

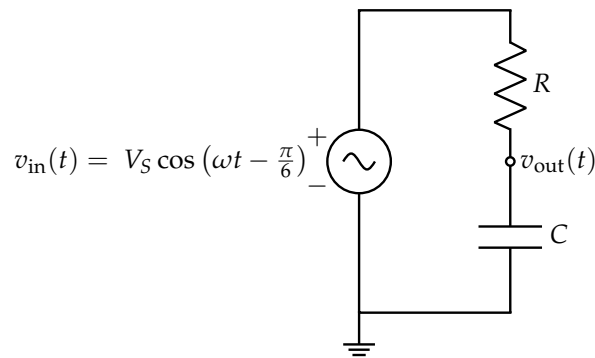
Note 6: Circuit Filters

Overview

In the previous note, we developed the method of Phasor Domain Analysis to analyze the steady state behavior of circuits with sinusoidal voltage and current sources. In this note, we will consider some applications of these circuits, and begin to explore techniques for designing these circuits to fit a set of requirements. *NOTE:* While this note may seem long, much of the space is consumed by figures, and each example provided comes with a complete set of steps.

1 Circuit Example and Transfer Functions

In the previous note, we analyzed a circuit that resembled a voltage divider, but replaced the first (top) resistor with a capacitor. Below, we consider a similar circuit where the roles of the capacitor and resistor are swapped. Our notation has changed, but only very slightly ($\tilde{U} \rightarrow \tilde{V}_{in}$). We will also assign specific values to the input voltage and circuit elements to get a feel for a "complete" solution.



Let's say that $V_S = 5\text{ V}$, $\omega = \sqrt{3} \cdot 10^5 \frac{\text{rad}}{\text{s}}$, $R = 1\text{ k}\Omega$, and $C = 10\text{ nF}$. We can calculate that the input voltage phasor \tilde{V}_{in} corresponding to $v_{in}(t) = V_S \cos(\omega t - \frac{\pi}{6})$ is:

$$\tilde{V}_{in} = \frac{V_S e^{-j\frac{\pi}{6}}}{2} = \frac{5e^{-j\frac{\pi}{6}}}{2} \quad (1)$$

Applying the equation for a voltage divider, we find the voltage phasor at $v_{out}(t)$ to be:

$$\tilde{V}_{out} = \frac{\frac{1}{j\omega C}}{R + \frac{1}{j\omega C}} \tilde{V}_{in}. \quad (2)$$

Keeping our results symbolic just a little while, we will look at the ratio of the output phasor to the input phasor. This term, $\frac{\tilde{V}_{out}}{\tilde{V}_{in}}$, is called the circuit's *transfer function* (denoted $H(j\omega)$). It relates the output of the circuit to the input while capturing the frequency dependence.

$$H(j\omega) = \frac{\tilde{V}_{out}}{\tilde{V}_{in}} = \frac{\frac{1}{j\omega C}}{R + \frac{1}{j\omega C}}. \quad (3)$$

More broadly, transfer functions map an angular frequency ω to a ratio of two phasors — an input and an output — and are used to characterize the behavior of a system across a range of frequencies.¹

Since transfer functions are the ratio of two phasors, they are complex-valued, and we can write

$$H(j\omega) = |H(j\omega)|e^{j\angle H(j\omega)}, \quad (4)$$

where $|H(j\omega)|$ is the magnitude of the transfer function, and $\angle H(j\omega)$ is the phase of the transfer function. In this example, we can rearrange $H(j\omega)$ to become:

$$H(j\omega) = \frac{\frac{1}{j\omega C}}{R + \frac{1}{j\omega C}} = \frac{1}{1 + j\omega RC} \quad (5)$$

$$= \frac{1}{1 + j\sqrt{3}} \quad (6)$$

Now, we can compute the information we need about $H(j\omega)$: First, the magnitude²:

$$|H(j\omega)| = \frac{1}{\sqrt{1 + (\omega RC)^2}} = \frac{1}{2} \quad (7)$$

Now, the phase³:

$$\angle H(j\omega) = \angle(1) - \angle(1 + j\omega RC) \quad (8)$$

$$= -\text{atan2}(\omega RC, 1) \quad (9)$$

$$= -\text{atan2}(\sqrt{3}, 1) \quad (10)$$

$$= -\frac{\pi}{3} \quad (11)$$

Now, we can compute the voltage phasor \tilde{V}_{out} using $H(j\omega)$, \tilde{V}_{in} as follows:

$$\tilde{V}_{\text{out}} = H(j\omega)\tilde{V}_{\text{in}} \quad (12)$$

$$= |H(j\omega)|e^{j\angle H(j\omega)}\tilde{V}_{\text{in}} \quad (13)$$

$$= \frac{1}{2}e^{-j\frac{\pi}{3}} \frac{5e^{-j\frac{\pi}{6}}}{2} \quad (14)$$

$$= \frac{5}{4}e^{-j\frac{\pi}{2}} \quad (15)$$

Now that we have \tilde{V}_{out} , we can solve for $v_{\text{out}}(t)$ using the Inverse Phasor Transform⁴:

$$v_{\text{out}}(t) = \tilde{V}_{\text{out}}e^{j\omega t} + \overline{\tilde{V}_{\text{out}}}e^{-j\omega t} \quad (16)$$

$$= \frac{5}{4}e^{-j\frac{\pi}{2}}e^{j\omega t} + \frac{5}{4}e^{j\frac{\pi}{2}}e^{-j\omega t} \quad (17)$$

$$= \frac{5}{4}\left(e^{j\left(-\frac{\pi}{2} + \omega t\right)} + e^{-j\left(-\frac{\pi}{2} + \omega t\right)}\right) \quad (18)$$

$$= \frac{5}{2}\cos\left(\sqrt{3} \cdot 10^5 t - \frac{\pi}{2}\right). \quad (19)$$

We see here that the transfer function will scale the input signal by a factor of $|H(j\omega)|$, and shift its phase by an angle $\angle H(j\omega)$. Equipped with the notion of a transfer function that scales and delays an input sinusoid, we can follow the following circuit analysis procedure:

¹Here, our transfer function relates two voltage phasors, but they can also relate two current phasors, or a voltage phasor with a current phasor, in a similar manner.

²Recall that $\left|\frac{z_1}{z_2}\right| = \frac{|z_1|}{|z_2|}$, where z_1, z_2 are both complex numbers ($a + jb$) with magnitudes as $|z| = \sqrt{a^2 + b^2}$

³ $\angle\left(\frac{z_1}{z_2}\right) = \angle(z_1) - \angle(z_2)$, where $\angle(z) = \text{atan2}(b, a)$

⁴If you can arrive the final answer by inspection, that's great! The steps are here for completeness.

1. Express the sinusoidal input (current or voltage) as a phasor.
2. Solve for the output phasor(s), which can be voltages or currents, as functions of the input phasor, and use the output to determine the transfer function.
3. Observe the magnitude and phase shift of the transfer function at the specific input frequency to determine the behavior of the circuit.

Note that we have previously solved this same example involving the output of an RC circuit with a cosine input, purely in the time domain. That process involved far more calculation and painful trigonometry to simplify the end result. The usage of phasors simplifies our calculations greatly.

2 Filters

One very common use for AC circuits is as a *filter*, which blocks certain categories of frequencies. The key idea behind filters is that of *superposition*. In EECS16A, we saw that in circuits with independent sources, the circuit could be analyzed with only 1 independent source one at a time (with the rest zeroed out).⁵ Then, the final signal was the sum of the intermediate responses from each source.

Here, rather than looking at the superposition of independent sources, we will look at the superposition of input signals with different frequencies. Specifically, let's represent our generalized input signal as follows:

$$v_{\text{in}}(t) = \sum_i \left(\tilde{U}_i e^{j\omega_i t} + \overline{\tilde{U}_i} e^{-j\omega_i t} \right) \quad (20)$$

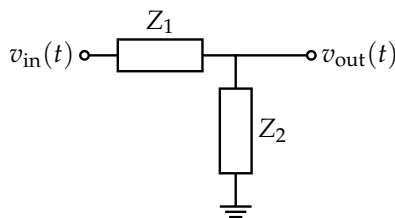
where each ω_i is some frequency with an associated phasor \tilde{U}_i . Applying this input to a circuit with transfer function $H(j\omega)$, the output voltage will be:

$$v_{\text{out}}(t) = \sum_i \left(H(j\omega_i) \tilde{U}_i e^{j\omega_i t} + H(-j\omega_i) \overline{\tilde{U}_i} e^{-j\omega_i t} \right) \quad (21)$$

That is, we can treat the superposition of signals of two different frequencies as if the two signals were separate. Note that the conjugate phasor voltage term is multiplied by the *conjugate* of the transfer function (or equivalently, with the substitution of $-j\omega_i$ as the argument to the transfer function for a given sinusoid).

We can see this by noting $\tilde{V}_{\text{out}} = H(j\omega) \tilde{V}_{\text{in}}$ which implies also that $\overline{\tilde{V}_{\text{out}}} = \overline{H(j\omega) \tilde{V}_{\text{in}}} = H(-j\omega) \overline{\tilde{V}_{\text{in}}}$. Each frequency sinusoid is affected by the circuit as governed by the transfer function evaluated at that frequency.

Now, we will consider various filter configurations, and study their behavior. For now, we will look at *elementary filters*, which are filters consisting of two elements placed in series, as shown:



Recall from the impedance divider equation that the transfer function of this circuit is:

$$H(j\omega) = \frac{Z_2}{Z_1 + Z_2}. \quad (22)$$

We will now consider all the possible forms of impedances that we can substitute into this equation. There are nine possible configurations, and we divide them into 3 main categories: Frequency-Independent, Low-Pass, High-Pass, and LC Tank.

⁵If there are dependent sources, this doesn't hold!

2.1 Frequency-Independent (RR, CC, LL)

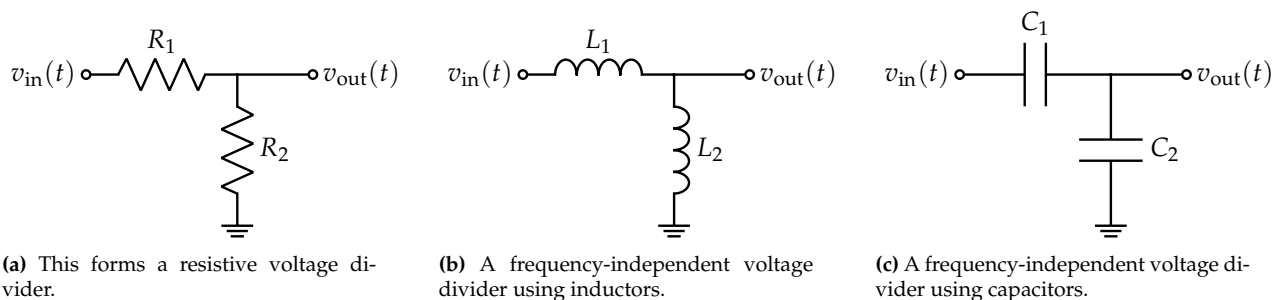


Figure 1: Frequency-Independent L -filters.

R-R: Both components are resistors. We can compute:

$$H(j\omega) = \frac{R_2}{R_1 + R_2} \quad (23)$$

Notice there's no frequency dependence here, which makes sense since this circuit only contains resistors.

L-L: Both components are inductors. Substituting the impedance of an inductor into our equation for the transfer function, we obtain:

$$H(j\omega) = \frac{j\omega L_2}{j\omega L_1 + j\omega L_2} = \frac{L_2}{L_1 + L_2} \quad (24)$$

So, this circuit exhibits behavior similar to a resistive voltage divider (once again, independently of frequency). Notice, however, that unlike resistors, inductors do not dissipate energy.⁶

C-C: Both components are capacitors. Substituting in the impedance of a capacitor:

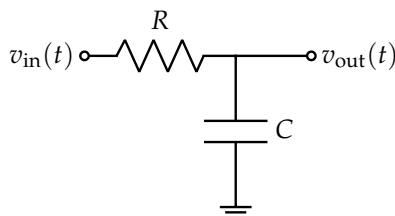
$$H(j\omega) = \frac{\frac{1}{j\omega C_2}}{\frac{1}{j\omega C_1} + \frac{1}{j\omega C_2}} = \frac{C_1}{C_1 + C_2}. \quad (25)$$

Again, we get something very similar to a voltage divider. Here, however, when $\omega = 0$, no current is permitted to flow since the capacitors act as open circuits. A charge-sharing argument would nevertheless demonstrate that $v_{\text{out}}(t)$ will remain as we have claimed.

Looking at these circuits' transfer functions, we see why we call them frequency-independent. Because the elements' interaction with frequency is of the same nature, it is cancelled in the transfer function expression. So we end up with transfer functions that don't contain ω terms.

2.2 Low-Pass (RC, LR)

R-C: We have a circuit with a resistor and capacitor, and we take the voltage across the capacitor.



⁶A natural question to ask would be: what happens to the energy if we supply a DC voltage ($\omega = 0$) to this circuit? Surely there will be some current flowing between the two potentials, which would cause energy to be dissipated, but where can this energy go? The resolution to this issue relies on the observation that our expression for $H(j\omega)$ resolves to $0/0$ for $\omega = 0$, and since our simplifications assumed that $\omega \neq 0$ (allowing us to divide by $j\omega$), the pure voltage-divider formula doesn't quite hold. In the case of $\omega = 0$, we can no longer model the inductors as being ideal, and their internal resistance will dissipate energy

We can compute the transfer function (the same as we calculated at the start of this note):

$$H(j\omega) = \frac{\frac{1}{j\omega C}}{R + \frac{1}{j\omega C}} = \frac{1}{1 + j\omega RC} \quad (26)$$

Now we see some more interesting behavior, since the transfer function depends on frequency. But how can we see that this is a low-pass filter? As a quick intuition check, if $\omega = 0$, $H(j\omega) = 1$ and if $\omega \rightarrow \infty$, then $H(j\omega) \rightarrow 0$. This makes sense, but what happens in between 0 and ∞ ? Let's take a look at a couple of values⁷ around a special frequency value, $\omega_c = \frac{1}{RC}$:

ω	$H(j\omega)$	$ H(j\omega) $	$\angle H(j\omega)$
$0.1\omega_c$	$\frac{1}{1 + j 0.1}$	0.995	-6°
ω_c	$\frac{1}{1 + j}$	0.71	-45°
$10\omega_c$	$\frac{1}{1 + j 10}$	0.1	-84°

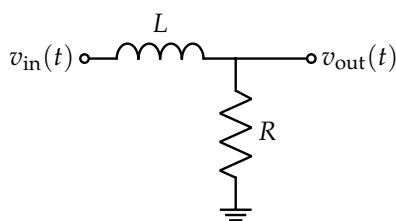
We see here that $\omega_c = \frac{1}{RC}$ is a very important frequency to look at since this is around the frequency where the behavior of the filter starts to qualitatively change.⁸ In fact, this is so important that we call this the **cutoff** frequency. Signals far below this frequency are allowed fully through, while signals above this frequency are largely blocked. So, this RC circuit constitutes a low-pass filter. We can write the low-pass RC transfer function in terms of ω_c : $H(j\omega) = \frac{1}{1 + \frac{j\omega}{\omega_c}}$.

We define the cutoff frequency ω_c ,⁹ as the point where:

$$|H(j\omega_c)| = \frac{\max_{\omega} |H(j\omega)|}{\sqrt{2}} \quad (27)$$

Here, $\max_{\omega} |H(j\omega)|$ is the maximum magnitude of $H(j\omega)$ over the full frequency range. For passive circuits containing passive components like R, L, C elements, this will usually be 1.

L-R: We have a circuit with a resistor and inductor, and we take the voltage across the resistor.



The above has the transfer function

$$H(j\omega) = \frac{R}{R + j\omega L} = \frac{1}{1 + j\omega \frac{L}{R}} = \frac{1}{1 + \frac{j\omega}{\omega_c}} \quad (28)$$

with $\omega_c = \frac{R}{L}$. Here, $\tau = \frac{L}{R}$ (as we saw in the transient analysis for an RL circuit).

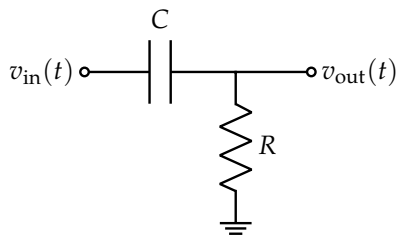
⁷We will soon plot such transfer functions over a continuum of frequencies.

⁸Note how this is the reciprocal of the time constant from back when we studied inverter transitions. We'll explore the significance of τ in a later section.

⁹ ω_c goes by many names, including cutoff frequency, corner frequency, critical frequency, and -3 dB frequency. If you see any of these, they all mean ω_c .

2.3 High-Pass (CR, RL)

C-R: We have a circuit with a resistor and capacitor, and we take the voltage across the resistor.



We compute:

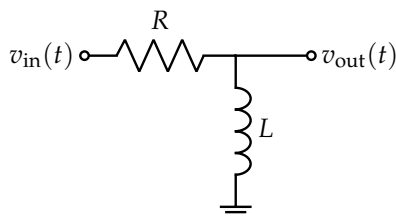
$$H(j\omega) = \frac{R}{R + \frac{1}{j\omega C}} = \frac{j\omega RC}{1 + j\omega RC} \quad (29)$$

The qualitative behavior is different from the low-pass filters; let's generate a table as before ($\omega_c = \frac{1}{RC}$):

ω	$H(j\omega)$	$ H(j\omega) $	$\angle H(j\omega)$
$0.1\omega_c$	$\frac{j 0.1}{1 + j 0.1}$	0.1	84°
ω_c	$\frac{j}{1 + j}$	0.71	45°
$10\omega_c$	$\frac{j 10}{1 + j 10}$	0.995	6°

Here, instead of passing low frequencies and blocking high frequencies, this circuit does the opposite!

R-L: We have a circuit with a resistor and inductor, and we take the voltage across the inductor.

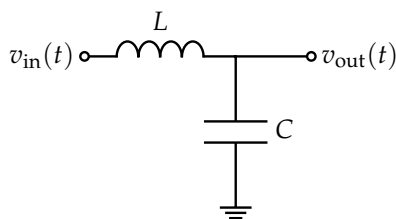


We compute ($\omega_c = \frac{R}{L}$):

$$H(j\omega) = \frac{j\omega L}{R + j\omega L} = \frac{j\omega \frac{L}{R}}{1 + j\omega \frac{L}{R}} = \frac{\frac{j\omega}{\omega_c}}{1 + \frac{j\omega}{\omega_c}} \quad (30)$$

2.4 LC Tank (LC, CL)

We introduced LC circuits like the one below in [Note 4](#), and observed how energy in such a circuit alternates between being stored in the inductor's magnetic field and the capacitor's electric field.



We can compute its transfer function in the standard way and arrive at the expression below, but we will study them more in the context of resonance in RLC circuits.

$$H(j\omega) = \frac{\frac{1}{j\omega C}}{\frac{1}{j\omega C} + j\omega L} = \frac{1}{1 - \omega^2 LC} \quad (31)$$

2.5 Summary of Filters

Well, that was a lot of filters! But looking at their transfer functions, we primarily see two distinct forms:

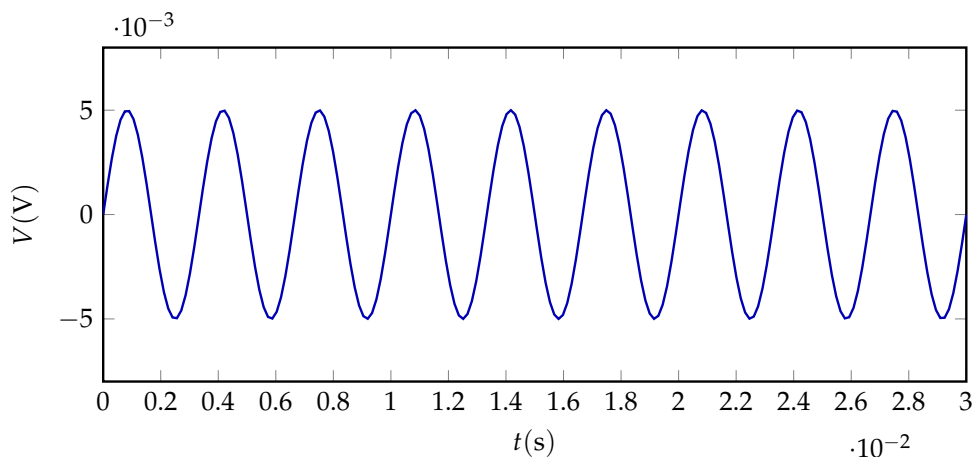
$$H_{LP}(\omega) = \frac{1}{1 + \frac{j\omega}{\omega_c}} \quad H_{HP}(\omega) = \frac{\frac{j\omega}{\omega_c}}{1 + \frac{j\omega}{\omega_c}} \quad (32)$$

The first of these two transfer functions describes a *low-pass filter*, since for values of $\omega \rightarrow 0$, $H(j\omega) \rightarrow 1$, while for values of $\omega \rightarrow \infty$, $H(j\omega) \rightarrow 0$. In essence, low frequencies can pass through, while high frequencies cannot. In a symmetric manner, the second of these transfer functions describes a *high-pass filter*, which has the opposite behavior.

3 Design

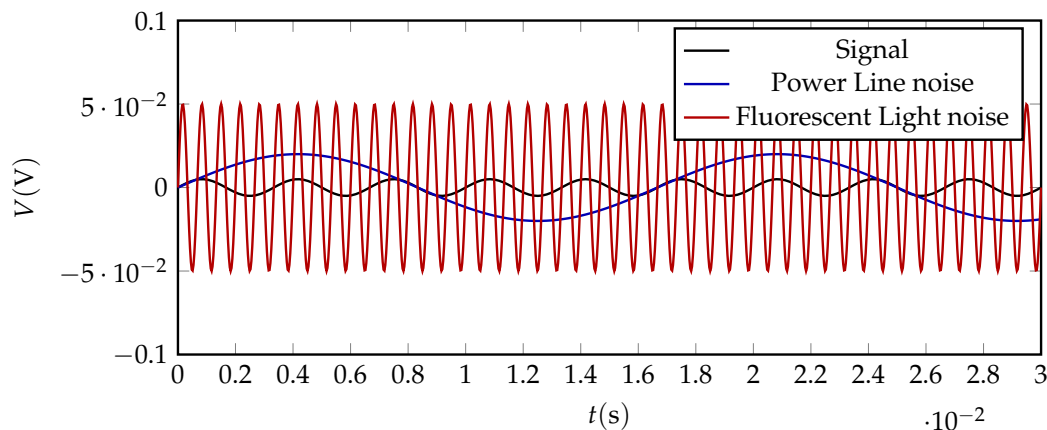
Now that we have a better understanding of filters, it's time to look at an example to understand why filters are important in practical applications. We also will recognize how useful it will be to have more powerful visualization tools than the tables of the previous section (we will cover these in later notes).

Imagine that we have a signal with amplitude 5 mV and frequency approximately $6 \frac{\text{krad}}{\text{s}}$ ¹⁰, as shown:

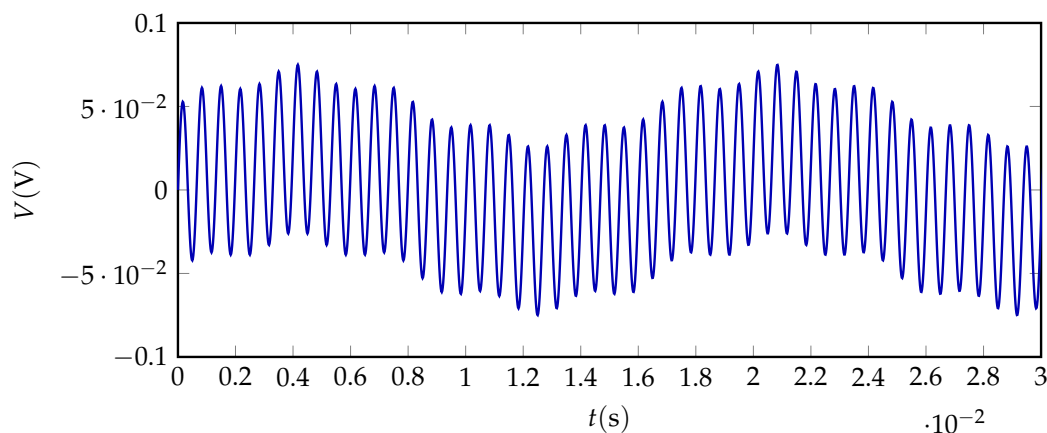


¹⁰Throughout this example, for simplicity, we will work purely with angular frequencies (conventionally denoted using ω). To convert to corresponding frequencies f in hertz (Hz), we would use $\omega = 2\pi f$ or $f = \frac{\omega}{2\pi}$.

Now, we introduce some $60 \frac{\text{rad}}{\text{s}}$ noise (a common contamination from power lines) at a stronger 20 mV, as well as some $200 \frac{\text{krad}}{\text{s}}$ noise (such as from a malfunctioning fluorescent light ballast) at 30 mV. These 3 waveforms (1 desired signal, and 2 undesired sources of noise) are shown below.¹¹



Adding these noise sources to our signal, we obtain the result:



At this point, our original signal is not even visible. But, there's still hope! There's a technique called the Fourier transform, which can break up a signal into its constituent sinusoids. We would need to record the signal precisely in a digital format, send it to a computer, and let a software library handle the rest.

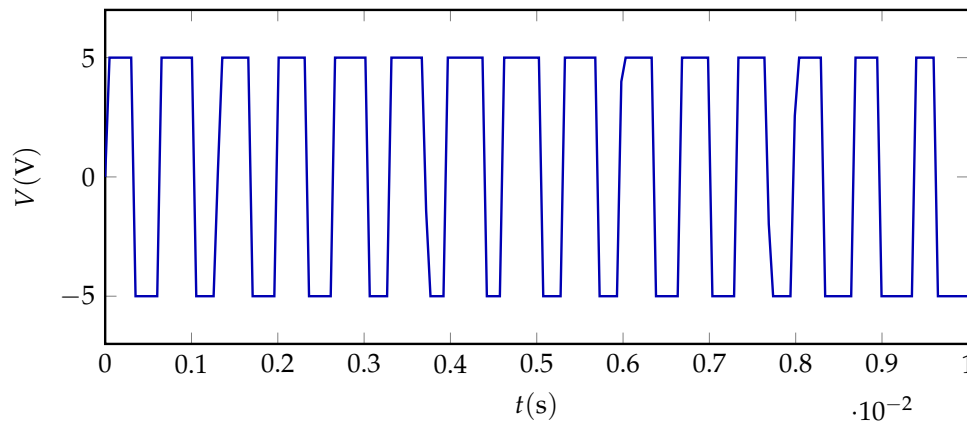
Unfortunately, we are likely encoding this signal using something akin to a 4-bit ADC, much like the one from lab. If our ADC has a maximum voltage of 3.3 V, then the least significant bit indicates a precision of:

$$\frac{3.3 \text{ V}}{2^4} \approx 0.2 \text{ V}. \quad (33)$$

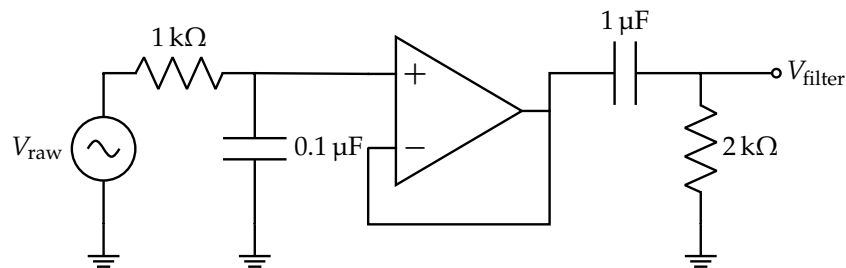
But our signal only has an amplitude of 5 mV! What can we do? One natural option is to amplify our signal, such as by using an op-amp circuit, until the amplitude of our signal is on the same order as our measurement ability. Suppose we amplify our signal by a factor of $\frac{3.3 \text{ V}}{5 \text{ mV}} = 660$. Unfortunately, when we do this, our noise signals will also become amplified (we can't choose just one sinusoid of the input to amplify)! They'll reach amplitudes of $30 \text{ mV} \cdot 660 \approx 20 \text{ V}$!

¹¹Throughout this example, the plots will be qualitative in terms of the specific frequencies, because frequencies of 60, 6000, and 200000 do not fit well on the same plot at a given timescale. However, the key pieces here will be the magnitudes of the waves, and these are correct as depicted. If you would like to replicate this example with full veracity, a plotting software like Matlab or [Desmos](#) will serve you well.

Of course, since our op-amps can only produce voltages between their rails (perhaps around $\pm 5\text{ V}$), our amplified signal will rail most of the time. Plotting the end result of this amplification, we obtain:



Notice that our signal is almost always railing, meaning that information has been lost before even arriving at a computer. *Therefore, analog processing is necessary to remove the noise.* We want to remove both the high and low frequency noise while maintaining the amplitude of our signal frequency. We could then amplify the signal, without worrying about our op-amp railing. To do so, consider the following circuit:



The final amplification of V_{signal} can be constructed using an non-inverting amplifier.

This example shows one of the key concepts in analog filtering, which is *frequency selectivity*. That is, we know our goal is to attenuate the undesired frequencies while maintaining as much of the desired signal as possible. How can we do so when our desired signal has a frequency *between* those of the undesired signals? We could consider simpler approaches, such as just a low-pass filter, or just a high-pass filter. But, while using just a low-pass filter could certainly block the light source noise very well, it would do nothing to the power line noise. The high-pass filter has the opposite problem of being able to block the power line noise, but not the light source noise.

Our approach here is to combine the low-pass and high-pass filter to create a *band-pass filter*, which allows "middle" frequencies through, but not low or high frequencies.¹² This is not one of the standard elementary filters, but demonstrates the power of a basic set of building blocks. This concept of cascading filter stages to achieve custom frequency selectivity is described in more detail in Notes 7 and 8. At a basic level, when filters with two (or more) stages are connected via unity-gain buffers (as we have here), the transfer functions' magnitudes multiply and their phases add. Here, we are more concerned with the magnitude.

Although we will later see how to choose the component values in our filtering circuits, let's do some preliminary analysis. We first have an RC low-pass filter. The cutoff frequency $\omega_{c,LP}$ here is $\frac{1}{RC} = 10 \times 10^3 \frac{\text{rad}}{\text{s}}$. Next, we have an RC high-pass filter, with cutoff frequency $\omega_{c,HP} = 500 \frac{\text{rad}}{\text{s}}$. Does this make sense for our application? Let's take a systematic look on how the filter impacts each of our input sinusoids (the tables in sections 2.2 and 2.3 will be useful here.).

¹²As we will see, what frequencies qualify as being in the "middle" will depend on the cutoff frequencies of the low-pass and high-pass filter components. We can design these too!

- Power Line, $60 \frac{\text{rad}}{\text{s}}$: The low-pass filter's cutoff is at $\omega_{c,LP} = 10 \times 10^3 \frac{\text{rad}}{\text{s}}$. This is many times larger than our power line noise frequency, this filter will effectively leave the power line noise unaffected.

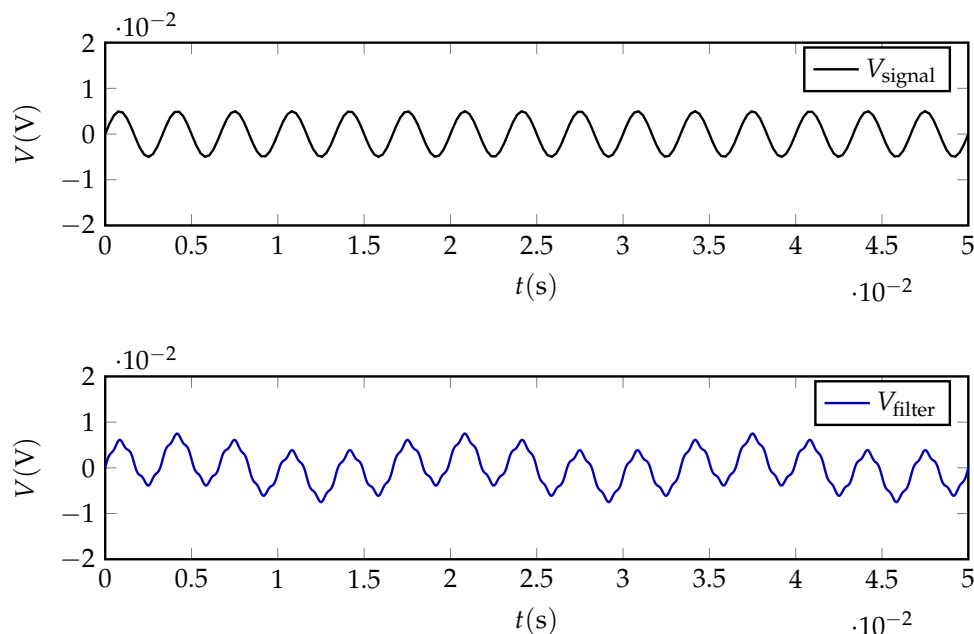
The high-pass filter, however, has a cutoff at $\omega_{c,HP} = 500 \frac{\text{rad}}{\text{s}}$. If we estimate that 60 is approximately $\frac{1}{10}$ of the cutoff (such estimation is very common for initial calculations), then the HP filter will attenuate the power line noise by a factor of about 10. This makes the power line noise magnitude 2 mV overall. This isn't great since it's still 40% of our input signal magnitude, but it's better than being 4 times as large.

- Signal, $6 \frac{\text{krad}}{\text{s}}$: The low-pass filter's cutoff is at $\omega_{c,LP} = 10 \times 10^3 \frac{\text{rad}}{\text{s}}$. This filter will therefore attenuate our signal by $\sim 14\%$; we can plug it into the formula for the magnitude of a low-pass filter's transfer function, estimating that $\omega = \frac{1}{1.7}\omega_c$. So, $|H_{LP}| = \left| \frac{1}{1+j1.7} \right| \approx 0.862$. The high-pass filter's cutoff is much lower than the signal frequency, so it doesn't affect the signal.

- Light Source Noise, $200 \frac{\text{krad}}{\text{s}}$: The same low-pass filter ($\omega_{c,LP} = 10 \times 10^3 \frac{\text{rad}}{\text{s}}$) will attenuate our signal by a factor of ~ 20 , so it becomes about 1.5 mV (about a third of our signal's magnitude).

The high-pass filter's cutoff is much lower than the light noise frequency, so it doesn't affect this input.

Looking at the ultimate end-result V_{filter} versus V_{signal} , we observe:



Notice that we have successfully used our low- and high-pass filters to reject undesired frequencies! There is still some jaggedness in the output signal, due to the higher frequency component which didn't become fully erased. Similarly, you may notice the output value slowly rises up and down; this is a manifestation of the imperfect removal of the low-frequency noise. Despite these imperfections, overall, we can now actually measure our signal with just a 4-bit DAC! This issue of noise sources of different frequencies contaminating the desired signal is ubiquitous in circuit design, so it's quite nice that we can develop a solution to this problem using our knowledge about filters!

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