

# Note j: Complex Numbers

## 1 Complex Numbers Introduction

### Definition 1 (Complex Numbers)

Consider an arbitrary complex number  $a \in \mathbb{C}$ . We can write this complex number as  $a = x + jy$  where  $j = \sqrt{-1}$  and  $x, y \in \mathbb{R}$ .

### Definition 2 (Complex Number Operations)

Consider two complex numbers  $a, b \in \mathbb{C}$ . Let  $a = x + jy$  and  $b = u + jv$  where  $x, y, u, v \in \mathbb{R}$ . Addition is defined as follows:

$$a + b = (x + jy) + (u + jv) = (x + u) + j(y + v) \quad (1)$$

and multiplication is defined as follows:

$$a \cdot b = (x + jy) \cdot (u + jv) = xu - yv + j(xv + uy) \quad (2)$$

*Note: this uses the "FOIL" technique for multiplication of real quantities.*

### Definition 3 (Complex Conjugate and Magnitudes)

Consider an arbitrary complex number  $a \in \mathbb{C}$  where we can equivalently write  $a = x + jy$  for  $x, y \in \mathbb{R}$ . The complex conjugate of  $a$  is

$$\bar{a} = x - jy \quad (3)$$

The magnitude of  $a$  is

$$|a| = \sqrt{a\bar{a}} \quad (4)$$

## 2 Polar Form

We will investigate another method to write complex numbers.

### Theorem 4 (Euler's Identity)

Consider an arbitrary complex number  $a \in \mathbb{C}$  which we can write as  $a = x + jy$ . We can equivalently write this as  $a = |a|e^{j\theta}$  where  $x = |a| \cos(\theta)$  and  $y = |a| \sin(\theta)$  (equivalently,  $\theta = \text{atan2}(y, x)$ <sup>a</sup>).

<sup>a</sup>Here,  $\text{atan2}(y, x)$  is a function that returns the angle from the positive x-axis to the vector from the origin to the point  $(x, y)$ . See <https://en.wikipedia.org/wiki/Atan2>.

*Proof.* Let us write  $a = |a|e^{j\theta}$ . We can show that  $x = |a| \cos(\theta)$  and  $y = |a| \sin(\theta)$ , using the Taylor expansion of  $f(x) = e^x$ :

$$a = |a|e^{j\theta} \quad (5)$$

$$= |a| \left( 1 + j\theta + \frac{(j\theta)^2}{2!} + \frac{(j\theta)^3}{3!} + \cdots + \frac{(j\theta)^{2n}}{2n!} + \frac{(j\theta)^{2n+1}}{(2n+1)!} + \cdots \right) \quad (6)$$

$$= |a| \left( 1 + j\theta - \frac{\theta^2}{2!} - j\frac{\theta^3}{3!} + \cdots + (-1)^n \frac{\theta^{2n}}{2n!} + j(-1)^n \frac{\theta^{2n+1}}{(2n+1)!} + \cdots \right) \quad (7)$$

$$= |a| \left[ \left( 1 - \frac{\theta^2}{2!} + \frac{\theta^4}{4!} - \cdots + (-1)^n \frac{\theta^{2n}}{2n!} + \cdots \right) + j \left( \theta - \frac{\theta^3}{3!} + \cdots + (-1)^n \frac{\theta^{2n+1}}{(2n+1)!} + \cdots \right) \right] \quad (8)$$

$$= |a| (\cos(\theta) + j \sin(\theta)) \quad (9)$$

$$= \underbrace{|a| \cos(\theta)}_x + j \underbrace{|a| \sin(\theta)}_y \quad (10)$$

To show that  $\theta = \text{atan2}(y, x)$ , consider that

$$\frac{y}{x} = \frac{|a| \sin(\theta)}{|a| \cos(\theta)} \quad (11)$$

$$\implies \theta = \arctan \frac{y}{x} \quad (12)$$

Instead of using regular arctan, we will use atan2, two argument arctan, which protects against sign errors (i.e., to differentiate the cases when  $x$  and  $y$  are both positive or both negative) and division by zero (i.e., when  $x = 0$ ). Hence, we write

$$\theta = \text{atan2}(y, x) \quad (13)$$

□

The plot in Figure 1 visually describes the conversion from rectangular (i.e.,  $x + jy$ ) form to polar form (i.e.,  $|a|e^{j\theta}$ )

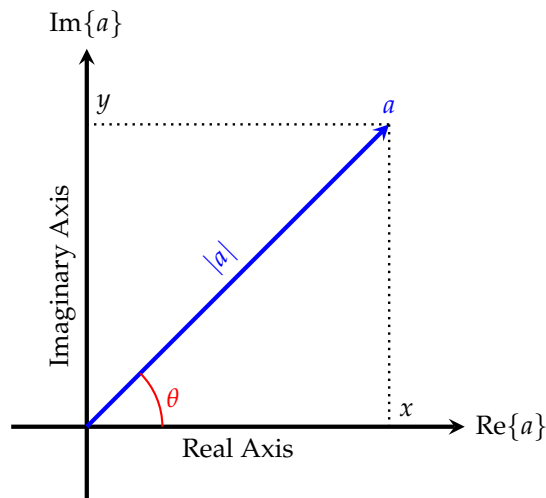


Figure 1: Complex number  $a \in \mathbb{C}$  depicted as a vector in the complex plane.

#### Corollary 5 (Complex Exponential Representations of Sine and Cosine)

Using Theorem 4, we have that

$$\cos(\theta) = \frac{e^{j\theta} + e^{-j\theta}}{2} \quad (14)$$

$$\sin(\theta) = \frac{e^{j\theta} - e^{-j\theta}}{2j} \quad (15)$$

*Proof.* Using Theorem 4 and the even/odd nature of cosine/sine respectively, we have the following direct results:

$$e^{j\theta} = \cos(\theta) + j \sin(\theta) \quad (16)$$

$$e^{-j\theta} = \cos(\theta) - j \sin(\theta) \quad (17)$$

From this, we have that

$$2 \cos(\theta) = e^{j\theta} - j \sin(\theta) + e^{-j\theta} + j \sin(\theta) \quad (18)$$

$$\cos(\theta) = \frac{e^{j\theta} + e^{-j\theta}}{2} \quad (19)$$

and

$$2j \sin(\theta) = e^{j\theta} - \cos(\theta) - e^{-j\theta} + \cos(\theta) \quad (20)$$

$$2j \sin(\theta) = e^{j\theta} - e^{-j\theta} \quad (21)$$

$$\sin(\theta) = \frac{e^{j\theta} - e^{-j\theta}}{2j} \quad (22)$$

□

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