# Note 5: Phasors and AC Circuits

## 1 Introduction to Phasors

### 1.1 (OPTIONAL) Motivation

So far in this course, we have been looking at inhomogeneous differential equations of the following form:

$$\frac{\mathrm{d}}{\mathrm{d}t}x(t) = \lambda x(t) + bu(t) \tag{1}$$

for some constant $b \in \mathbb{R}$ and a function $u(t) : \mathbb{R}_+ \to \mathbb{R}$ (where $\mathbb{R}_+$ denotes nonnegative real numbers). In the context of circuits, we often consider the function $u(t)$ term to be some sort of "input" into our circuit. In practice, we often encounter inputs $u(t)$ that fall under a specific class of functions, namely periodic functions.

**Definition 1** (Periodic Function)
A function $f(t)$ is periodic if there exists some constant $T \in \mathbb{R}$ such that $f(t) = f(t+T)$ for all $t$ in the domain of $f$.

An example of a periodic function is $f(t) = \sin(t)$. In this case, $T = 2\pi$. There exists a theorem that says any periodic function can be written as a sum of sinusoidal functions with period $T$. This theorem and its proof are out of scope for this class, but it is stated below for completeness.

**Theorem 2** (Dirichlet's Theorem)
Let $f$ be a periodic, well-behaved[a] function with period $T$. We can write $f$ as a superposition of sinusoidal functions, namely

$$f(t) = \sum_{n=-\infty}^{\infty} c_n \mathrm{e}^{\frac{\mathrm{j}2\pi n t}{T}} \tag{2}$$

where

$$c_n = \frac{1}{T}\int_{-\frac{T}{2}}^{\frac{T}{2}} f(t)\mathrm{e}^{-\frac{\mathrm{j}2\pi n t}{T}}\,\mathrm{d}t \tag{3}$$

Furthermore, it should be noted that $c_{-n} = \overline{c_n}$.

[a] Satisfies the Dirichlet conditions

As we will explore in more depth in this note, it is often easier to reason about periodic functions in terms of the coefficients, $c_n$. Each $c_n$ (for $n \geq 0$) are the *phasors* for the corresponding sinusoid with frequency $\frac{2\pi n}{T}$. It should be noted that phasors have **no time dependence**. We call this process of analyzing circuits through the phasors as *phasor domain analysis*. When dealing with phasors in this class, we will only calculate phasors for sine or cosine inputs (so $f(t) = V_0 \cos(\omega t + \phi)$ or $f(t) = V_0 \sin(\omega t + \phi)$).

### 1.2 Magnitude-Phase Representations of Complex Numbers

We can represent any complex number with a magnitude and phase. That is, we can write any $c_n$ in the form $A\mathrm{e}^{\mathrm{j}\phi}$ for real values $A, \phi$.

**Theorem 3** (Magnitude-Phase Representation)
Given a complex number $x = a + \mathrm{j}b$, we can equivalently represent it in the form $x = A\mathrm{e}^{\mathrm{j}\phi}$ where $A = |x| = \sqrt{x\overline{x}} = \sqrt{a^2 + b^2}$ and $\phi = \mathrm{atan2}(b, a)$.

*Proof.* We can set the two representations equal and solve for $A$ and $\phi$. That is,

$$a + \mathrm{j}b = A\mathrm{e}^{\mathrm{j}\phi} \tag{4}$$
$$= A\cos(\phi) + \mathrm{j}A\sin(\phi) \tag{5}$$

so we have that $a = A\cos(\phi)$ and $b = A\sin(\phi)$. We have that

$$A^2 = A^2\cos^2(\phi) + A^2\sin^2(\phi) \tag{6}$$
$$= a^2 + b^2 \tag{7}$$

so $A = \sqrt{a^2 + b^2}$. Next, we have that

$$\frac{b}{a} = \frac{\sin(\phi)}{\cos(\phi)} \tag{8}$$
$$= \tan(\phi) \tag{9}$$

so $\phi = \mathrm{atan2}(b, a)$[1]. □

### 1.3 Determining Phasors for Sine and Cosine Functions

First, we should note a corollary of Euler's formula.

**Theorem 4** (Euler's Theorem)
The following identities hold:

$$\sin(\theta) = \frac{\mathrm{e}^{\mathrm{j}\theta} - \mathrm{e}^{-\mathrm{j}\theta}}{2\mathrm{j}} \tag{10}$$
$$\cos(\theta) = \frac{\mathrm{e}^{\mathrm{j}\theta} + \mathrm{e}^{-\mathrm{j}\theta}}{2} \tag{11}$$

**Concept Check:** Show that these identities hold, using the fact that $\mathrm{e}^{\mathrm{j}\theta} = \cos(\theta) + \mathrm{j}\sin(\theta)$.

Now, we can derive a formula for the phasor representation of a sine/cosine. Note that $\cos\left(x - \frac{\pi}{2}\right) = \sin(x)$, so it suffices to derive a phasor representation for an arbitrary cosine function.

[1] We choose to use two argument atan (i.e., atan2) because this preserves the sign of the angles and we will not encounter division by 0 this way.

  

**Theorem 5** (Cosine Phasors)
Suppose we are given an arbitrary, time-varying cosine function of the form $v(t) = V_0 \cos(\omega t + \phi)$, where $V_0$ is the amplitude, $\omega$ is the frequency, and $\phi$ is a phase shift. The function $v(t)$'s phasor for the frequency $\omega$ is given by $\widetilde{V} = \frac{V_0 e^{j\phi}}{2}$[a].

[a] We denote the phasor for $v(t)$ as $\widetilde{V}$, dropping the time input, capitalizing, and putting a tilde on top.

*Proof.* Using Theorem 4, we have that

$$V_0 \cos(\omega t + \phi) = V_0 \frac{e^{j(\omega t+\phi)} + e^{-j(\omega t+\phi)}}{2} \tag{12}$$

$$= \underbrace{\frac{V_0 e^{j\phi}}{2}}_{c_\omega} e^{j\omega t} + \underbrace{\frac{V_0 e^{-j\phi}}{2}}_{\overline{c_\omega}} e^{-j\omega t} \tag{13}$$

so we have that the phasor, i.e., $c_\omega$, is $\frac{V_0 e^{j\phi}}{2}$ as desired. □

**Corollary 6** (Sine Phasors)
Suppose we are given an arbitrary, time-varying cosine function of the form $v(t) = V_0 \sin(\omega t + \phi)$, where $V_0$ is the amplitude, $\omega$ is the frequency, and $\phi$ is a phase shift. The function $v(t)$'s phasor for the frequency $\omega$ is given by $\widetilde{V} = \frac{V_0 e^{j\phi}}{2j}$.

**Concept Check:** Prove this corollary, using the fact that $\sin(x) = \cos\left(x - \frac{\pi}{2}\right)$ and that $e^{-j\frac{\pi}{2}} = -j = \frac{1}{j}$.

*Example:*
Suppose $v(t) = 10\cos\left(20t + \frac{3\pi}{2}\right)$. To find the phasor for this function, we can begin by pattern matching $V_0 = 10$ and $\phi = \frac{3\pi}{2}$. Applying this to the resulf of Theorem 5, we have $\widetilde{V} = 5e^{j\frac{3\pi}{2}} = -5j$.

## 2 Computing Impedances in Phasor Domain

We can look at the phasor domain "resistances" of all passive circuit elements we have learned so far. The technical term for these "resistances" is impedance. Formally, we denote this as

$$Z = \frac{\widetilde{V}}{\widetilde{I}} \tag{14}$$

We are leveraging the I-V relationship of each circuit element in phasor domain so that we can derive their phasor domain impedances.

**Theorem 7** (Impedance of a Capacitor)
Suppose we applied an input voltage $v_C(t) = V_0 \cos(\omega t + \phi)$ across a capacitor with capacitance $C$. Its phasor domain impedance is given by $Z_C = \frac{1}{j\omega C}$.

*Proof.* We can find $i_C(t)$ and then find its phasor domain representation, i.e., $\widetilde{I}_C$. We can apply the equation relating current and voltage across a capacitor, namely

$$i_C(t) = C\frac{d}{dt}v_C(t) \tag{15}$$

  

$$= C\frac{\mathrm{d}}{\mathrm{d}t}(V_0 \cos(\omega t + \phi)) \tag{16}$$

$$= -\omega C V_0 \sin(\omega t + \phi) \tag{17}$$

Using Corollary 6, we have that

$$\widetilde{I}_C = \frac{-\omega C V_0 e^{j\phi}}{2j} \tag{18}$$

$$= j\omega C \frac{V_0 e^{j\phi}}{2} \tag{19}$$

and by Theorem 5, we have that

$$\widetilde{V}_C = \frac{V_0 e^{j\phi}}{2} \tag{20}$$

Hence,

$$Z_C = \frac{\widetilde{V}_C}{\widetilde{I}_C} = \frac{1}{j\omega C} \tag{21}$$

□

**Theorem 8** (Impedance of a Resistor)
Suppose we applied an input voltage $v_R(t) = V_0 \cos(\omega t + \phi)$ across a resistor with resistance $R$. Its phasor domain impedance is given by $Z_R = R$.

*Proof.* Using the same technique as the proof of Theorem 7, we find $i_R(t)$ as follows:

$$i_R(t) = \frac{1}{R} v_R(t) = \frac{V_0}{R}\cos(\omega t + \phi) \tag{22}$$

The phasor domain representation of this is

$$\widetilde{I}_R = \frac{\frac{V_0}{R} e^{j\phi}}{2} = \frac{1}{R}\frac{V_0 e^{j\phi}}{2} \tag{23}$$

The expression for $\widetilde{V}_R$ remains the same as the expression for $\widetilde{V}_C$ in Theorem 7. Hence,

$$Z_R = \frac{\widetilde{V}_R}{\widetilde{V}_C} = R \tag{24}$$

□

**Theorem 9** (Impedance of an Inductor)
Suppose we applied an input current $i_L(t) = V_0 \cos(\omega t + \phi)$ through an inductor with inductance $L$. Its phasor domain impedance is given by $Z_L = j\omega L$.

*Proof.* We can find $\widetilde{V}_L$ by first finding $v_L(t)$ as follows:

$$v_L(t) = L\frac{\mathrm{d}}{\mathrm{d}t} i_L(t) \tag{25}$$

$$= L\frac{\mathrm{d}}{\mathrm{d}t}(V_0 \cos(\omega t + \phi)) \tag{26}$$

$$= -L\omega V_0 \sin(\omega t + \phi) \tag{27}$$

Now, we can use Corollary 6 to find $\widetilde{V}_L$:

$$\widetilde{V}_L = \frac{-\omega L V_0 e^{j\phi}}{2j} \tag{28}$$

$$= j\omega L \frac{V_0 e^{j\phi}}{2} \tag{29}$$

Here, we have that $\widetilde{I}_L = \frac{V_0 e^{j\phi}}{2}$ so

$$Z_L = \frac{\widetilde{V}_L}{\widetilde{I}_L} = j\omega L \tag{30}$$

□

**Key Idea 10** (Using Phasor Impedances)
Since the phasor impedance represent an I-V relationship in phasor domain, and since the impedance is constant with respect to time, we can treat all components' phasor domain representations as time domain resistors. That is, we can apply the same rules for KCL, NVA, and parallel/series combinations of resistors.

*Example:*
We can solve for $v_{\text{out}}(t)$ in Figure 1 by using phasor domain conversions.

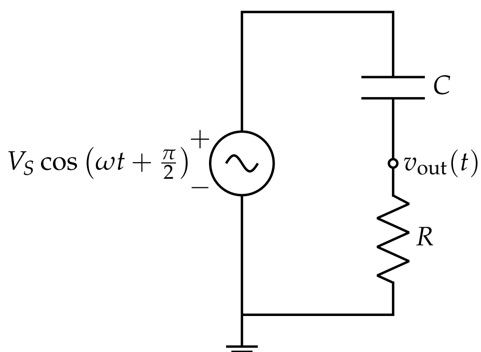


**Figure 1:** Example Circuit

Here, we can perform the phasor domain conversion on the input voltage since it is a single sinusoid. That is, we have that $v_{\text{in}}(t) := V_S \cos(\omega t + \frac{\pi}{2})$ so $\widetilde{V}_{\text{in}} = \frac{V_S e^{j\frac{\pi}{2}}}{2}$. Using the fact that $Z_C = \frac{1}{j\omega C}$ and $Z_R = R$, we can treat these components as "resistors" in phasor domain. That is, we can apply the resistor voltage divider formula to obtain

$$\widetilde{V}_{\text{out}} = \frac{Z_R}{Z_C + Z_R} \widetilde{V}_{\text{in}} \tag{31}$$

$$= \frac{R}{\frac{1}{j\omega C} + R} \left( \frac{V_S e^{j\frac{\pi}{2}}}{2} \right) \tag{32}$$

$$= \frac{j\omega RC}{1 + j\omega RC} \left( \frac{V_S e^{j\frac{\pi}{2}}}{2} \right) \tag{33}$$

$$= \frac{\omega RC e^{j\frac{\pi}{2}}}{\sqrt{1 + (\omega RC)^2} e^{j\,\text{atan2}(\omega RC, 1)}} \left( \frac{V_S e^{j\frac{\pi}{2}}}{2} \right) \tag{34}$$

$$= \frac{V_S \omega RC}{2\sqrt{1 + (\omega RC)^2}} e^{j(\pi - \text{atan2}(\omega RC, 1))} \tag{35}$$

$$= \frac{1}{2}\left(\frac{V_S \omega RC}{\sqrt{1 + (\omega RC)^2}}\right) e^{j(\pi - \text{atan2}(\omega RC, 1))} \tag{36}$$

where we convert to the magnitude-phase representation of the numerator and denominator in eq. (34). Next, we can reverse the steps of Theorem 5 to obtain the time domain output. We can pattern match $V_0 = \frac{V_S \omega RC}{\sqrt{1+(\omega RC)^2}}$ and $\phi = \pi - \text{atan2}(\omega RC, 1)$, so

$$v_{\text{out}}(t) = \frac{V_S \omega RC}{\sqrt{1 + (\omega RC)^2}} \cos(\omega t + \pi - \text{atan2}(\omega RC, 1)) \tag{37}$$

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