

1 Matrix Form

1.1 Introduction

Last time we saw how to solve simple scalar first order linear differential equations.

Now, we are going to expand our tool set and learn how to tackle multivariable differential equations. (You will see in homeworks that these exact same ideas will also equip us to deal with higher-order differential equations where we have second and third derivatives involved.) Let's motivate the need to understand two dimensional systems with the circuit in fig. 1.

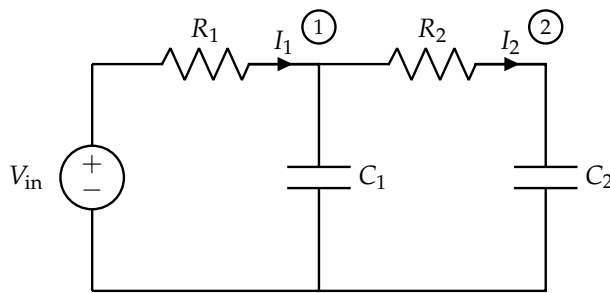


Figure 1: Circuit forming a two-dimensional system.

Let $C_1 = C_2 = 1 \mu\text{F}$, $R_1 = \frac{1}{3}\text{M}\Omega$ and $R_2 = \frac{1}{2}\text{M}\Omega$.¹

Concept Check: Before we begin, how would you solve for the above system if $R_2 = 0$?

Answer: If $R_2 = 0$, then C_1 and C_2 are connected in parallel and can be combined into a single effective capacitance of $C_1 + C_2$. This happened earlier when we had two gate capacitances in parallel.

We first consider the discharging case. In the above system, let $V_{in} = 1\text{V}$ for time $t < 0$, and $V_{in} = 0\text{V}$ for time $t \geq 0$. With this in mind, we have two steady state conditions:

1. Initial condition $t = 0$: The voltage has been charging the capacitors for an infinite amount of time. Hence, both capacitors have voltage $V_{C_1} = V_{C_2} = 1\text{V}$ and current $I_1 = I_2 = 0\text{A}$.
2. As $t \rightarrow \infty$: After the capacitors have been allowed to discharge for a long period of time, they carry no charges on their plates, hence $V_{C_1} = V_{C_2} = 0\text{V}$.

Now, we solve for the transients. That is, how does our system go from condition (a) to condition (b)? First, we need to set up the circuit equations. We will define $V_1 = V_{C_1}$ and $V_2 = V_{C_2}$ as the voltages across our capacitors.

$$V_2 = V_1 - I_2 R_2 \tag{1}$$

$$I_2 = C_2 \frac{d}{dt} V_2 \tag{2}$$

$$0 - V_1 = I_1 R_1 \Rightarrow I_1 = -\frac{V_1}{R_1} \tag{3}$$

¹The SI prefixes 'M' and 'μ' stand for "mega" and "micro," and correspond to the decimal multiples of 10^6 and 10^{-6} respectively.

$$I_1 = I_2 + C_1 \frac{d}{dt} V_1 \quad (4)$$

For eq. (1) and eq. (2), we look at the current through Node (2), using Ohm's law for the former and voltage-current relation for the capacitor in the latter. Similarly, we find eq. (3) and eq. (4) by looking at Node (1). Similar to what we had seen in a single capacitor system, we have effectively introduced two new variables: $\frac{d}{dt} V_1$ and $\frac{d}{dt} V_2$. Next, to solve for the transients, we need to first define our system variables. **The standard approach that we will always take is to make anything that gets differentiated into a state variable.** Hence, we will need two state variables: V_1 and V_2 .

1.2 Systems of Differential Equations

We get a system of differential equations by isolating the derivative terms and solving for them in terms of their non-differentiated selves.² In principle, every other circuit quantity of interest can be found once we know how the state variables and their derivatives behave.

$$\frac{d}{dt} V_1(t) = -\left(\frac{1}{R_1 C_1} + \frac{1}{R_2 C_1}\right) V_1(t) + \frac{V_2(t)}{R_2 C_1} \quad (5)$$

$$\frac{d}{dt} V_2(t) = \frac{V_1(t)}{R_2 C_2} - \frac{V_2(t)}{R_2 C_2} \quad (6)$$

The steps outlining how we obtained eq. (5) and eq. (6) are in section [Appendix A](#). The fact that every linear circuit involving capacitors (and as we will see inductors as well) can be expressed in this form is effectively a consequence of Gaussian Elimination.

Concept Check: Take a second to work out setting up the above differential equations. It is important to try to reduce all your branch equations from the circuit analysis to as few variables as possible.

As seen in EECS16A, when encountering a system of equations, we try to put it into vector/matrix form to try to solve it. So, let's put eq. (5) and eq. (6) in the matrix differential form. We define $\vec{x}(t) = \begin{bmatrix} V_1(t) \\ V_2(t) \end{bmatrix}$, and:

$$\begin{bmatrix} \frac{dV_1}{dt} \\ \frac{dV_2}{dt} \end{bmatrix} = \begin{bmatrix} -\left(\frac{1}{R_1 C_1} + \frac{1}{R_2 C_1}\right) V_1 + \frac{V_2}{R_2 C_1} \\ \frac{V_1}{R_2 C_2} - \frac{V_2}{R_2 C_2} \end{bmatrix} \quad (7)$$

$$= \begin{bmatrix} -\left(\frac{1}{R_1 C_1} + \frac{1}{R_2 C_1}\right) & \frac{1}{R_2 C_1} \\ \frac{1}{R_2 C_2} & -\frac{1}{R_2 C_2} \end{bmatrix} \begin{bmatrix} V_1(t) \\ V_2(t) \end{bmatrix} \quad (8)$$

$$\frac{d}{dt} \vec{x}(t) = \begin{bmatrix} -5 & 2 \\ 2 & -2 \end{bmatrix} \vec{x}(t) \quad (9)$$

In eq. (9), we have substituted in component values to get a matrix with concrete numbers.

Quick Aside: You may come across a dot over the variable of interest (e.g. \dot{V}_1) as short hand to represent the $\frac{d}{dt}$ operator. This is alternatively known as Newton's notation and is sometimes used for conciseness.

In general, we want to set up our systems in the following generic form:

$$\frac{d}{dt} \vec{x}(t) = A \vec{x}(t) + \vec{b} \quad (10)$$

So, why do we choose this form? Because it most closely resembles the first order scalar differential equation we studied in the previous note, and our goal in this note is to convert this equation into a collection of first

²You can view this as doing the downward pass of Gaussian Elimination using an ordering of the variables so that the $\frac{d}{dt}(\cdot)$ variables come second-to-last and their non-differentiated counterparts come last. Because there will be more unknowns than equations, Gaussian elimination will stop on the last $\frac{d}{dt}$ -ed variable. Then, we can do a limited upward-pass (back-substitution pass) of Gaussian elimination to purge any dependence of one $\frac{d}{dt}$ -ed variable on any other $\frac{d}{dt}$ -ed variable. This gives rise to the system of differential equations in a systematic way — it is the lower block that remains after doing this back-substitution.

order differential equations. In the context of our circuit example, $A = \begin{bmatrix} -5 & 2 \\ 2 & -2 \end{bmatrix}$ and $\vec{b} = \vec{0}$ (because there is no external voltage or current being applied for $t \geq 0$).

2 Diagonalization by Means of an Eigenbasis

2.1 Eigenvalues and Eigenvectors (EECS16A Review)

Before we delve into solving our system of differential equations, let's review a core concept from EECS16A - eigenvalues and eigenvectors. While this will be an abridged version of EECS16A content, the [note from EECS16A](#) has more detail.

An eigenvector is a vector that is only scaled by a value (the corresponding eigenvalue) when acted upon by a matrix:

$$A\vec{v}_\lambda = \lambda\vec{v}_\lambda \quad (11)$$

Here, $\vec{v}_\lambda \neq \vec{0}$ is an eigenvector of A corresponding to the scalar λ eigenvalue. If we look at all the eigenvectors of A corresponding to a single λ , these form a subspace known as the λ -eigenspace. Each distinct eigenvalue is associated with its own nontrivial eigenspace³. We can see this subspace property by rearranging the above equation into:

$$(A - \lambda I_n)\vec{v} = \vec{0}. \quad (12)$$

Here, I_n is the $n \times n$ identity matrix. Since $A\vec{v}_\lambda = \lambda\vec{v}_\lambda$, we know that $A - \lambda I_n$ has a nullspace (namely the eigenspace corresponding to the eigenvalue λ of A). Consequently, $A - \lambda I_n$ must be non-invertible and thus $\det(A - \lambda I_n) = 0$. Remember, the determinant of a square matrix measures the oriented volume⁴ of the unit hypercube as transformed by that matrix. Non-invertible matrices destroy information by squashing the space along at least one direction (the nullspace directions), and therefore result in something with volume zero.

This gives us the **characteristic equation, or polynomial** and solving this equation will give us the eigenvalues λ . As an example, let's compute the eigenvalues of $A = \begin{bmatrix} 1 & 2 \\ 4 & 3 \end{bmatrix}$:

$$\det(A - \lambda I_2) = \det\left(\begin{bmatrix} 1 & 2 \\ 4 & 3 \end{bmatrix} - \begin{bmatrix} \lambda & 0 \\ 0 & \lambda \end{bmatrix}\right) \quad (13)$$

$$= \det\left(\begin{bmatrix} 1 - \lambda & 2 \\ 4 & 3 - \lambda \end{bmatrix}\right) \quad (14)$$

Setting the determinant equal to zero, we get the characteristic equation:

$$0 = (1 - \lambda)(3 - \lambda) - 4 \cdot 2 = \lambda^2 - 4\lambda - 5 = (\lambda - 5)(\lambda + 1) \quad (15)$$

³In 16A, you also saw some interesting properties of these eigenspaces. For example, any collection of nonzero vectors drawn from eigenspaces corresponding to distinct eigenvalues is linearly independent. This is proved by contradiction—the simplest case is for two distinct eigenvalues. (A vector can't simultaneously be two different multiples of the same thing!) From there, the proof continues to three vectors. If one is expressible as a combination of the others, then those two must be expressible as a multiple of each other, which we have already ruled out, and so on. As a result, there cannot be more than n distinct eigenvalues for an $n \times n$ matrix since each one must be associated with its own eigenspace, and there can only be n linearly independent directions in an n -dimensional space.

⁴Recall how the determinant is computed by doing Gaussian elimination. Multiplication of a row by -1 incurs a factor of -1 because it is like looking through a mirror, since a mirror flips orientation. Every time we swap two rows, we also incur a factor of -1 . This is because swapping two rows is like rotating and then looking through a mirror. Rotating doesn't change volume, but the mirror flips the orientation. Adding a multiple of one row to another does nothing to the oriented volume because it is like shearing a cube—just as pushing over a deck of cards doesn't change the volume of the deck, shearing doesn't change volume either. Scaling a row by a positive number has that scaling effect on the volume, and so by reversing the scalings done in Gaussian Elimination, we can get the volume effect of the original matrix.

Setting eq. (15) to 0 and solving the quadratic equation, we find $\lambda_1 = 5, \lambda_2 = -1$. Each eigenvalue has its own eigenspace, defining the nullspace of the matrix $A - \lambda I_n$. Hence to find the eigenspace, we can just find the relevant nullspace⁵

$$(A - \lambda_1 I_2)\vec{v}_1 = (A - 5I_2)\vec{v}_1 = \begin{bmatrix} -4 & 2 \\ 4 & -2 \end{bmatrix} \begin{bmatrix} v_{11} \\ v_{12} \end{bmatrix} = \vec{0} \quad (16)$$

We see that both rows indicate that $4v_{11} - 2v_{12} = 0$, and hence an eigenvector is $\vec{v}_1 = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$.

Concept Check: Find an eigenvector \vec{v}_2 corresponding to λ_2 .

Answer: A second eigenvector is $\vec{v}_2 = \alpha \begin{bmatrix} 1 \\ -1 \end{bmatrix}$. Note that any $\alpha \neq 0$ would still be a valid eigenvector.

2.2 Repeated and Complex Eigenvalues

2.2.1 Repeated Eigenvalues

For a 2×2 matrix, it's possible that the two eigenvalues that you end up with have the same value, leading to **repeated eigenvalues**. This repeated eigenvalue can have one or two-dimensional eigenspace (unlike a single, unrepeated eigenvalue, which will only have a one-dimensional eigenspace).

For example, the following matrix has a repeated eigenvalue of λ .

$$A = \begin{bmatrix} \lambda & 0 \\ 0 & \lambda \end{bmatrix} \quad (17)$$

The λ -eigenspace of this matrix is \mathbb{R}^2 since for any vector $\vec{v} \in \mathbb{R}^2$, $A\vec{v} = \lambda\vec{v}$.

We can also have examples like

$$A = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} \quad (18)$$

that have a single eigenvalue $\lambda = 0$. In this case, the relevant eigenspace is one-dimensional, only $\begin{bmatrix} 1 \\ 0 \end{bmatrix}$, and its multiples are eigenvectors here.⁶

2.2.2 Complex Eigenvalues

It can also be the case that when we solve $\det(A - \lambda I_n) = 0$, there will be no real solutions to λ . Consider the rotation matrix (described in [Note j](#)):

$$R_\theta = \begin{bmatrix} \cos(\theta) & -\sin(\theta) \\ \sin(\theta) & \cos(\theta) \end{bmatrix} \quad (19)$$

Concept Check: When does R_θ have real eigenvalues? Why? *Answer:* For $\theta = 0^\circ$ or $\theta = 180^\circ$.

However, when θ does not correspond to 0° or 180° rotation, there are no real vectors that are scaled versions of themselves after the transformation. This results in complex eigenvalues. For example, let's look

⁵You know from EECS16A how you can find nullspaces systematically using Gaussian Elimination. Simply run Gaussian elimination until it terminates—you will have at least one row of zeros. Then work your way backward from the end finding free variables (variables that you did not “pivot” on, or variables that you never explicitly tried to eliminate from lower equations. These are variables that got eliminated on their own!) and then expressing others in terms of them. This will give you a basis for the nullspace.

⁶If there were any other linearly independent eigenvectors corresponding to the 0 eigenvalue, then everything would have to be an eigenvector and that would mean the matrix mapped everything to the zero vector. But the matrix is definitely not the zero matrix, so that isn't what is going on.

at 45° rotation:

$$R = \frac{1}{\sqrt{2}} \begin{bmatrix} 1 & -1 \\ 1 & 1 \end{bmatrix}$$

$$\det(R - \lambda I) = \frac{1}{2}(1 - \lambda)(1 - \lambda) + \frac{1}{2}$$

Setting this determinant equal to zero and solving yields the complex eigenvalues $\lambda = \frac{1}{\sqrt{2}}(1 + j)$ and $\lambda = \frac{1}{\sqrt{2}}(1 - j)$, which makes sense because there are no physical (real) eigenvectors for a rotation transformation in two dimensions.

2.3 Change of Basis and Diagonalization

Let's start with a vector $\vec{v} \in \mathbb{R}^n$. This vector represents a point in space. When you think about this vector written out using the coordinates, you are scaling the vectors in the standard basis (i.e. the columns of the identity matrix, I_n) by the components of \vec{v} and then adding them up:

$$\vec{v} = I_n \vec{v} = v_1 \begin{bmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{bmatrix} + v_2 \begin{bmatrix} 0 \\ 1 \\ \vdots \\ 0 \end{bmatrix} + \cdots + v_n \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 1 \end{bmatrix} \quad (20)$$

But, suppose that I think about this vector in terms of a different set of directions. We can define a new coordinate system using a basis, i.e. n linearly independent vectors $\vec{v}_1, \vec{v}_2, \dots, \vec{v}_n$. Now, let's say I have a

vector \vec{u} which I am representing as $\begin{bmatrix} u_1 \\ u_2 \\ \vdots \\ u_n \end{bmatrix}$ with respect to my new coordinate system. How can I translate

this to the coordinates you are familiar with? Well, instead of scaling the vectors of the standard basis, we are now scaling the vectors defining my new basis (the new basis vectors). Assuming that both of us were thinking of the same physical point in space, then the vector \vec{v} in the standard basis is:

$$\vec{v} = u_1 \vec{v}_1 + u_2 \vec{v}_2 + \cdots + u_n \vec{v}_n \quad (21)$$

$$= \begin{bmatrix} | & | & \cdots & | \\ \vec{v}_1 & \vec{v}_2 & \cdots & \vec{v}_n \\ | & | & \cdots & | \end{bmatrix} \begin{bmatrix} u_1 \\ u_2 \\ \vdots \\ u_n \end{bmatrix} \quad (22)$$

$$= V \vec{u} \quad (23)$$

Hence, to transform a vector from my basis to your standard basis involves just a matrix multiplication. This can be expressed in a diagram given in fig. 2 using the up and down arrows.

Next, let's see what happens if we have a matrix A that transforms vectors. Consider $A = \begin{bmatrix} 1 & 2 \\ 4 & 3 \end{bmatrix}$. This matrix performs a linear transformation, $\vec{y} = A\vec{x}$. Is there another basis within which this transformation is much simpler to understand?

Let's suppose we had our new basis V so that $\vec{x} = V\vec{\tilde{x}}$ and correspondingly, $\vec{\tilde{x}} = V^{-1}\vec{x}$. Similarly, $\vec{y} = V\vec{\tilde{y}}$ and correspondingly, $\vec{\tilde{y}} = V^{-1}\vec{y}$. Then:

$$A\vec{x} = AV\vec{\tilde{x}} \quad (24)$$

$$= A(\tilde{x}_1 \vec{v}_1 + \tilde{x}_2 \vec{v}_2) \quad (25)$$

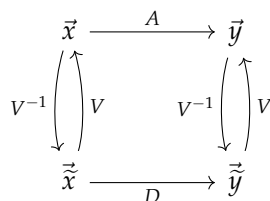


Figure 2: Change of Basis Mapping. The top row has everything in the standard basis. The bottom row is in V -basis. The matrix D is supposed to represent the same linear transformation as A , except that it does so for vectors expressed in V -basis. It turns out $D = V^{-1}AV$ since vectors are columns. We use tilde \sim coordinates for things in the lower basis.

$$= \tilde{x}_1 A \vec{v}_1 + \tilde{x}_2 A \vec{v}_2 \quad (26)$$

Now, if we chose our new basis to be the ones defined by the eigenvectors of A , then we can simplify:

$$= \tilde{x}_1 \lambda_1 \vec{v}_1 + \tilde{x}_2 \lambda_2 \vec{v}_2 \quad (27)$$

$$= \begin{bmatrix} | & | \\ \vec{v}_1 & \vec{v}_2 \\ | & | \end{bmatrix} \begin{bmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{bmatrix} \begin{bmatrix} \tilde{x}_1 \\ \tilde{x}_2 \end{bmatrix} \quad (28)$$

$$= VD\tilde{x} \quad (29)$$

$$= VDV^{-1}\vec{x}, \quad (30)$$

where D is the diagonal matrix of eigenvalues and V is a matrix with the corresponding eigenvectors as its columns. Thus we have proved that $A = VDV^{-1}$. Furthermore, this also means that $D = V^{-1}AV$.

In general, the pattern we have used above will hold *whenever we can find an eigenbasis*, which is a full basis consisting of eigenvectors. Why? From the eigenvalue-eigenvector relationship we have $AV = [A\vec{v}_1 \ A\vec{v}_2 \ \cdots \ A\vec{v}_n] = [\lambda_1\vec{v}_1 \ \lambda_2\vec{v}_2 \ \cdots \ \lambda_n\vec{v}_n] = VD$. If V is full rank and thus invertible, we can left multiply both sides by V^{-1} to get:

$$V^{-1}AV = V^{-1}VD = D = \begin{bmatrix} \lambda_1 & 0 & \cdots & 0 \\ 0 & \lambda_1 & \cdots & 0 \\ \vdots & \ddots & \ddots & \vdots \\ 0 & \cdots & 0 & \lambda_n \end{bmatrix} \quad (31)$$

But, we know that there also exist matrices for which an eigenbasis does not exist. For those, this trick won't work.

3 How to Solve?

Now that we understand how to convert our systems of differential equations into vector/matrix form, let's understand how to solve them. We would ideally like to use the tools we developed for the scalar case. But how can we do this when our equations seem to be fundamentally dependent on two independent variables? We wouldn't have this problem of "coupling" if our A matrix were diagonal. So, let's try to *change our basis to one where the A matrix does become diagonal*.

Note that if our system of differential equations were "uncoupled" to begin with, our A matrix would have been diagonal. Hence, we could proceed to solve the first order differential equations independently of each other, as seen before. Consider the circuit in fig. 3.

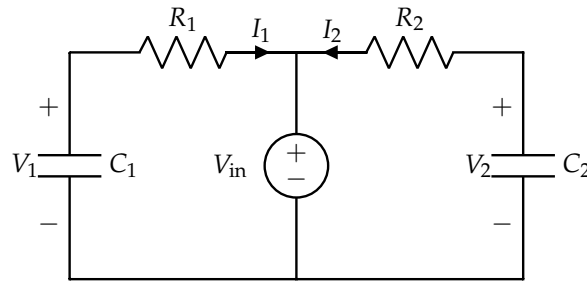


Figure 3: This circuit is expressible as a diagonal system.

Both the capacitors have been charged to V_{in} . At $t = 0$, we set $V_{in} = 0V$, and allow the capacitors to discharge. Hence our initial conditions are $V_1(0) = V_2(0) = V_{in}$. We find the branch equations eqs. (32) to (33):

$$I_1 = -C_1 \frac{d}{dt} V_1 = \frac{V_1}{R_1} \quad (32)$$

$$I_2 = -C_2 \frac{d}{dt} V_2 = \frac{V_2}{R_2} \quad (33)$$

Hence from eq. (32) and eq. (33), we arrive at the following uncoupled differential equation:

$$\frac{d}{dt} \begin{bmatrix} V_1(t) \\ V_2(t) \end{bmatrix} = \begin{bmatrix} -\frac{1}{R_1 C_1} & 0 \\ 0 & -\frac{1}{R_2 C_2} \end{bmatrix} \begin{bmatrix} V_1(t) \\ V_2(t) \end{bmatrix} \quad (34)$$

Concept Check: We can easily solve the above system of equations by separately solving for V_1 and V_2 . Review [Note 1](#) if you are unsure about how to solve for the voltages here.

Coming back to our original system, we define $\vec{x}(t) = \begin{bmatrix} V_1(t) \\ V_2(t) \end{bmatrix}$. Then, in eq. (9), we set up the differential equations as follows:

$$\frac{d}{dt} \vec{x}(t) = \begin{bmatrix} -5 & 2 \\ 2 & -2 \end{bmatrix} \vec{x}(t) \quad (35)$$

As discussed, let's begin by finding a coordinate system in which the transformation represented by this matrix is diagonal (which we know how to solve). First, we must find its eigenvalues:

$$\det(\lambda I - A) = \det \left(\begin{bmatrix} \lambda + 5 & -2 \\ -2 & \lambda + 2 \end{bmatrix} \right) \quad (36)$$

$$= (\lambda + 5)(\lambda + 2) - 4 \quad (37)$$

$$= \lambda^2 + 7\lambda + 6 = (\lambda + 6)(\lambda + 1) \quad (38)$$

Solving the above characteristic equation, we get the eigenvalues as $\lambda_1 = -6$ and $\lambda_2 = -1$ and hence we can find a set of eigenvectors as $\vec{v}_1 = \begin{bmatrix} -\frac{2}{\sqrt{5}} \\ \frac{1}{\sqrt{5}} \end{bmatrix}$ and $\vec{v}_2 = \begin{bmatrix} \frac{1}{\sqrt{5}} \\ \frac{2}{\sqrt{5}} \end{bmatrix}$. Defining the new basis as $V = [\vec{v}_1 \quad \vec{v}_2]$, we get the diagonal matrix $\Lambda = \begin{bmatrix} -6 & 0 \\ 0 & -1 \end{bmatrix}$.

Hence, we can write $A = V\Lambda V^{-1}$ and rewrite eq. (9) as follows:

$$\frac{d}{dt} \vec{x}(t) = V\Lambda V^{-1} \vec{x}(t) \quad (39)$$

$$= \begin{bmatrix} -\frac{2}{\sqrt{5}} & \frac{1}{\sqrt{5}} \\ \frac{1}{\sqrt{5}} & \frac{2}{\sqrt{5}} \end{bmatrix} \begin{bmatrix} -6 & 0 \\ 0 & -1 \end{bmatrix} \begin{bmatrix} -\frac{2}{\sqrt{5}} & \frac{1}{\sqrt{5}} \\ \frac{1}{\sqrt{5}} & \frac{2}{\sqrt{5}} \end{bmatrix}^{-1} \vec{x}(t) \quad (40)$$

Let's perform a change of basis by setting $\vec{\tilde{x}}(t) = \begin{bmatrix} \tilde{x}_1(t) \\ \tilde{x}_2(t) \end{bmatrix} = V^{-1}\vec{x}(t)$. By left multiplying V^{-1} on both sides of eq. (39), we get the following:

$$V^{-1} \frac{d}{dt} \vec{x}(t) = V^{-1} A \vec{x}(t) = V^{-1} V \Lambda V^{-1} \vec{x}(t) \quad (41)$$

$$\implies \frac{d}{dt} V^{-1} \vec{x}(t) = \Lambda \vec{\tilde{x}}(t) \quad (42)$$

$$\implies \frac{d}{dt} \vec{\tilde{x}} = \begin{bmatrix} -6 & 0 \\ 0 & -1 \end{bmatrix} \vec{\tilde{x}}(t) \quad (43)$$

Because V^{-1} is a constant matrix, we can pull it inside the derivative to go from eq. (41) to eq. (42). In eq. (43), we have successfully uncoupled our equations and we can proceed to solve them independently as mentioned earlier:

$$\frac{d}{dt} \tilde{x}_1(t) = -6\tilde{x}_1(t) \implies \tilde{x}_1 = k_1 e^{-6t} \quad (44)$$

$$\frac{d}{dt} \tilde{x}_2(t) = -\tilde{x}_2(t) \implies \tilde{x}_2 = k_2 e^{-t} \quad (45)$$

Next, we need to solve for our constants k_1 and k_2 . Recall our initial conditions, $V_1(0) = V_2(0) = 1V$. Hence, $\tilde{x}_1(0)$ and $\tilde{x}_2(0)$ are given by⁷:

$$\vec{\tilde{x}}(0) = V^{-1} \begin{bmatrix} V_1(0) \\ V_2(0) \end{bmatrix} \quad (46)$$

$$\implies \begin{bmatrix} \tilde{x}_1(0) \\ \tilde{x}_2(0) \end{bmatrix} = \begin{bmatrix} -\frac{2}{\sqrt{5}} & \frac{1}{\sqrt{5}} \\ \frac{1}{\sqrt{5}} & \frac{2}{\sqrt{5}} \end{bmatrix} \begin{bmatrix} 1 \\ 1 \end{bmatrix} \quad (47)$$

$$= \begin{bmatrix} -\frac{1}{\sqrt{5}} \\ \frac{3}{\sqrt{5}} \end{bmatrix} \quad (48)$$

Hence, $k_1 = -\frac{1}{\sqrt{5}}$, $k_2 = \frac{3}{\sqrt{5}}$. Now, we can transform back into our original basis to find $V_1(t)$ and $V_2(t)$:

$$\vec{x} = V \vec{\tilde{x}} \quad (49)$$

$$= \begin{bmatrix} -\frac{2}{\sqrt{5}} & \frac{1}{\sqrt{5}} \\ \frac{1}{\sqrt{5}} & \frac{2}{\sqrt{5}} \end{bmatrix} \begin{bmatrix} -\frac{1}{\sqrt{5}} e^{-6t} \\ \frac{3}{\sqrt{5}} e^{-t} \end{bmatrix} \quad (50)$$

$$= \begin{bmatrix} \frac{2}{5} e^{-6t} + \frac{3}{5} e^{-t} \\ -\frac{1}{5} e^{-6t} + \frac{6}{5} e^{-t} \end{bmatrix} \quad (51)$$

For $t \geq 0$, we find that $V_1 = \frac{2}{5} e^{-6t} + \frac{3}{5} e^{-t}$ and $V_2 = -\frac{1}{5} e^{-6t} + \frac{6}{5} e^{-t}$. In fig. 4 is a plot of our solutions: Using the same argument, we can see how the voltage will vary with different initial conditions (fig. 5)

⁷Note that in this case, $V^{-1} = V^T = V$ so when we plug in V^{-1} below, it's the same as plugging in V .

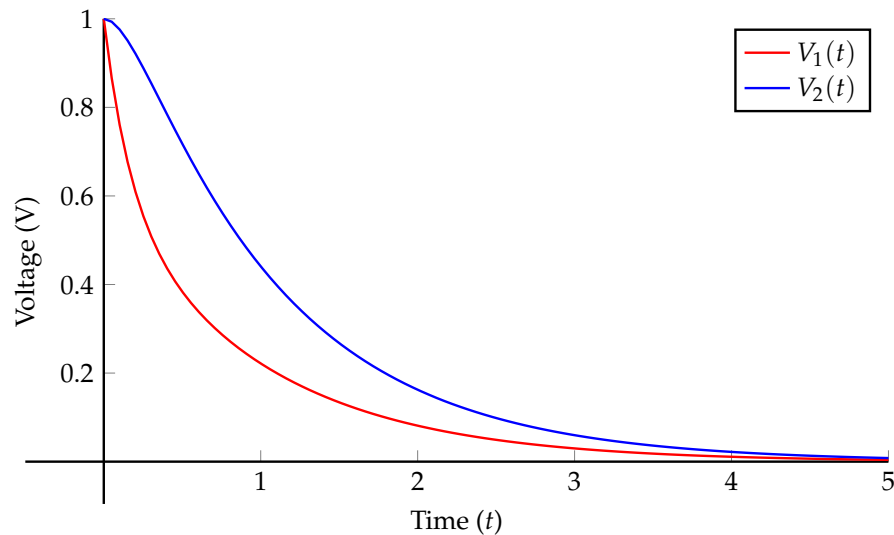
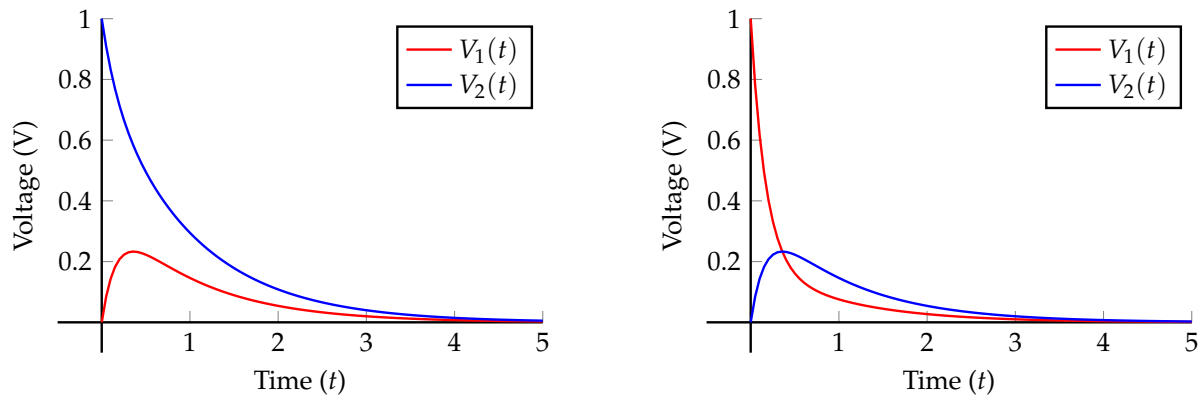


Figure 4: Initial Conditions: $V_1(0) = 1V$ and $V_2(0) = 1V$. Homogeneous Case Solution.



(a) Initial Conditions: $V_1(0) = 0V$ and $V_2(0) = 1V$. Here, $V_1(t) = -\frac{2}{5}e^{-6t} + \frac{2}{5}e^{-t}$, and $V_2(t) = \frac{1}{5}e^{-6t} + \frac{4}{5}e^{-t}$.

(b) Initial Conditions: $V_1(0) = 1V$ and $V_2(0) = 0V$. Here, $V_1(t) = \frac{4}{5}e^{-6t} + \frac{1}{5}e^{-t}$, and $V_2(t) = -\frac{2}{5}e^{-6t} + \frac{2}{5}e^{-t}$.

Figure 5: Voltage transients for different initial conditions on the capacitors. The specific solutions are in the subcaptions for your reference.

Concept Check: Take a minute to qualitatively reason about the initial increase in voltages in Figure 5.

Now that we have a good understanding of the homogenous case, let's look at the voltage transients of *charging up* our two capacitor system. Then, we have uncharged capacitors to start ($V_1(0) = V_2(0) = 0V$), and we apply a voltage $V_{in} = 1V$ for time $t > 0$. We get the following equations:

$$V_2 = V_1 - I_2 R_2 \quad (52)$$

$$I_2 = C_2 \frac{d}{dt} V_2 \quad (53)$$

$$V_{in} - V_1 = I_1 R_1 \Rightarrow I_1 = \frac{V_{in} - V_1}{R_1} \quad (54)$$

$$I_1 = I_2 + C_1 \frac{d}{dt} V_1 \quad (55)$$

Hence, our matrix differential equation is:

$$\frac{d}{dt} \vec{x}(t) = \frac{d}{dt} \begin{bmatrix} V_1(t) \\ V_2(t) \end{bmatrix} = \begin{bmatrix} -\left(\frac{1}{R_1 C_1} + \frac{1}{R_2 C_1}\right) & \frac{1}{R_2 C_1} \\ \frac{1}{R_2 C_2} & -\frac{1}{R_2 C_2} \end{bmatrix} \begin{bmatrix} V_1 \\ V_2 \end{bmatrix} + \begin{bmatrix} \frac{V_{in}}{R_1 C_1} \\ 0 \end{bmatrix} \quad (56)$$

$$= \begin{bmatrix} -5 & 2 \\ 2 & -2 \end{bmatrix} \begin{bmatrix} V_1 \\ V_2 \end{bmatrix} + \begin{bmatrix} 3 \\ 0 \end{bmatrix} \quad (57)$$

$$= A\vec{x}(t) + \vec{b}. \quad (58)$$

Now the system has a non-homogeneous term which we haven't seen before, but let's see if diagonalization can help us simplify the problem. Letting $\vec{\tilde{x}}(t) = V^{-1}\vec{x}(t)$ and using $A = V\Lambda V^{-1}$, we plug in:

$$\frac{d}{dt} \vec{\tilde{x}}(t) = \frac{d}{dt} V^{-1}\vec{x}(t) = V^{-1} \left(\frac{d}{dt} \vec{x}(t) \right) \quad (59)$$

$$= V^{-1}(A\vec{x}(t) + \vec{b}) \quad (60)$$

$$= V^{-1}A\vec{x}(t) + V^{-1}\vec{b} \quad (61)$$

$$= V^{-1}AV\vec{\tilde{x}}(t) + V^{-1}\vec{b} \quad (62)$$

$$= \Lambda\vec{\tilde{x}}(t) + V^{-1}\vec{b}. \quad (63)$$

In this case, recall from earlier in the notes that

$$A = \begin{bmatrix} -5 & 2 \\ 2 & -2 \end{bmatrix} = \begin{bmatrix} -\frac{2}{\sqrt{5}} & \frac{1}{\sqrt{5}} \\ \frac{1}{\sqrt{5}} & \frac{2}{\sqrt{5}} \end{bmatrix} \begin{bmatrix} -6 & 0 \\ 0 & -1 \end{bmatrix} \begin{bmatrix} -\frac{2}{\sqrt{5}} & \frac{1}{\sqrt{5}} \\ \frac{1}{\sqrt{5}} & \frac{2}{\sqrt{5}} \end{bmatrix}^{-1} \quad (64)$$

so then

$$\frac{d}{dt} \vec{\tilde{x}}(t) = \Lambda\vec{\tilde{x}}(t) + V^{-1}\vec{b} \quad (65)$$

$$= \begin{bmatrix} -6 & 0 \\ 0 & -1 \end{bmatrix} \vec{\tilde{x}}(t) + \begin{bmatrix} -\frac{2}{\sqrt{5}} & \frac{1}{\sqrt{5}} \\ \frac{1}{\sqrt{5}} & \frac{2}{\sqrt{5}} \end{bmatrix}^{-1} \begin{bmatrix} 3 \\ 0 \end{bmatrix} \quad (66)$$

$$= \begin{bmatrix} -6 & 0 \\ 0 & -1 \end{bmatrix} \vec{\tilde{x}}(t) + \begin{bmatrix} -\frac{6}{\sqrt{5}} \\ \frac{3}{\sqrt{5}} \end{bmatrix}. \quad (67)$$

We can thus write the following uncoupled system of differential equations:

$$\frac{d}{dt} \tilde{x}_1(t) = -6\tilde{x}_1(t) - \frac{6}{\sqrt{5}} \quad (68)$$

$$\frac{d}{dt} \tilde{x}_2(t) = -\tilde{x}_2(t) + \frac{3}{\sqrt{5}} \quad (69)$$

We already know how to solve these non-homogeneous differential equations! (see near the end of [Note 1](#)):

$$\tilde{x}_1(t) = e^{-6t} \left(\tilde{x}_1(0) + \frac{1}{\sqrt{5}} \right) - \frac{1}{\sqrt{5}} \quad (70)$$

$$\tilde{x}_2(t) = e^{-t} \left(\tilde{x}_2(0) - \frac{3}{\sqrt{5}} \right) + \frac{3}{\sqrt{5}} \quad (71)$$

To get a full function for $\vec{\tilde{x}}(t)$, we need to find $\vec{\tilde{x}}(0)$. By the definition of $\vec{\tilde{x}}(t)$,

$$\vec{\tilde{x}}(t) = V^{-1}\vec{x}(t) \implies \vec{\tilde{x}}(0) = V^{-1}\vec{x}(0) = \begin{bmatrix} -\frac{2}{\sqrt{5}} & \frac{1}{\sqrt{5}} \\ \frac{1}{\sqrt{5}} & \frac{2}{\sqrt{5}} \end{bmatrix}^{-1} \begin{bmatrix} 0 \\ 0 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \quad (72)$$

Thus plugging these values in for $\tilde{\vec{x}}(t)$,

$$\tilde{x}_1(t) = \frac{1}{\sqrt{5}}e^{-6t} - \frac{1}{\sqrt{5}} \quad (73)$$

$$\tilde{x}_2(t) = -\frac{3}{\sqrt{5}}e^{-t} + \frac{3}{\sqrt{5}} \quad (74)$$

or, putting it together,

$$\tilde{\vec{x}}(t) = \begin{bmatrix} \frac{1}{\sqrt{5}}e^{-6t} - \frac{1}{\sqrt{5}} \\ -\frac{3}{\sqrt{5}}e^{-t} + \frac{3}{\sqrt{5}} \end{bmatrix} \quad (75)$$

Remember, we require $\vec{x}(t)$, not $\tilde{\vec{x}}(t)$. By definition of $\tilde{\vec{x}}(t)$,

$$\vec{x}(t) = V\tilde{\vec{x}}(t) \quad (76)$$

$$= \begin{bmatrix} -\frac{2}{\sqrt{5}} & \frac{1}{\sqrt{5}} \\ \frac{1}{\sqrt{5}} & \frac{2}{\sqrt{5}} \end{bmatrix} \begin{bmatrix} \frac{1}{\sqrt{5}}e^{-6t} - \frac{1}{\sqrt{5}} \\ -\frac{3}{\sqrt{5}}e^{-t} + \frac{3}{\sqrt{5}} \end{bmatrix} \quad (77)$$

$$= \begin{bmatrix} -\frac{2}{5}e^{-6t} - \frac{3}{5}e^{-t} + 1 \\ \frac{1}{5}e^{-6t} - \frac{6}{5}e^{-t} + 1 \end{bmatrix}. \quad (78)$$

Finally, we have $V_1(t) = 1 - \frac{2}{5}e^{-6t} - \frac{3}{5}e^{-t}$ and $V_2(t) = 1 + \frac{1}{5}e^{-6t} - \frac{6}{5}e^{-t}$.

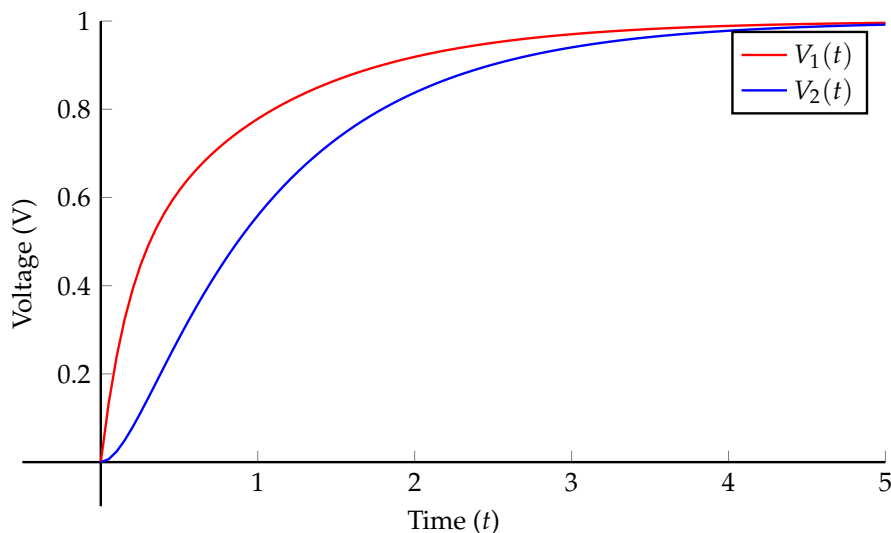


Figure 6: Initial Conditions: $V_1(0) = 0V$ and $V_2(0) = 0V$. Nonhomogeneous Case Solution.

Appendix A Algebra for Formation of Matrix Equation

We start with the following equations:

$$V_2 = V_1 - I_2 R_2 \quad (79)$$

$$I_2 = C_2 \frac{d}{dt} V_2 \quad (80)$$

$$0 - V_1 = I_1 R_1 \Rightarrow I_1 = -\frac{V_1}{R_1} \quad (81)$$

$$I_1 = I_2 + C_1 \frac{d}{dt} V_1 \quad (82)$$

Our goal is to rearrange this into a form where the derivatives of our state quantities (voltages in this case) are expressed in terms of the voltages themselves and the input voltage.⁸ Keeping this in mind, we start with the simpler equation to derive:

$$\text{eq. (80)} \implies \frac{d}{dt} V_2(t) = \frac{I_2}{C_2} \quad (83)$$

$$\text{eq. (79)} \implies I_2 = \frac{V_1(t)}{R_2} - \frac{V_2(t)}{R_2} \quad (84)$$

$$\frac{d}{dt} V_2(t) = \frac{V_1(t)}{R_2 C_2} - \frac{V_2(t)}{R_2 C_2} \quad (85)$$

And now:

$$\text{eq. (82)} \implies \frac{d}{dt} V_1(t) = \frac{I_1}{C_1} - \frac{I_2}{C_1} \quad (86)$$

$$\text{eq. (84), eq. (81)} \implies \frac{d}{dt} V_1(t) = \frac{I_1}{C_1} - \frac{\frac{V_1(t)}{R_2} - \frac{V_2(t)}{R_2}}{C_1} \quad (87)$$

$$= \frac{-\frac{V_1(t)}{R_1}}{C_1} - \frac{\frac{V_1(t)}{R_2} - \frac{V_2(t)}{R_2}}{C_1} \quad (88)$$

Simplifying:

$$\frac{d}{dt} V_1(t) = -\frac{V_1(t)}{R_1 C_1} - \frac{V_1(t)}{R_2 C_1} + \frac{V_2(t)}{R_2 C_1} \quad (89)$$

$$\frac{d}{dt} V_1(t) = \left(-\frac{1}{R_1 C_1} - \frac{1}{R_2 C_1} \right) V_1(t) + \frac{1}{R_2 C_1} V_2(t) \quad (90)$$

And these two equations (eq. (90), eq. (85)) exactly match eq. (5), eq. (6).

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⁸There are multiple ways to arrive at eq. (5) and eq. (6); this is just one valid sequence of steps.