ u_2[i] \end{bmatrix} \in \mathbb{R}^2$$

To simplify the system further, you run Schur Decomposition on A to upper-triangularize it. This results in the system below:

$$\vec{\tilde{x}}[i+1] = \begin{bmatrix} 0.8 & 5 & 2 \\ 0 & -0.5+j & -0.1 \\ 0 & 0 & -0.5-j \end{bmatrix} \vec{\tilde{x}}[i] + \begin{bmatrix} 0 & 0 \\ 1 & 0 \\ 0 & 1 \end{bmatrix} \vec{u}[i]$$

where $\vec{\tilde{x}}[i] = \begin{bmatrix} \tilde{h}[i] & \tilde{s}[i] & \tilde{p}[i] \end{bmatrix}^T$

(a) (2 pts.) **Is the original system BIBO stable? Why or why not?**

Solution: The original system is unstable. Since Schur Decomposition is a similar transform, the eigenvalues of A are on the diagonal of $\tilde{A}$. The two complex eigenvalues have norm greater than one which means that the system is unstable as this is a discrete-time system.

(b) (6 pts.) Now, we want to apply closed-loop feedback to the system such that:

$$\tilde{s}[i+1] = \tilde{s}[i] \text{ and } \tilde{p}[i+1] = \tilde{p}[i]$$

Using these equations, **apply a closed loop feedback and solve for the coefficients f_1, f_2, and f_3 in**

$$F = \begin{bmatrix} 0 & f_1 & f_2 \\ 0 & 0 & f_3 \end{bmatrix}$$

Solution:

$$\vec{\tilde{x}}[i+1] = \tilde{A}\vec{\tilde{x}}[i] + \tilde{B}\vec{u}[i]$$

$$= \tilde{A}\vec{\tilde{x}}[i] + \tilde{B}F\vec{\tilde{x}}[i]$$

$$= (\tilde{A} + \tilde{B}F)\vec{\tilde{x}}[i]$$

$$= \begin{bmatrix} 0.8 & 5 & 2 \\ 0 & -0.5+j+f_1 & -0.1+f_2 \\ 0 & 0 & -0.5-j+f_3 \end{bmatrix} \vec{\tilde{x}}[i]$$

since

$$\tilde{B}F = \begin{bmatrix} 0 & 0 \\ 1 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 0 & f_1 & f_2 \\ 0 & 0 & f_3 \end{bmatrix} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & f_1 & f_2 \\ 0 & 0 & f_3 \end{bmatrix}$$

Then, the characteristic polynomial of $\tilde{A} + \tilde{B}F$ is

$$p(\lambda) = (0.8 - \lambda)(-0.5 + j + f_1 - \lambda)(-0.5 - j + f_3 - \lambda)$$

Solving for the roots, we get: $\lambda_1 = 0.8$, $\lambda_2 = -0.5 + j + f_1$, and $\lambda_3 = -0.5 - j + f_3$. Since we want the state parameters to have constant values, we should set $\lambda_2, \lambda_3 = 1$. Therefore, $f_1 = 1.5 - j$ and $f_3 = 1.5 + j$. Finally, we want to zero out the $p[i]$ term in $s[i+1]$ so $f_2 = 0.1$.

(c) (4 pts.) **(EXAM EXTRA CREDIT)** For this part, in addition to the closed-loop feedback from the previous part, we also add the open loop input vector

$$\vec{u}_{OL}[i] = \begin{bmatrix} u_{1,OL}[i] \\ u_{2,OL}[i] \end{bmatrix} \in \mathbb{R}^2$$

Regardless of your answers for the previous part, suppose that you found, $f_1 = -j, f_2 = 1.1, f_3 = j$. **Is the system with closed-loop feedback and open loop input controllable? Why or why not?** *(HINT: First calculate $A_{CL}\tilde{B}$ and $\tilde{B}$. Do you need to calculate $A_{CL}^2\tilde{B}$ as well?)*

Solution: By adding the open loop control, the system becomes:

$$\vec{x}[i+1] = \tilde{A}\vec{x}[i] + \tilde{B}\vec{u}[i]$$

$$= \tilde{A}\vec{x}[i] + \tilde{B}(F\vec{x}[i] + \vec{u}_{OL}[i])$$

$$= (\tilde{A} + \tilde{B}F)\vec{x}[i] + \tilde{B}\vec{u}_{OL}[i]$$

$$= A_{CL}\vec{x}[i] + \tilde{B}\vec{u}_{OL}[i]$$

Then, the system is controllable because

$$\tilde{C}_3 = \begin{bmatrix} A_{CL}^2\tilde{B} & A_{CL}\tilde{B} & \tilde{B} \end{bmatrix}$$

$$= \begin{bmatrix} 1.5 & 5.6 & 5 & 2 & 0 & 0 \\ 0.25 & -1 & -0.5 & 1 & 1 & 0 \\ 0 & 0.25 & 0 & -0.5 & 0 & 1 \end{bmatrix}$$

is full rank, using $f_1 = -j, f_2 = 1.1, f_3 = j$ to calculate A_{CL}. In terms of the hint, it was sufficient to calculate $A_{CL}\tilde{B}$ and $\tilde{B}$ because these four column vectors already form a basis for $\mathbb{R}^3$ which implies that our controllability matrix is full rank.

5. **Designing a Controllable Aircraft (8 pts.)**

Continuing from the previous problem, we have a similar setup where an aircraft's state can be described by its vertical speed and pitch only. Therefore, the new system will look like this:

$$\begin{bmatrix} s[i+1] \\ p[i+1] \end{bmatrix} = \begin{bmatrix} a_1 & a_2 \\ a_3 & a_4 \end{bmatrix} \begin{bmatrix} s[i] \\ p[i] \end{bmatrix} + \begin{bmatrix} b_1 \\ b_2 \end{bmatrix} u[i]$$

You are working with aeronautical engineers to come up with a design for an aircraft that can always recover from a stall.

(a) (3 pts.) Suppose that these are the designed system parameters.

$$\begin{bmatrix} s[i+1] \\ p[i+1] \end{bmatrix} = \begin{bmatrix} 4 & 7 \\ -2 & 3 \end{bmatrix} \begin{bmatrix} s[i] \\ p[i] \end{bmatrix} + \begin{bmatrix} 1 \\ 2 \end{bmatrix} u[i]$$

Is the system controllable?

Solution: The system is controllable because

$$C_2 = \begin{bmatrix} A\vec{b} & \vec{b} \end{bmatrix}$$

$$= \begin{bmatrix} 18 & 1 \\ -4 & 2 \end{bmatrix}$$

is full rank.

(b) (3 pts.) Based on your answer for the previous part, **is it possible for the aircraft to recover from a stall (ie. an underisable state)? Why or why not?** Note: Recovering means that the aircraft leaves an undesirable state and gets steered to (or eventually reaches) a desirable state.

Solution: Yes. Since the system that describes the aircraft's state is controllable, the system can reach any final state from any initial state. Therefore, if we started from a state that is a stall, which is an undesirable state, the aircraft will be able to recover from the stall by reaching a desirable state.

(c) (2 pts.) Regardless of the previous parts, you found that the system was actually not controllable. The engineers re-design the parts and you find the following system parameters.

$$\begin{bmatrix} s[i+1] \\ p[i+1] \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ -2 & 3 \end{bmatrix} \begin{bmatrix} s[i] \\ p[i] \end{bmatrix} + \begin{bmatrix} 0 \\ 1 \end{bmatrix} u[i]$$

Given an initial state where the aircraft has stalled (an undesirable state), does there exist a sequence of inputs that can recover the aircraft? Why or why not? Note: Recall that recovering means that the aircraft leaves an undesirable state and gets steered to (or eventually reaches) a desirable state.

Solution: This system is in CCF form which means that the system is controllable. Moreover, controllability implies reachability since the column space of A is all of $\mathbb{R}^2$. Therefore, there does exist a sequence of inputs that can recover the aircraft. It is also correct to calculate the controllability matrix and showing that it is full rank.

6. **SVD (30 pts.)**

Consider the following matrix $A = \begin{bmatrix} 3 & 2 & 2 \\ 2 & 3 & -2 \end{bmatrix}$. In this problem, we will be finding the SVD of the matrix A.

(a) (4 pts.) **First, find the eigenvalues and eigenvectors of AA^T.**

Solution: Multiplying, we find that

$$AA^T = \begin{bmatrix} 17 & 8 \\ 8 & 17 \end{bmatrix} \tag{12}$$

From inspection, we can then determine that $\lambda_1 = 25$, $\vec{v}_1 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$, $\lambda_2 = 9$, and $\vec{v}_1 = \begin{bmatrix} 1 \\ -1 \end{bmatrix}$

(b) (2 pts.) **Now, what are the singular values σ_1 and σ_2 as well as the corresponding columns $\vec{u}_1$ and $\vec{u}_2$ of U (note: $\sigma_1 >= \sigma_2$) in the SVD of A?**

Solution: Using our answer to the previous question, the singular values of A are simply $\sigma_1 = \sqrt{\lambda_1} = 5$ and $\sigma_2 = \sqrt{\lambda_2} = 3$.

Similarly, the corresponding columns $\vec{u}_1$ and $\vec{u}_2$ of U are $\vec{u}_1 = \begin{bmatrix} \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} \end{bmatrix}$ and $\vec{u}_2 = \begin{bmatrix} \frac{1}{\sqrt{2}} \\ -\frac{1}{\sqrt{2}} \end{bmatrix}$

(c) (4 pts.) **Using your answer to the previous question, find $\vec{v}_1$ and $\vec{v}_2$, the first two columns of V in SVD. Show all your work.**

(HINT: If we know that $V_r^T = \Sigma_r^{-1} U_r^T A$, how can we write this matrix expression for the compact SVD in terms of the individual vectors $\vec{v}_i^T$ (and thus solving for $\vec{v}_i$)?)

Solution: To solve for $\vec{v}_i$, we can rearrange the equation in the hint to as follows:

$$\vec{v}_i^T = \frac{1}{\sigma_i} \vec{u}_i^T A \tag{13}$$

Therfore, we find that $\vec{v}_1 = \left(\vec{v}_1^T\right)^T = \left(\frac{1}{\sigma_1} \vec{u}_1^T A\right)^T = \left(\frac{1}{5} \begin{bmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{bmatrix} \begin{bmatrix} 3 & 2 & 2 \\ 2 & 3 & -2 \end{bmatrix}\right)^T = \begin{bmatrix} \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} \\ 0 \end{bmatrix}$

And $\vec{v}_2 = \left(\vec{v}_2^T\right)^T = \left(\frac{1}{\sigma_1} \vec{u}_2^T A\right)^T = \left(\frac{1}{3} \begin{bmatrix} \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} \end{bmatrix} \begin{bmatrix} 3 & 2 & 2 \\ 2 & 3 & -2 \end{bmatrix}\right)^T = \begin{bmatrix} \frac{1}{3\sqrt{2}} \\ -\frac{1}{3\sqrt{2}} \\ \frac{4}{3\sqrt{2}} \end{bmatrix}$

(d) (5 pts.) Regardless of your answer to the previous question, suppose $\vec{v}_1 = \begin{bmatrix} \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} \\ 0 \end{bmatrix}$ and $\vec{v}_2 = \begin{bmatrix} \frac{1}{3\sqrt{2}} \\ -\frac{1}{3\sqrt{2}} \\ \frac{4}{3\sqrt{2}} \end{bmatrix}$. **Find $\vec{v}_3$ using Gram-Schmidt basis extension with the standard basis vector $\vec{e}_1 = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$. Show all your work.**

Solution: In order to extend $\vec{v}_1$ and $\vec{v}_2$ using basis extension, we need to append the standard basis vectors. Thus, we are performing Gram-Schmidt on the following set of vectors:

$$\left\{ \begin{bmatrix} \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} \\ 0 \end{bmatrix}, \begin{bmatrix} \frac{1}{3\sqrt{2}} \\ -\frac{1}{3\sqrt{2}} \\ \frac{4}{3\sqrt{2}} \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \right\}$$

Since $\vec{v}_1$ and $\vec{v}_2$ are already orthonormal, we start with $\vec{v}_2$ and $\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$.

$$\vec{z}_3 = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} - \left(\begin{bmatrix} 1 & 0 & 0 \end{bmatrix}^T \begin{bmatrix} \frac{1}{3\sqrt{2}} \\ -\frac{1}{3\sqrt{2}} \\ \frac{4}{3\sqrt{2}} \end{bmatrix} \right) \begin{bmatrix} \frac{1}{3\sqrt{2}} \\ -\frac{1}{3\sqrt{2}} \\ \frac{4}{3\sqrt{2}} \end{bmatrix} - \left(\begin{bmatrix} 1 & 0 & 0 \end{bmatrix}^T \begin{bmatrix} \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} \\ 0 \end{bmatrix} \right) \begin{bmatrix} \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} \\ 0 \end{bmatrix} \tag{14}$$

$$\vec{z}_3 = \begin{bmatrix} \frac{4}{9} \\ -\frac{4}{9} \\ -\frac{2}{9} \end{bmatrix} \tag{15}$$

Normalizing, we find that $\vec{v}_3 = \frac{\vec{z}_3}{\|\vec{z}_3\|} = \begin{bmatrix} \frac{2}{3} \\ -\frac{2}{3} \\ -\frac{1}{3} \end{bmatrix}$.

(e) (6 pts.) Regardless of your answers to all previous parts, suppose $\sigma_1 = 4$, $\sigma_2 = 2$, $\vec{u}_1 = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$,

$\vec{u}_2 = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$, $\vec{v}_3 = \begin{bmatrix} \frac{2}{3} \\ -\frac{2}{3} \\ -\frac{1}{3} \end{bmatrix}$, and $\vec{v}_1$ and $\vec{v}_2$ are defined as in part (d). **Using these definitions, write out the SVD of A *numerically* in compact form.**

Solution: The full SVD is given by:

$$A = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 4 & 0 & 0 \\ 0 & 2 & 0 \end{bmatrix} \begin{bmatrix} \frac{1}{\sqrt{2}} & \frac{1}{3\sqrt{2}} & \frac{2}{3} \\ \frac{1}{\sqrt{2}} & -\frac{1}{3\sqrt{2}} & -\frac{2}{3} \\ 0 & \frac{4}{3\sqrt{2}} & -\frac{1}{3} \end{bmatrix}^T \tag{16}$$

The compact SVD is thus given by:

$$A = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 4 & 0 \\ 0 & 2 \end{bmatrix} \begin{bmatrix} \frac{1}{\sqrt{2}} & \frac{1}{3\sqrt{2}} \\ \frac{1}{\sqrt{2}} & -\frac{1}{3\sqrt{2}} \\ 0 & \frac{4}{3\sqrt{2}} \end{bmatrix}^T \tag{17}$$

(f) (3 pts.) In the next several parts, consider the compact SVD of the same wide A matrix *symbolically* (i.e. $A = U_r \Sigma_r V_r^T$). Suppose you are told to solve the problem $A\vec{x} = \vec{b}$. **If $\vec{b} \in \text{Col}(A)$, are there zero, one, or many solutions for $\vec{x}$? Explain your answer briefly in one to two sentences referring to the $\text{Col}(A)$ and the $\text{Null}(A)$.**

Solution: If $\vec{b} \in \text{Col}(A)$, there are infinitely many solutions for $\vec{x}$. This is because A, being a wide matrix, has a nontrivial null space ($\text{Null}(A)$) due to more columns than rows. Any solution $\vec{x}_0$

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can be modified by adding a vector from $\text{Null}(A)$ (i.e., $\vec{x}_0 + \vec{n}$, where $\vec{n} \in \text{Null}(A)$) to get another solution.

(g) (3 pts.) **Symbollically, solve for $\vec{x}$ using the SVD of A. Your answer should only contain the terms U_r, Σ_r, and/or V_r and any necessary transposes or inverses. In one sentence or less, what is special about this solution to the equation?**

Solution:

$$\vec{x} = V_r \Sigma_r^{-1} U_r^T \vec{b} \tag{18}$$

This is the minimum norm solution out of all of the possible solutions for $\vec{x}$. An alternative answer is that this is the psuedo-inverse and it returns a unique solution for $\vec{x}$ regardless of the characteristics of A.

Note: $\vec{x} = V_r \Sigma_r^{\dagger} U_r^T \vec{b}$ is an equivalent answer.

(h) (3 pts.) **Simplify the quantity $AA^{\dagger}A$ as much as possible using the compact SVD of A ($A^{\dagger}$ is the psuedo-inverse of A). Your final answer should only be in terms of A. Show your work.**

Solution: Using the compact SVD of A, where $A = U_r \Sigma_r V_r^T$, and the inverse of Σ_r, denoted as Σ_r^{-1}, we can expand $AA^{\dagger}A$ as follows:

$$AA^{\dagger}A = U_r \Sigma_r V_r^T (V_r \Sigma_r^{-1} U_r^T) U_r \Sigma_r V_r^T \tag{19}$$

$$= U_r \Sigma_r (V_r^T V_r) \Sigma_r^{-1} (U_r^T U_r) \Sigma_r V_r^T \tag{20}$$

$$= U_r \Sigma_r I \Sigma_r^{-1} I \Sigma_r V_r^T \tag{21}$$

$$= U_r \Sigma V_r^T \tag{22}$$

$$= A \tag{23}$$

Here, we use the facts that $V_r^T V_r = I$ and $U_r^T U_r = I$ (since U_r and V_r are orthogonal matrices), and that $\Sigma_r \Sigma_r^{-1} = I$. Thus, $AA^{\dagger}A$ simplifies to A.

7. **PCA and LS Prediction (12 pts.)**

In this problem, you will be helping to analyze some clinical data for patients using PCA and least squares. Suppose the following is a data matrix consisting of scaled *zero-mean* patient data. The patients (i.e. data points) are the *columns*, while the rows from top to bottom represent scores for the height, weight, and blood pressure of the 4 patients.

$$A = \begin{bmatrix} -3 & 0 & 2 & 1 \\ -1 & -2 & 0 & 3 \\ -1 & -1 & 1 & 1 \end{bmatrix} \tag{24}$$

Using this information, we would like to predict a risk index for heart disease for each patient (negative if low-risk and positive if high risk). But unfortunately, there appears to be some noise mixed into the data. Let's try to mitigate this by *performing PCA to embed the training data into a lower dimension first*.

(a) (4 pts.) Let us perform PCA on A to embed the data into a single dimension. Suppose you know that $\sigma_1 = \sqrt{8}$ and $\vec{v}_1 = \begin{bmatrix} \frac{1}{2} \\ -\frac{1}{2} \\ -\frac{1}{2} \\ \frac{1}{2} \end{bmatrix}$ from the first element of Σ and first column of V for the SVD of A.

What is the first principal component $\vec{u}_1$ of A? Show all your work.

Solution: Using an intermediate step of solving for the SVD, we can find $\vec{u}_1$ as follows:

$$\vec{u}_1 = \frac{1}{\sigma_1} A \vec{v}_1 \tag{25}$$

$$= \begin{bmatrix} -3 & 0 & 2 & 1 \\ -1 & -2 & 0 & 3 \\ -1 & -1 & 1 & 1 \end{bmatrix} \begin{bmatrix} \frac{1}{2} \\ -\frac{1}{2} \\ -\frac{1}{2} \\ \frac{1}{2} \end{bmatrix} \tag{26}$$

$$= \begin{bmatrix} -\frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} \\ 0 \end{bmatrix} \tag{27}$$

(b) (4 pts.) Regardless of your answer to the previous question, suppose $\vec{u}_1 = \begin{bmatrix} \frac{2}{3} \\ \frac{2}{3} \\ \frac{1}{3} \end{bmatrix}$. **Project your three-dimensional datapoints from A into a one-dimensional data. Express your answer as the vector $\vec{z} \in \mathbf{R}^{1\times 4}$, and show all your work.**

Solution: Recall that we can use our principal component to project our 3D data into 1D by doing $\vec{u}_1^T A$

$$\vec{u}_1^T A = \left(\begin{bmatrix} \frac{2}{3} & \frac{2}{3} & \frac{1}{3} \end{bmatrix} \begin{bmatrix} -3 & 0 & 2 & 1 \\ -1 & -2 & 0 & 3 \\ -1 & -1 & 1 & 1 \end{bmatrix} \right) \tag{28}$$

$$= \begin{bmatrix} -3 & -\frac{5}{3} & \frac{5}{3} & 3 \end{bmatrix} \tag{29}$$

(c) (4 pts.) Now, suppose that for this training data we have known risk index scores already (with a scalar index for each patient):

$$\vec{b} = \begin{bmatrix} -3 & -1 & 4 & 6 \end{bmatrix} \tag{30}$$

We would like to have a way to predict risk scores. Let's try to use least squares on our scenario. In one dimension, we can set up our least squares problem using data $\vec{d} = \vec{z}^T$ and target $\vec{s} = \vec{b}^T = \begin{bmatrix} -3 \\ -1 \\ 4 \\ 6 \end{bmatrix}$ to estimate a scalar parameter p_{LS}.

Perform least squares on the system $\vec{d} p_{\text{LS}} = \vec{s}$ to estimate p_{LS}, and show your work.

Solution: We perform least squares using the data points from the previous part. Thus, $\vec{d} = \vec{z}^T = \begin{bmatrix} -3 \\ -\frac{5}{3} \\ \frac{5}{3} \\ 3 \end{bmatrix}$ is your data for least squares.

$$p_{\text{LS}} = \left(\vec{d}^T \vec{d} \right)^{-1} \vec{d}^T \vec{s} \tag{31}$$

$$= \frac{3}{2} \tag{32}$$

8. **Pulse Response (14 pts.)**

In this problem, we will explore the pulse response associated with RL circuits.

Suppose you have the following circuit:
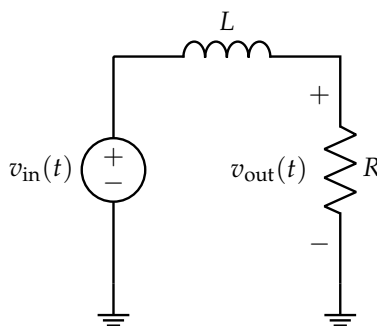
(a) (4 pts.) **Write the differential equation for $v_{\text{out}}(t)$ in terms of $v_{\text{in}}(t)$, R, and L.**

Solution: We will start with KCL at the node between the inductor and resistor:

$$i_L(t) = \frac{v_{\text{out}}(t)}{R} \tag{33}$$

We have $i_L(t)$, but we want to describe this in terms of voltage, so we will have to make use of the inductor equation $v_L(t) = L\frac{di_L(t)}{dt}$ (here, we will take the derivative of both sides so we can replace $\frac{di_L(t)}{dt}$, but you could also replace $i_L(t)$ with an integral and then take the derivative later):

$$\frac{di_L(t)}{dt} = \frac{1}{R}\frac{dv_{\text{out}}(t)}{dt} \tag{34}$$

$$\frac{v_{\text{in}}(t) - v_{\text{out}}(t)}{L} = \frac{1}{R}\frac{dv_{\text{out}}(t)}{dt} \tag{35}$$

$$\frac{dv_{\text{out}}(t)}{dt} = -\frac{R}{L}v_{\text{out}} + \frac{R}{L}v_{\text{in}}(t) \tag{36}$$

(b) (2 pts.) **What is the time constant τ of the circuit in terms of R and L?** Recall that the time constant will show up in the exponential term $e^{-\frac{t}{\tau}}$ that is characteristic of first order differential equations as the one obtained in the previous part.

Solution: If we have our differential equation in the form $\frac{dv(t)}{dt} = \lambda v(t) + u(t)$, the time constant is $\tau = -\frac{1}{\lambda}$ (this is because we know that $e^{\lambda t}$ will be the characteristic exponential for this standard form of our first order differential equation).

From the previous part, we can see that $\lambda = -\frac{R}{L}$, so $\tau = \frac{L}{R}$ is the time constant for this circuit. This makes sense because this is the standard time constant for RL circuits, as the one from this problem.

(c) (8 pts.) Suppose the plot of input $v_{\text{in}}(t)$ is a pulse as shown below. Assume $v_R(0) = 0$.
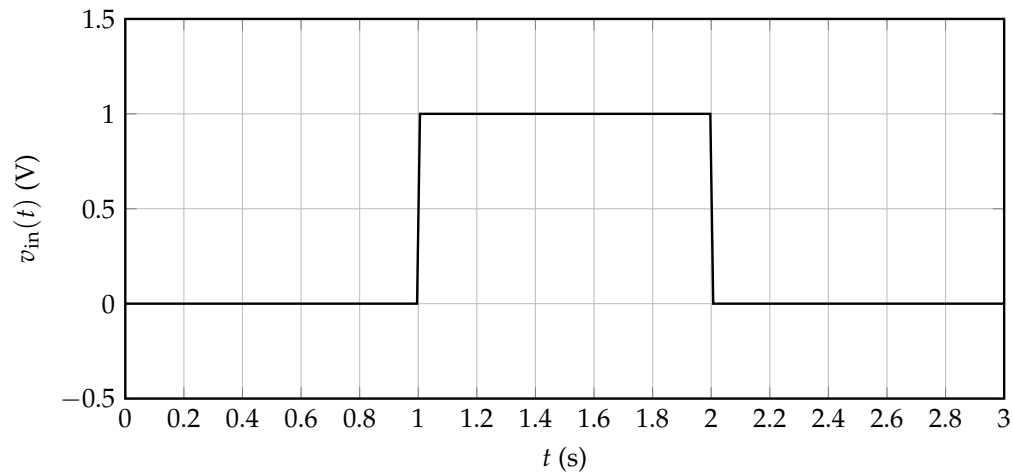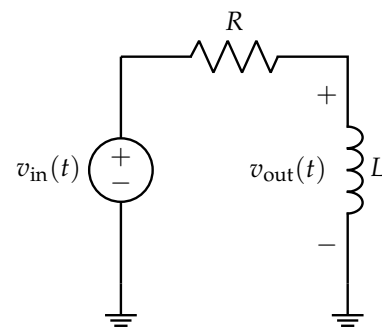
For this part, we will try and plot the output waveform of this circuit and the same circuit if the resistor and inductor were swapped with varying time constants.
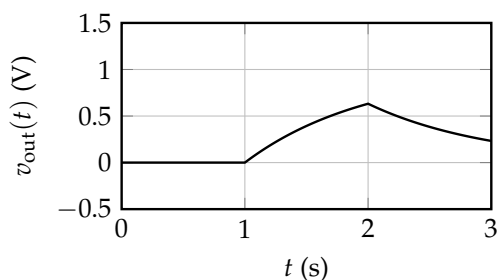
Here are the two circuits for reference:
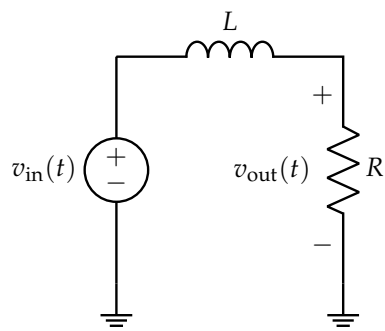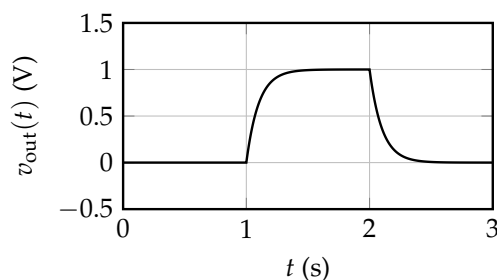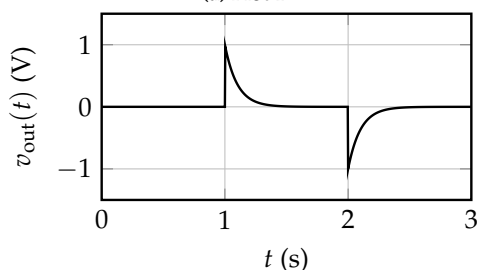

(a) Circuit 1

(b) Circuit 2

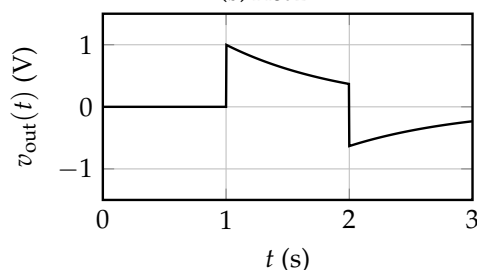The following four plots will be used for this part:

13

(a) Plot 1

(b) Plot 2

(c) Plot 3

(d) Plot 4

Fill out the provided table to match each combination of circuit and time constant to one of the provided plots. You should have one answer per row, but each column could have multiple selections.

	Plot 1	Plot 2	Plot 3	Plot 4
Circuit 1, $\tau = 0.1\,\text{s}$	○	○	○	○
Circuit 1, $\tau = 1\,\text{s}$	○	○	○	○
Circuit 2, $\tau = 0.1\,\text{s}$	○	○	○	○
Circuit 2, $\tau = 1\,\text{s}$	○	○	○	○

(HINT: Use your knowledge of the differential equation for Circuit 1. For Circuit 2, it may be helpful to use the KVL equation $v_{in}(t) = v_L(t) + v_R(t)$. You should also think about the steady state voltage for each circuit.)

Solution: From the hint, we can examine the differential equation from the previous parts to know that the response of Circuit 1 should be the standard exponential response we have studied a lot with RC circuits. The steady state voltage for Circuit 1 (which involves shorting the inductor for each DC segment of the input) is just $v_{in}(t)$ so Circuit 1's response will follow the input voltage. Both of these characteristics are observed with the first two plots (Plot 1 and Plot 2).

The shorter the time constant, the faster the response will approach the steady state value over each segment of the input, so the $\tau = 0.1\,\text{s}$ scenario should correspond to Plot 2 (which asymptotes quickly) while the $\tau = 1\,\text{s}$ scenario will correspond to Plot 1.

For Circuit 2, the response will also involve exponentials, but the situation is a bit more complicated because the output is now taken on the inductor. With the hint though, we can say that $v_L(t) = v_{in}(t) - v_R(t)$, where $v_R(t)$ is the response of Circuit 1 (where the input was taken across the resistor). Thus, we can perform this pointwise subtraction between the plots of the input voltage and output voltage of Circuit 1 to see that the plot of Circuit 2 will look like Plots 3 and

4. (This also matches the idea that the steady state output voltage for Circuit 2 is always 0 since the output voltage is across the inductor, which is shorted for DC steady state.)

The role of the time constant is the exact same as for Circuit 1 so the $\tau = 0.1\,\text{s}$ scenario should correspond to Plot 3 (which asymptotes quickly) while the $\tau = 1\,\text{s}$ scenario will correspond to Plot 4.

Another perspective of this problem is to look at the frequency responses of each circuit; Circuit 1 is a low pass filter and thus it makes sense that the low frequency flat sections of the input are preserved with the steady state voltage while the high frequency spikes are attenuated (become exponential).

Circuit 2 is a high pass filter so the high frequency spikes are preserved (appear in the output) while the low frequency flat sections are attenuated (since the DC steady state voltage is always 0, regardless of the DC level of the input voltage.)

The completed table is shown below:

|	Plot 1	Plot 2	Plot 3	Plot 4
Circuit 1, $\tau = 0.1\,\text{s}$	○	●	○	○
Circuit 1, $\tau = 0.1\,\text{s}$	●	○	○	○
Circuit 2, $\tau = 1\,\text{s}$	○	○	●	○
Circuit 2, $\tau = 1\,\text{s}$	○	○	○	●

9. **Amplifier Analysis (10 pts.)**

For this problem, we will examine the transfer function for an amplifier.

Suppose that we have the following circuit:
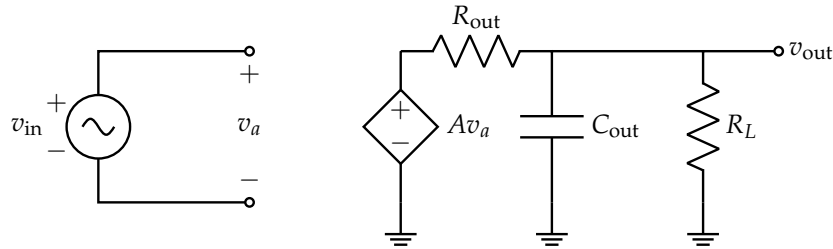
We want to find the transfer function $H(j\omega) = \frac{\tilde{V}_{out}}{\tilde{V}_{in}}$ and analyze its behavior.

(a) (6 pts.) To simplify the analysis, we will find the Thevenin equivalent circuit as seen by the capacitor C_{out} as follows:
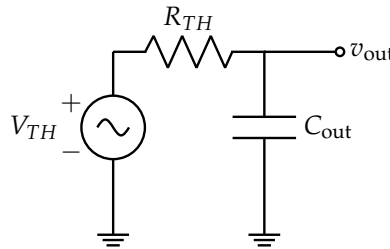
Find $V_{TH} = A_0 v_{in}$ and R_{TH}, where A_0 and R_{TH} are in terms of A and the resistances R_{out} and R_L. (HINT: For this scenario, V_{TH} will be voltage at the load (the capacitor) when the load (the capacitor) is replaced with an open circuit, and R_{TH} will be the equivalent resistance seen from the load (the capacitor) perspective with independent sources set to 0, which will be easiest found with series and parallel equivalence.)

Solution: The effective circuit we need to analyze is (the capacitor will be the load for this circuit):
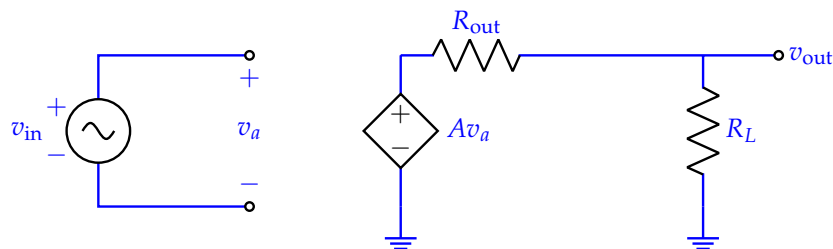

The Thevenin voltage V_{TH} can be found as the voltage at v_{out} when the load is open (which is already displayed by the diagram). We can notice that $v_a = v_{in}$ so the dependent voltage source outputs Av_{in}, and v_{out} in this case is just a voltage divider of Av_{in} between R_{out} and R_L:

$$V_{TH} = \frac{R_L}{R_{out} + R_L}(Av_{in}) = \frac{AR_L}{R_{out} + R_L}v_{in} \tag{37}$$

Thus, $A_0 = \frac{AR_L}{R_{\text{out}}+R_L}$.

To find the Thevenin resistance R_{TH}, we zero out the independent voltage source v_{in} (which will also cause the dependent voltage source to zero out and act like a short circuit) and find the equivalent resistance seen by the v_{out} output port. The equivalent circuit for this analysis is shown below:
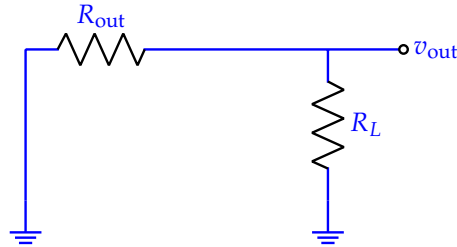
With our knowledge of series and parallel resistance, the equivalent resistance in this case is:

$$R_{TH} = R_{\text{out}}||R_L = \frac{R_{\text{out}}R_L}{R_{\text{out}} + R_L} \tag{38}$$

(b) (4 pts.) **Find** $H(j\omega) = \frac{\widetilde{V}_{\text{out}}}{\widetilde{V}_{\text{in}}}$ in terms of A_0, R_{TH}, and C_{out}.

Solution: We can observe that the equivalent circuit is the same as a low pass filter, with the difference being that the input voltage is $A_0 v_{\text{in}}$. Thus, the overall transfer function for H will be the same as a low pass filter with $R = R_{TH}$ and $C = C_{\text{out}}$, scaled by A_0 (the same result would be obtained if solved from scratch such as with a voltage divider or by KCL/KVL):

$$H(j\omega) = \frac{A_0}{1 + j\omega R_{TH}C_{\text{out}}} \tag{39}$$

10. **Stability Analysis with Bode Plots (13 pts.)**

    For this problem, we will use Bode plots to examine the stability of an amplifier.

    Suppose that we analyze our amplifier circuit and are able to approximate the overall transfer function as a two-pole system:

    $$T(j\omega) = \frac{T_0}{(1 + j\frac{\omega}{\omega_{p1}})(1 + j\frac{\omega}{\omega_{p2}})} \tag{40}$$

   We will assume that $\omega_{p1} \ll \omega_{p2}$ for the entire problem.

    Important: Your answers should be made with the Bode straight-line approximation in place; with this approximation, your answers should all be relatively simple.

    (a) (5 pts.) For $\omega \ll \omega_{p2}$, we can approximate the transfer function as a one-pole system with pole ω_{p1}:
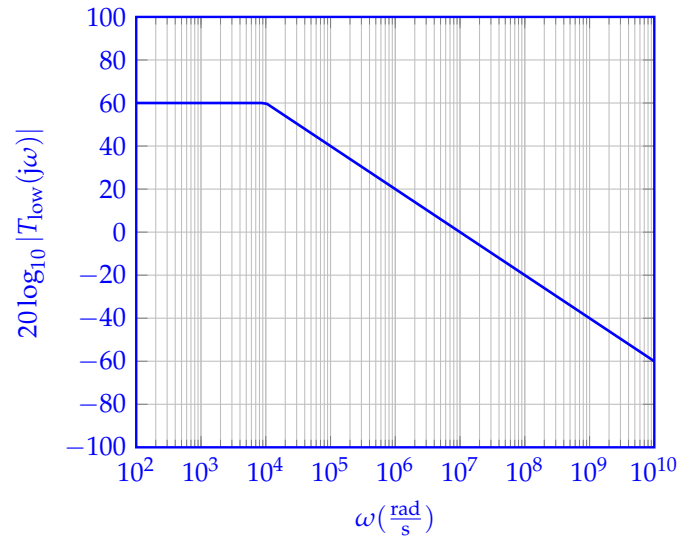
    $$T(j\omega) \approx T_{\text{low}}(j\omega) = \frac{T_0}{1 + j\frac{\omega}{\omega_{p1}}} \tag{41}$$

    Let $T_0 = 10^3$ and $\omega_{p1} = 10^4 \frac{\text{rad}}{\text{s}}$. **Plot the Bode plots for $|T_{\text{low}}(j\omega)|$ and $\angle T_{\text{low}}(j\omega)$.**

    Based on the Bode plots, what is ω_u, defined as the frequency such that $|T_{\text{low}}(j\omega_u)| = 1$?
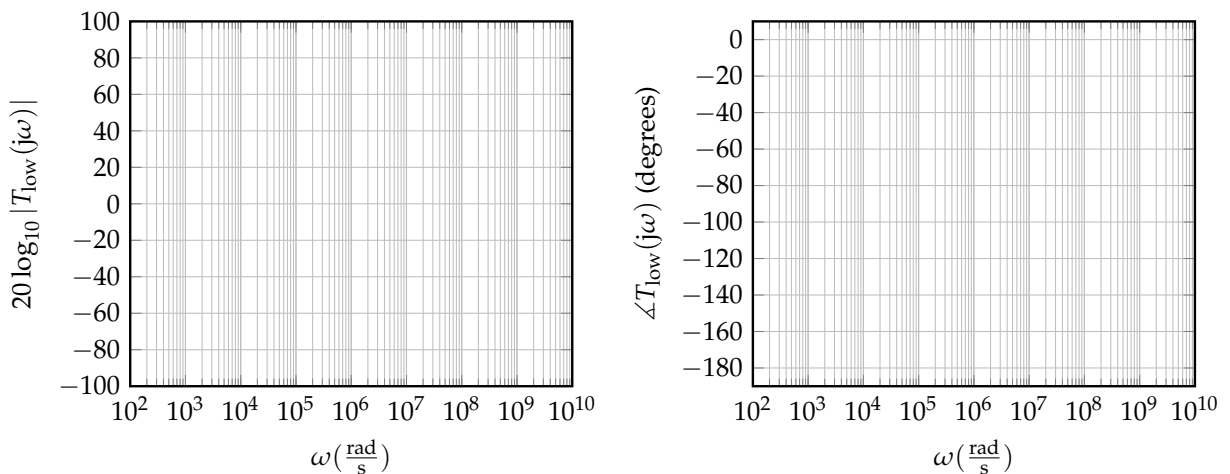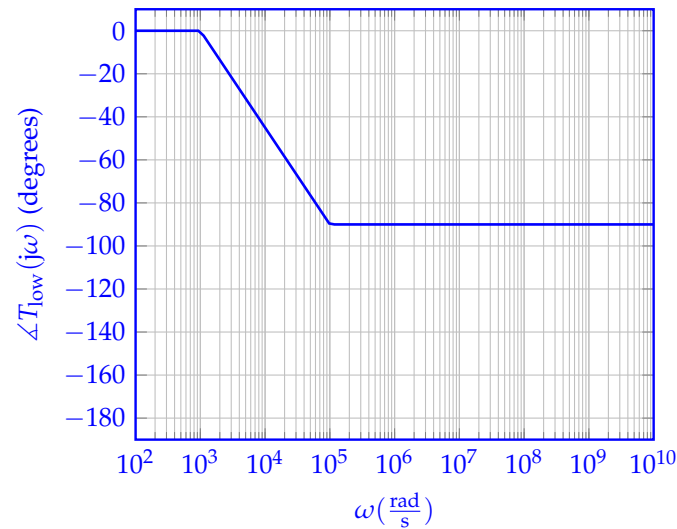

    Solution: For the magnitude plot:

The plot starts at $|T_{\text{low}}(j0)| = 10^3 = 60\text{dB}$ and starts to decrease by $-20\frac{\text{dB}}{\text{dec}}$ due to the pole at $\omega_{p1} = 10^4$.

For the phase plot:

The plot starts at $0°$ since there are no poles or zeros at $\omega = 0$ and then decreases to $-90°$ from $\omega = 10^3$ to $\omega = 10^5$ due to the pole at $\omega_{p1} = 10^4$.

From the magnitude Bode plot, $|T_{\text{low}}(j\omega)| = 1 = 0\text{dB}$ at $\omega = 10^7$ so $\omega_u = 10^7\frac{\text{rad}}{\text{s}}$.

(b) (8 pts.) Now, we will work with the overall two-pole transfer function $T(j\omega)$ (assume T_0 and ω_{p1} are the same as from the previous part).

Suppose that we are able to choose what frequency we want the second pole ω_{p2} at. For stability purposes, we want $\angle T(j\omega_u)$ to be reasonably above $-180°$, where ω_u is the same frequency as the one found in the previous problem; for this problem, we will choose $\angle T(j\omega_u) = -135°$ to be the target phase.

Find the pole frequency ω_{p2} that causes $\angle T(j\omega_u) = -135°$. Justify your answer by plotting the Bode plots for the two-pole transfer function $T(j\omega) = \frac{T_0}{(1+j\frac{\omega}{\omega_{p1}})(1+j\frac{\omega}{\omega_{p2}})}$, with ω_{p2} as your chosen second pole. (Use your answer for the previous part for ω_u; even if your answer to that part is incorrect, you can still receive credit for this part if your answer is consistent with ω_u from the previous part.) *(HINT: You should have found that $10\omega_{p1} < \omega_u$ from the previous part, which means that the first pole ω_{p1} will have already contributed its $-90°$ of phase by ω_u, which means that to have a total phase of $-135°$ at ω_u, the second pole ω_{p2} must contribute $-45°$ phase at ω_u. Think about at what frequency a pole contributes $-45°$ phase; if you are unsure, draw the Bode plot, check to see if you pole frequency works, and adjust as necessary.)*
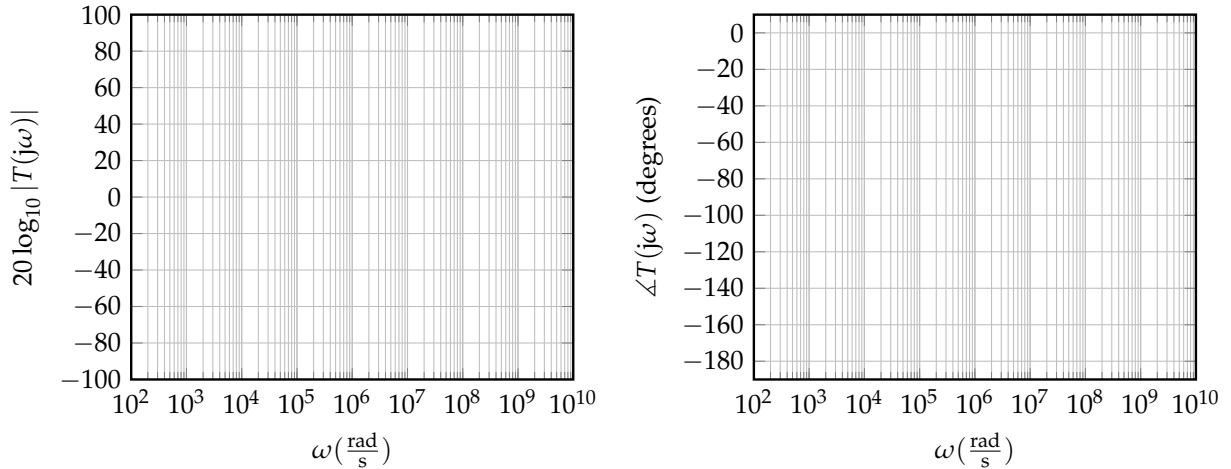
Solution: As the hint advises, we should think about at what frequency a pole contributes $-45°$ phase; this occurs at the pole frequency itself. Thus, if we want ω_{p2} to contribute $-45°$ phase at ω_u, we should set $\omega_{p2} = \omega_u = 10^7 \frac{\text{rad}}{\text{s}}$ (if you did not find the correct ω_u in the previous part, but you did find that $\omega_{p2} = \omega_u$, this will receive credit). With this ω_{p2}, the overall transfer function would be:
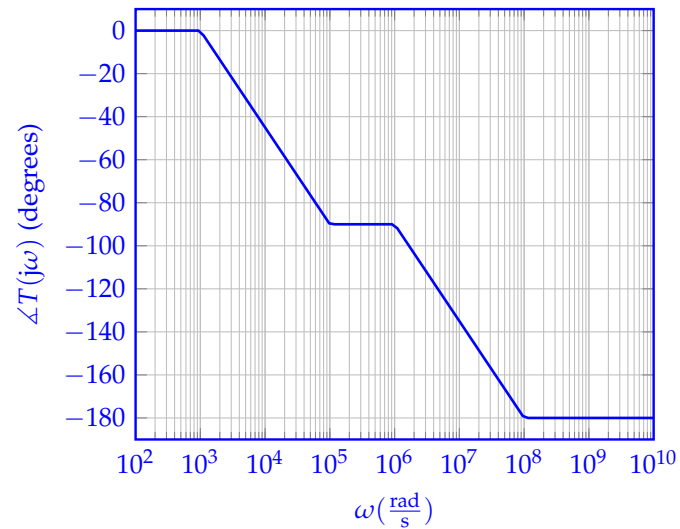
$$T(j\omega) = \frac{10^3}{(1+j\frac{\omega}{10^4})(1+j\frac{\omega}{10^7})} \tag{42}$$

For the magnitude plot:
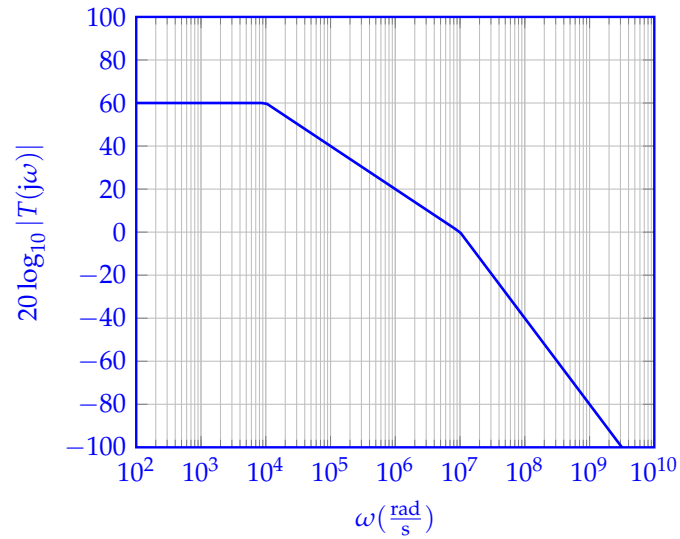
The plot starts at $|T(j0)| = 10^3 = 60$dB. The slope starts at $0\frac{\text{dB}}{\text{dec}}$ due to no poles or zeros at $\omega = 0$, decreases to $-20\frac{\text{dB}}{\text{dec}}$ due to the pole at $\omega_{p1} = 10^4$ and then to $-40\frac{\text{dB}}{\text{dec}}$ due to the pole at $\omega_{p2} = 10^7$.

For the phase plot:

The plot starts at $0°$ since there are no poles or zeros at $\omega = 0$ and then decreases to $-90°$ from $\omega = 10^3$ to $\omega = 10^5$ due to the pole at $\omega_{p1} = 10^4$ and then to $-180°$ from $\omega = 10^6$ to $\omega = 10^8$ due to the pole at $\omega_{p2} = 10^7$.

Based on the Bode plots, we can see that the phase at $\omega_u = 10^7$ is $-135°$ as desired.