

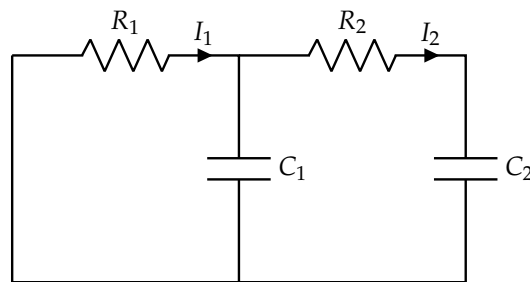
# Note 8: Vector Differential Equations

## 1 Introduction to Vector Differential Equations

### 1.1 Motivation

Last time we saw how to solve simple scalar first order linear differential equations.

Now, we are going to expand our tool set and learn how to tackle multivariable differential equations. (You will see in homeworks that these exact same ideas will also equip us to deal with higher-order differential equations where we have second and third derivatives involved.) Let's motivate the need to understand two dimensional systems with the circuit in fig. 1.



**Figure 1:** Circuit forming a two-dimensional system.

Suppose we want to model the voltage across both  $C_1$  and  $C_2$  simultaneously. As you may notice (through KCL and NVA), it is not possible to model the two capacitors independently. Let  $V_1(t)$  denote the voltage across capacitor  $C_1$  and  $V_2(t)$  denote the voltage across capacitor  $C_2$ . Applying NVA and KCL, we obtain the following two differential equations:

$$\frac{dV_1}{dt} = -\left(\frac{1}{R_1C_1} + \frac{1}{R_2C_1}\right)V_1 + \frac{1}{R_2C_1}V_2 \quad (1)$$

$$\frac{dV_2}{dt} = \frac{1}{R_2C_2}V_1 - \frac{1}{R_2C_2}V_2 \quad (2)$$

**Concept Check:** Derive these differential equations using KCL and NVA. Ensure to simplify your equations to only include  $V_1$ ,  $V_2$ ,  $R_1$ , and  $R_2$ .

The differential equation for  $V_1$  includes both  $V_1$  and  $V_2$  terms, and likewise for the differential equation for  $V_2$ <sup>1</sup>. We cannot use techniques we have previously covered to solve these kinds of differential equations, so we will have to derive new techniques.

### 1.2 Matrix Form for Differential Equations

We will refer to a collection of differential equations, similar to eq. (1) and eq. (2), as a *system* of differential equations. The terminology is analogous to systems of linear equations, which you have covered in EECS

<sup>1</sup>We typically refer to this as *coupled* differential equations

16A.

**Definition 1** (Vector Differential Equations)

Suppose we are given a system of linear differential equations of the form

$$\frac{dx_1}{dt} = a_{11}x_1(t) + a_{12}x_2(t) + \dots + a_{1n}x_n(t) \quad (3)$$

$$\frac{dx_2}{dt} = a_{21}x_1(t) + a_{22}x_2(t) + \dots + a_{2n}x_n(t) \quad (4)$$

$$\vdots \quad (5)$$

$$\frac{dx_n}{dt} = a_{n1}x_1(t) + a_{n2}x_2(t) + \dots + a_{nn}x_n(t) \quad (6)$$

$$(7)$$

with initial conditions  $x_1(t_0) = k_1, x_2(t_0) = k_2, \dots, x_n(t_0) = k_n$  for some constants  $a_{ij} \in \mathbb{R}$  and  $k_i \in \mathbb{R}$ .We may combine them into a single *vector differential equation* of the form

$$\frac{d}{dt}\vec{x}(t) = A\vec{x}(t) \quad (8)$$

with initial condition  $\vec{x}(t_0) = \vec{k}$  where

$$\vec{x}(t) = \begin{bmatrix} x_1(t) \\ x_2(t) \\ \vdots \\ x_n(t) \end{bmatrix} \quad (9)$$

$$A = \begin{bmatrix} a_{11} & a_{12} & \dots & a_{1n} \\ a_{21} & a_{22} & \dots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{n1} & a_{n2} & \dots & a_{nn} \end{bmatrix} \quad (10)$$

$$\vec{k} = \begin{bmatrix} k_1 \\ k_2 \\ \vdots \\ k_n \end{bmatrix} \quad (11)$$

We refer to  $\vec{x}(t)$  as our “state”.

Note that

$$\frac{d}{dt}\vec{x}(t) = \frac{d}{dt} \begin{bmatrix} x_1(t) \\ x_2(t) \\ \vdots \\ x_n(t) \end{bmatrix} = \begin{bmatrix} \frac{dx_1(t)}{dt} \\ \frac{dx_2(t)}{dt} \\ \vdots \\ \frac{dx_n(t)}{dt} \end{bmatrix} \quad (12)$$

*Example:*We may turn the system of differential equations in eq. (1) and eq. (2) into a vector differential equation by pattern matching  $x_1(t) := V_1(t)$ ,  $x_2 := V_2(t)$ ,  $a_{11} := -\left(\frac{1}{R_1C_1} + \frac{1}{R_2C_1}\right)$ ,  $a_{12} := \frac{1}{R_2C_1}$ ,  $a_{21} := \frac{1}{R_2C_2}$ , and

$a_{22} := -\frac{1}{R_2 C_2}$ . Applying the symbolic simplifications in Definition 1, we have

$$\underbrace{\frac{d}{dt} \begin{bmatrix} V_1 \\ V_2 \end{bmatrix}}_{\vec{x}(t)} = \underbrace{\begin{bmatrix} -\left(\frac{1}{R_1 C_1} + \frac{1}{R_2 C_1}\right) & \frac{1}{R_2 C_1} \\ \frac{1}{R_2 C_2} & -\frac{1}{R_2 C_2} \end{bmatrix}}_A \underbrace{\begin{bmatrix} V_1 \\ V_2 \end{bmatrix}}_{\vec{x}(t)} \quad (13)$$

In the next section, we will go over how to solve these kinds of vector differential equations, for a special case where  $A$  is *diagonalizable*.

## 2 Solving Diagonalizable Vector Differential Equations

### 2.1 Motivation

Suppose that, from the previous section, we had

$$A := \begin{bmatrix} a_1 & 0 & 0 & \cdots & 0 \\ 0 & a_2 & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & 0 & a_n \end{bmatrix} \quad (14)$$

for some constants  $a_i \in \mathbb{R}$ . Indeed, this would make our differential equation easier to solve since we would have

$$\frac{d}{dt} \vec{x}(t) = A \vec{x}(t) \quad (15)$$

$$\implies \begin{bmatrix} \frac{dx_1}{dt} \\ \frac{dx_2}{dt} \\ \vdots \\ \frac{dx_n}{dt} \end{bmatrix} = \begin{bmatrix} a_1 x_1(t) \\ a_2 x_2(t) \\ \vdots \\ a_n x_n(t) \end{bmatrix} \quad (16)$$

which we could solve row-by-row. That is, we can solve  $\frac{dx_1}{dt} = a_1 x_1(t)$  first, then  $\frac{dx_2}{dt} = a_2 x_2(t)$ , etc. We will use the help of diagonalization to obtain this end goal.

### 2.2 Diagonalizing Matrices

#### Definition 2 (Diagonalizability)

A square matrix  $A \in \mathbb{R}^{n \times n}$  is diagonalizable (i.e., it can be diagonalized) if it admits  $n$  linearly independent eigenvectors<sup>a</sup>. If  $A$  is diagonalizable, it can be diagonalized as follows:

$$A = V \Lambda V^{-1} \quad (17)$$

where

$$V := \begin{bmatrix} \vec{v}_1 & \vec{v}_2 & \cdots & \vec{v}_n \end{bmatrix} \quad (18)$$

$$\Lambda := \begin{bmatrix} \lambda_1 & 0 & 0 & \cdots & 0 \\ 0 & \lambda_2 & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & 0 & \lambda_n \end{bmatrix} \quad (19)$$

<sup>a</sup>Note that it is not always the case for a matrix to have linearly independent eigenvectors. Consider  $A = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$ , for example.

### Theorem 3 (Diagonalization)

We will use the definition of  $V$  and  $\Lambda$  from Definition 2. If  $\vec{v}_1, \dots, \vec{v}_n$  are the eigenvectors of  $A$ , then  $\lambda_1, \dots, \lambda_n$  are the corresponding eigenvalues.

*Proof.* We can show that, if the columns of  $V$  are the eigenvectors of  $A$ , then  $\Lambda$  will be a diagonal matrix with the corresponding eigenvalues along the diagonal. We know that we can write  $A = V\Lambda V^{-1}$  from Definition 2. This is equivalent to writing  $\Lambda = V^{-1}AV$ . Let  $\ell_i$  be the eigenvalue of  $A$  corresponding to  $\vec{v}_i$ . Now,

$$\Lambda = V^{-1}AV \quad (20)$$

$$= V^{-1}A \begin{bmatrix} \vec{v}_1 & \vec{v}_2 & \cdots & \vec{v}_n \end{bmatrix} \quad (21)$$

$$= V^{-1} \begin{bmatrix} A\vec{v}_1 & A\vec{v}_2 & \cdots & A\vec{v}_n \end{bmatrix} \quad (22)$$

$$= V^{-1} \begin{bmatrix} \ell_1\vec{v}_1 & \ell_2\vec{v}_2 & \cdots & \ell_n\vec{v}_n \end{bmatrix} \quad (23)$$

$$= V^{-1} \begin{bmatrix} \vec{v}_1 & \vec{v}_2 & \cdots & \vec{v}_n \end{bmatrix} \begin{bmatrix} \ell_1 & 0 & 0 & \cdots & 0 \\ 0 & \ell_2 & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & 0 & \ell_n \end{bmatrix} \quad (24)$$

$$= V^{-1}V \begin{bmatrix} \ell_1 & 0 & 0 & \cdots & 0 \\ 0 & \ell_2 & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & 0 & \ell_n \end{bmatrix} \quad (25)$$

$$= \begin{bmatrix} \ell_1 & 0 & 0 & \cdots & 0 \\ 0 & \ell_2 & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & 0 & \ell_n \end{bmatrix} \quad (26)$$

and hence,  $\lambda_i = \ell_i$ . Note that in steps eq. (22) and eq. (24), we use facts of the matrix multiplication algorithm to distribute the  $A$  inside  $V$  and to split up the algebraic multiplication into matrix multiplication, respectively.  $\square$

*Note:* This result also holds if  $\lambda_i$  and  $\vec{v}_i$  are complex, i.e.,  $\lambda_i \in \mathbb{C}$  and  $\vec{v}_i \in \mathbb{C}^n$ .

*Example:*

Consider again the circuit in Figure 1. Let  $C_1 = C_2 = 1 \mu\text{F}$ ,  $R_1 = \frac{1}{3}\text{M}\Omega$  and  $R_2 = \frac{1}{2}\text{M}\Omega$ .<sup>2</sup> This means that  $A = \begin{bmatrix} -5 & 2 \\ 2 & -2 \end{bmatrix}$ . We can find its eigenvalues by solving for  $\lambda$  in  $\det\{A - \lambda I_2\} = 0$ . From this, we obtain  $\lambda_1 = -6$  and  $\lambda_2 = -1$ . To find  $\vec{v}_1$ , the corresponding eigenvector for  $\lambda_1$ , we can find a basis for  $\text{Null}(A - \lambda_1 I_2)$ . Doing this yields  $\vec{v}_1 = \begin{bmatrix} -\frac{2}{\sqrt{5}} \\ \frac{1}{\sqrt{5}} \end{bmatrix}$ . Applying a similar process for  $\lambda_2$ , we obtain  $\vec{v}_2 = \begin{bmatrix} \frac{1}{\sqrt{5}} \\ \frac{2}{\sqrt{5}} \end{bmatrix}$ . These eigenvectors are linearly independent, so we can diagonalize this matrix. Hence,

$$V = \begin{bmatrix} \vec{v}_1 & \vec{v}_2 \end{bmatrix} = \begin{bmatrix} -\frac{2}{\sqrt{5}} & \frac{1}{\sqrt{5}} \\ \frac{1}{\sqrt{5}} & \frac{2}{\sqrt{5}} \end{bmatrix} \quad (27)$$

$$\Lambda = \begin{bmatrix} -6 & 0 \\ 0 & -1 \end{bmatrix} \quad (28)$$

Notice that, with this definition of  $\Lambda$ , we can try to use it to obtain the goal mentioned in Section 2.1. For this idea to work, we need to introduce a *change of basis*.

## 2.3 Change of Basis

### Key Idea 4 (Bases and Basis Coefficients of Vectors)

Any vector in  $\mathbb{R}^n$  is fundamentally written in a certain basis that spans  $\mathbb{R}^n$ . Typically, we implicitly

use the *standard basis*. If we have  $\vec{x} = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix}$ , this is equivalent to writing

$$\vec{x} = x_1 \vec{e}_1 + x_2 \vec{e}_2 + \dots + x_n \vec{e}_n \quad (29)$$

where  $\vec{e}_i$  is a vector of all zeros, except for a 1 in the  $i$ th entry. We also call  $\vec{e}_i$  the  $i$ th standard basis vector.

Let  $\{\vec{v}_1, \dots, \vec{v}_n\}$  represent a basis for  $\mathbb{R}^n$ , with each  $\vec{v}_i \in \mathbb{R}^n$ . This means that we can represent any vector (written in standard basis)  $\vec{x} \in \mathbb{R}^n$  as

$$\vec{x} = \tilde{x}_1 \vec{v}_1 + \tilde{x}_2 \vec{v}_2 + \dots + \tilde{x}_n \vec{v}_n \quad (30)$$

for some constants  $\tilde{x}_i \in \mathbb{R}$ . These constants are known as the *coefficients* of  $\vec{x}$  in the basis given by

$\{\vec{v}_1, \dots, \vec{v}_n\}$ . That is,  $\tilde{\vec{x}} := \begin{bmatrix} \tilde{x}_1 \\ \tilde{x}_2 \\ \vdots \\ \tilde{x}_n \end{bmatrix}$  is the representation of  $\vec{x}$  in this basis.

<sup>2</sup>The SI prefixes 'M' and 'μ' stand for "mega" and "micro," and correspond to the decimal multiples of  $10^6$  and  $10^{-6}$  respectively.

**Theorem 5** (Change of Basis)

Let  $\{\vec{v}_1, \dots, \vec{v}_n\}$  represent a basis for  $\mathbb{R}^n$ , with each  $\vec{v}_i \in \mathbb{R}^n$ . Define this as  $V$ -basis. Suppose we are given  $\vec{x}$  in standard basis, and further suppose we wish to find  $\vec{\tilde{x}}$ , the representation of  $\vec{x}$  in  $V$ -basis. We can accomplish this by computing

$$\vec{\tilde{x}} = V^{-1}\vec{x} \quad (31)$$

where  $V := \begin{bmatrix} \vec{v}_1 & \vec{v}_2 & \dots & \vec{v}_n \end{bmatrix}$ .

*Proof.* It first needs to be shown that  $V$  is invertible. We know that  $\vec{v}_1, \dots, \vec{v}_n$  must be a linearly independent collection of vectors, since they span  $\mathbb{R}^n$ . Hence,  $V := \begin{bmatrix} \vec{v}_1 & \vec{v}_2 & \dots & \vec{v}_n \end{bmatrix} \in \mathbb{R}^{n \times n}$  is a square matrix with linearly independent columns, meaning it is invertible. Now, we can equivalently show that  $\vec{x} = V\vec{\tilde{x}}$ :

$$\vec{x} = V\vec{\tilde{x}} \quad (32)$$

$$= \underbrace{\begin{bmatrix} \vec{v}_1 & \vec{v}_2 & \dots & \vec{v}_n \end{bmatrix}}_V \underbrace{\begin{bmatrix} \tilde{x}_1 \\ \tilde{x}_2 \\ \vdots \\ \tilde{x}_n \end{bmatrix}}_{\vec{\tilde{x}}} \quad (33)$$

$$= \tilde{x}_1\vec{v}_1 + \tilde{x}_2\vec{v}_2 + \dots + \tilde{x}_n\vec{v}_n \quad (34)$$

which is exactly what we have in eq. (30).  $\square$

*Example:*

Let  $V$  be given as in eq. (27). Consider the basis of  $\mathbb{R}^2$  that is spanned by the columns of  $V$  (recall that  $V$  has linearly independent columns). Let  $\vec{x} = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$ , and suppose we want to find the coefficients of  $\vec{x}$  in  $V$ -basis,

namely  $\vec{\tilde{x}}$ . Using the matrix inversion formula for a  $2 \times 2$  matrix, we find that  $V^{-1} = \begin{bmatrix} -\frac{2}{\sqrt{5}} & \frac{1}{\sqrt{5}} \\ \frac{1}{\sqrt{5}} & \frac{2}{\sqrt{5}} \end{bmatrix}$ . Hence,

$$\vec{\tilde{x}} = V^{-1} \begin{bmatrix} 1 \\ 1 \end{bmatrix} = \begin{bmatrix} -\frac{1}{\sqrt{5}} \\ \frac{3}{\sqrt{5}} \end{bmatrix} \quad (35)$$

Now we can use these results to derive a process to solve vector differential equations, where  $A$  is diagonalizable.

## 2.4 Solving Vector Differential Equations

**Theorem 6** (Diagonalizing a Vector Differential Equation)

Suppose we are given a vector differential equation of the form

$$\frac{d}{dt}\vec{x}(t) = A\vec{x}(t) \quad (36)$$

with initial condition  $\vec{x}(t_0) = \vec{k}$ . Further suppose  $A \in \mathbb{R}^{n \times n}$  is a diagonalizable matrix which can be

written as  $A = V\Lambda V^{-1}$ . The solution to eq. (36) is given by  $\vec{x}(t) = V\vec{\tilde{x}}(t)$ , where  $\vec{\tilde{x}}(t)$  is the solution to

$$\frac{d}{dt}\vec{\tilde{x}}(t) = \Lambda\vec{\tilde{x}}(t) \quad (37)$$

with initial condition  $\vec{\tilde{x}}(t_0) = V^{-1}\vec{k}$

*Proof.* First, we can rewrite eq. (36) as follows:

$$\frac{d}{dt}\vec{x}(t) = V\Lambda V^{-1}\vec{x}(t) \quad (38)$$

We can also notice that, since the derivative operator is linear,

$$M\frac{d}{dt}\vec{x}(t) = \frac{d}{dt}(M\vec{x}(t)) \quad (39)$$

for any matrix  $M \in \mathbb{R}^{m \times n}$ . Applying this fact,

$$\frac{d}{dt}\vec{x}(t) = V\Lambda V^{-1}\vec{x}(t) \quad (40)$$

$$V^{-1}\frac{d}{dt}\vec{x}(t) = \Lambda V^{-1}\vec{x}(t) \quad (41)$$

$$\frac{d}{dt}(V^{-1}\vec{x}(t)) = \Lambda V^{-1}\vec{x}(t) \quad (42)$$

Notice that  $\vec{\tilde{x}}(t) := V^{-1}\vec{x}(t)$  gives the coefficients of  $\vec{x}(t)$  in  $V$ -basis, as covered in Section 2.3. Hence, we may further simplify as

$$\frac{d}{dt}(V^{-1}\vec{x}(t)) = \Lambda V^{-1}\vec{x}(t) \quad (43)$$

$$\frac{d}{dt}\vec{\tilde{x}}(t) = \Lambda\vec{\tilde{x}}(t) \quad (44)$$

This is exactly the vector differential equation in eq. (37). To find the initial condition, we know that  $\vec{x}(t_0) = \vec{k}$ . We find the coefficients of this vector in  $V$ -basis and obtain  $\vec{\tilde{x}}(t_0) = V^{-1}\vec{x}(t_0) = V^{-1}\vec{k}$  as desired. To recover  $\vec{x}(t)$  from  $\vec{\tilde{x}}(t)$ , we undo the change of basis, i.e.,  $\vec{x}(t) = V\vec{\tilde{x}}(t)$ .  $\square$

### Key Idea 7 (Diagonalizing to Solve an Easier Vector Differential Equation)

Notice that the differential equation given in eq. (37) is a diagonal system, in that it matches the desired form in Section 2.1. That is, we can write out the differential equation more explicitly as

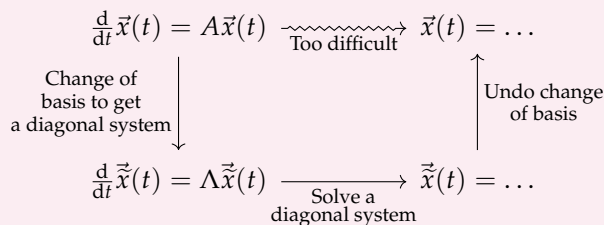
$$\frac{d}{dt}\tilde{x}_1(t) = \lambda_1\tilde{x}_1(t) \quad (45)$$

$$\frac{d}{dt}\tilde{x}_2(t) = \lambda_2\tilde{x}_2(t) \quad (46)$$

$$\vdots \quad (47)$$

$$\frac{d}{dt}\tilde{x}_n(t) = \lambda_n\tilde{x}_n(t) \quad (48)$$

with initial conditions  $\tilde{x}_1(t_0) = \tilde{k}_1, \tilde{x}_2(t_0) = \tilde{k}_2, \dots, \tilde{x}_n(t_0) = \tilde{k}_n$ , where  $\vec{\tilde{k}} := V^{-1}\vec{k}$  for  $\vec{k}$  as defined in Theorem 6. A graphical representation of this strategy to solve a vector differential equation is visualized in Figure 2.



**Figure 2:** A Strategy to Solve eq. (36)

*Example:*

Using this strategy, we can solve the circuit example in Figure 1, with initial condition  $\vec{x}(0) = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$ . Using the  $V$  and  $\Lambda$  from eq. (27) and eq. (28) respectively, we have the following differential equation for  $\vec{z}(t)$ :

$$\frac{d}{dt} \vec{z}(t) = \begin{bmatrix} -6 & 0 \\ 0 & -1 \end{bmatrix} \vec{z}(t) \quad (49)$$

with initial condition  $\vec{z}(0) = V^{-1}\vec{x}(0) = \begin{bmatrix} -\frac{1}{\sqrt{5}} \\ \frac{3}{\sqrt{5}} \end{bmatrix}$ , which is taken from eq. (35). We can explicitly write this out as

$$\frac{d}{dt} \tilde{x}_1(t) = -6\tilde{x}_1(t) \quad (50)$$

$$\frac{d}{dt} \tilde{x}_2(t) = -\tilde{x}_2(t) \quad (51)$$

with initial conditions  $\tilde{x}_1(0) = -\frac{1}{\sqrt{5}}$  and  $\tilde{x}_2(0) = \frac{3}{\sqrt{5}}$ . Notice that, unlike in eq. (1) and eq. (2), the system here is un-coupled, so we can solve it directly. Using techniques covered previously, we have

$$\tilde{x}_1(t) = -\frac{1}{\sqrt{5}}e^{-6t} \quad (52)$$

$$\tilde{x}_2(t) = \frac{3}{\sqrt{5}}e^{-t} \quad (53)$$

so  $\vec{z}(t) = \begin{bmatrix} -\frac{1}{\sqrt{5}}e^{-6t} \\ \frac{3}{\sqrt{5}}e^{-t} \end{bmatrix}$ . Now, to undo the change of basis and recover  $\vec{x}(t)$ , we compute

$$\vec{x}(t) = V\vec{z}(t) = \begin{bmatrix} -\frac{2}{\sqrt{5}} & \frac{1}{\sqrt{5}} \\ \frac{1}{\sqrt{5}} & \frac{2}{\sqrt{5}} \end{bmatrix} \begin{bmatrix} -\frac{1}{\sqrt{5}}e^{-6t} \\ \frac{3}{\sqrt{5}}e^{-t} \end{bmatrix} \quad (54)$$

$$= \begin{bmatrix} \frac{2}{5}e^{-6t} + \frac{3}{5}e^{-t} \\ -\frac{1}{5}e^{-6t} + \frac{6}{5}e^{-t} \end{bmatrix} \quad (55)$$

The voltage curves are plotted below in Figure 3.



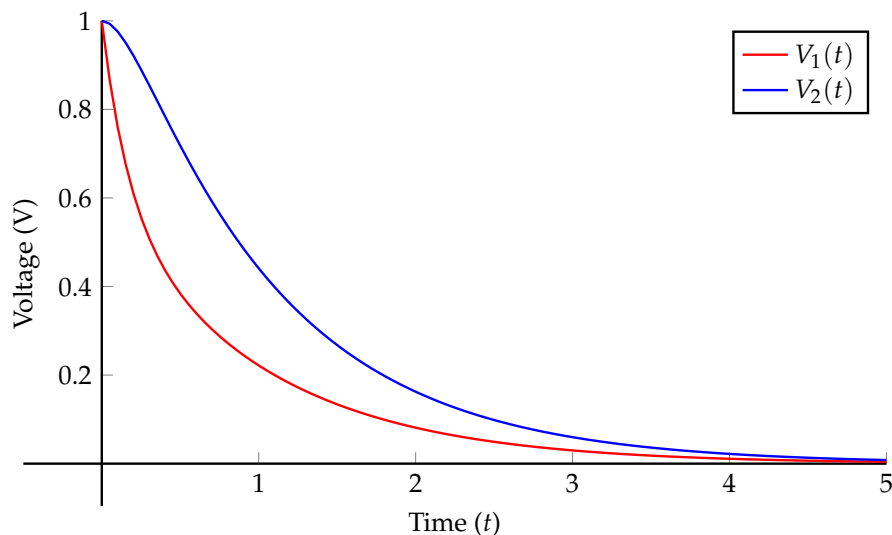


Figure 3: Initial Conditions:  $V_1(0) = 1$  V and  $V_2(0) = 1$  V.

### Corollary 8 (Inhomogeneous Vector Differential Equations)

Suppose we are given a vector differential equation of the form

$$\frac{d}{dt} \vec{x}(t) = A\vec{x}(t) + B\vec{u}(t) \quad (56)$$

with initial condition  $\vec{x}(t_0) = \vec{k}$  and with  $\vec{u}(t) : \mathbb{R}_+ \rightarrow \mathbb{R}^n$ . Further suppose  $A \in \mathbb{R}^{n \times n}$  is a diagonalizable matrix which can be written as  $A = V\Lambda V^{-1}$ , and  $B \in \mathbb{R}^n$ . The solution to eq. (56) is given by  $\vec{x}(t) = V\vec{\tilde{x}}(t)$ , where  $\vec{\tilde{x}}(t)$  is the solution to

$$\frac{d}{dt} \vec{\tilde{x}}(t) = \Lambda\vec{\tilde{x}}(t) + V^{-1}B\vec{u}(t) \quad (57)$$

with initial condition  $\vec{\tilde{x}}(t_0) = V^{-1}\vec{k}$

*Proof.* The proof of this is almost exactly the same as the proof of Theorem 6, and the additional algebraic portions have been left as an exercise to the reader. It should be noted that we can “relabel”  $B$  where  $\tilde{B} := V^{-1}B$ . In doing this, we can see that we still have a system of un-coupled equations, namely

$$\frac{d}{dt} \tilde{x}_1(t) = \lambda_1 \tilde{x}_1(t) + \tilde{b}_1 u(t) \quad (58)$$

$$\frac{d}{dt} \tilde{x}_2(t) = \lambda_2 \tilde{x}_2(t) + \tilde{b}_2 u(t) \quad (59)$$

$$\vdots \quad (60)$$

$$\frac{d}{dt} \tilde{x}_n(t) = \lambda_n \tilde{x}_n(t) + \tilde{b}_n u(t) \quad (61)$$

$$(62)$$

where  $\tilde{b}_i$  is the  $i$ th entry of  $\tilde{B}$ . We can solve each equation individually using either Theorem 4 or Theorem 6 of Note 2.  $\square$

### 3 LC Tank Example Revisited

This section is an extended example to demonstrate how to apply vector differential equations to the LC circuit from the previous note.

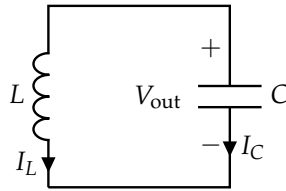


Figure 4: An LC Tank.

We can model  $V_{\text{out}}(t)$  using vector differential equations. Suppose that  $V_{\text{out}}(0) = 0$  and  $I_L(0) = 1$  A.

#### 3.1 Deriving the Differential Equations

We will use KCL and NVA to derive the system of differential equations that models this circuit. NVA gives us

$$V_L = V_C = V_{\text{out}} \quad (63)$$

KCL gives us

$$I_L = -I_C = -C \frac{dV_{\text{out}}}{dt} \quad (64)$$

$$\frac{dV_{\text{out}}}{dt} = -\frac{1}{C} I_L \quad (65)$$

and NVA again gives us

$$V_L = V_{\text{out}} = L \frac{dI_L}{dt} \quad (66)$$

$$L \frac{dI_L}{dt} = V_{\text{out}} \quad (67)$$

$$\frac{dI_L}{dt} = \frac{1}{L} V_{\text{out}} \quad (68)$$

Notice that we have derivatives of  $I_L(t)$  and  $V_L(t)$ , so we can make these state variables. Arranging this as a matrix differential equation, we have

$$\frac{d}{dt} \underbrace{\begin{bmatrix} V_{\text{out}} \\ I_L \end{bmatrix}}_{\vec{x}(t)} = \underbrace{\begin{bmatrix} 0 & -\frac{1}{C} \\ \frac{1}{L} & 0 \end{bmatrix}}_A \underbrace{\begin{bmatrix} V_{\text{out}} \\ I_L \end{bmatrix}}_{\vec{x}(t)} \quad (69)$$

#### 3.2 Solving the Matrix Differential Equation

It happens to be the case  $A$  is diagonalizable here. We can solve this matrix differential equation using the technique from the previous note: by first diagonalizing, performing a change of basis, solving a diagonal system, and then undoing the change of basis. We can find the eigenvalues by solving for  $\lambda$  in

$$\det\{A - \lambda I_2\} = 0 \quad (70)$$

which yields  $\lambda_1 = j\frac{1}{\sqrt{LC}}$  and  $\lambda_2 = -j\frac{1}{\sqrt{LC}}$ . We can find  $\vec{v}_1$ , the eigenvector for  $\lambda_1$ , by finding a basis for  $\text{Null}(A - \lambda_1 I)$ . Computing this gives  $\vec{v}_1 = \begin{bmatrix} j\sqrt{\frac{L}{C}} \\ 1 \end{bmatrix}$ . We perform a similar operation with  $\lambda_2$  and obtain  $\vec{v}_2 = \begin{bmatrix} -j\sqrt{\frac{L}{C}} \\ 1 \end{bmatrix}$ . Hence, we have

$$\Lambda = \begin{bmatrix} j\frac{1}{\sqrt{LC}} & 0 \\ 0 & -j\frac{1}{\sqrt{LC}} \end{bmatrix} \quad (71)$$

$$V = \begin{bmatrix} j\sqrt{\frac{L}{C}} & -j\sqrt{\frac{L}{C}} \\ 1 & 1 \end{bmatrix} \quad (72)$$

$$\implies V^{-1} = \frac{1}{2j\sqrt{\frac{L}{C}}} \begin{bmatrix} 1 & j\sqrt{\frac{L}{C}} \\ -1 & j\sqrt{\frac{L}{C}} \end{bmatrix} \quad (73)$$

The new differential equation for  $\vec{\tilde{x}}(t)$  is

$$\frac{d}{dt} \vec{\tilde{x}}(t) = \begin{bmatrix} j\frac{1}{\sqrt{LC}} & 0 \\ 0 & -j\frac{1}{\sqrt{LC}} \end{bmatrix} \vec{\tilde{x}}(t) \quad (74)$$

with initial condition  $\vec{\tilde{x}}(0) = V^{-1} \begin{bmatrix} 0 \\ 1 \end{bmatrix} = \frac{1}{2} \begin{bmatrix} 1 \\ 1 \end{bmatrix}$ . Solving this diagonal system, we see that

$$\vec{\tilde{x}}(t) = \begin{bmatrix} \frac{1}{2} e^{j\frac{t}{\sqrt{LC}}} \\ \frac{1}{2} e^{-j\frac{t}{\sqrt{LC}}} \end{bmatrix} \quad (75)$$

Undoing the change of variables to find  $\vec{x}(t)$ , we obtain

$$\vec{x}(t) = V \vec{\tilde{x}}(t) \quad (76)$$

$$= \begin{bmatrix} j\sqrt{\frac{L}{C}} & -j\sqrt{\frac{L}{C}} \\ 1 & 1 \end{bmatrix} \begin{bmatrix} \frac{1}{2} e^{j\frac{t}{\sqrt{LC}}} \\ \frac{1}{2} e^{-j\frac{t}{\sqrt{LC}}} \end{bmatrix} \quad (77)$$

$$= \begin{bmatrix} \sqrt{\frac{L}{C}} \left( \frac{j}{2} e^{j\frac{t}{\sqrt{LC}}} - \frac{j}{2} e^{-j\frac{t}{\sqrt{LC}}} \right) \\ \frac{1}{2} e^{j\frac{t}{\sqrt{LC}}} + \frac{1}{2} e^{-j\frac{t}{\sqrt{LC}}} \end{bmatrix} \quad (78)$$

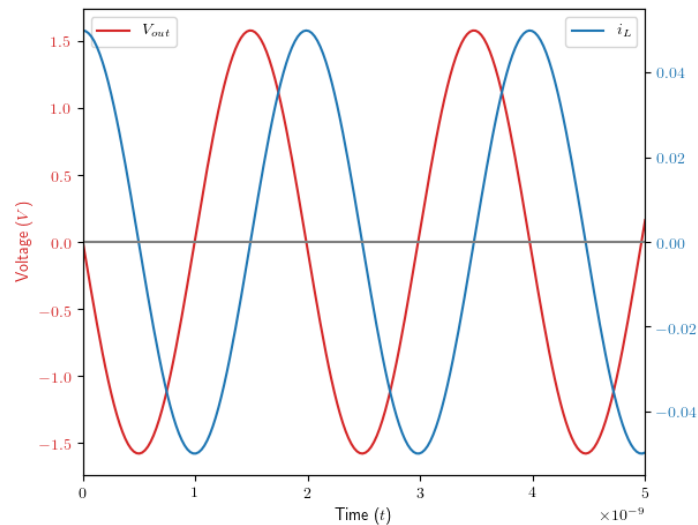
Using Euler's formula ( $e^{j\theta} = \cos(\theta) + j\sin(\theta)$ ), we can simplify the above to obtain

$$\vec{x}(t) = \begin{bmatrix} -\sqrt{\frac{L}{C}} \sin\left(\frac{t}{\sqrt{LC}}\right) \\ \cos\left(\frac{t}{\sqrt{LC}}\right) \end{bmatrix} \quad (79)$$

so we have  $V_{\text{out}}(t) = -\sqrt{\frac{L}{C}} \sin\left(\frac{t}{\sqrt{LC}}\right)$  and  $I_L(t) = \cos\left(\frac{t}{\sqrt{LC}}\right)$ .

### 3.3 Visualizing $V_{\text{out}}(t)$ , $I_L(t)$ , and Energy

A plot of  $I_L(t)$  and  $V_{\text{out}}(t)$  will resemble the graph in Figure 5.



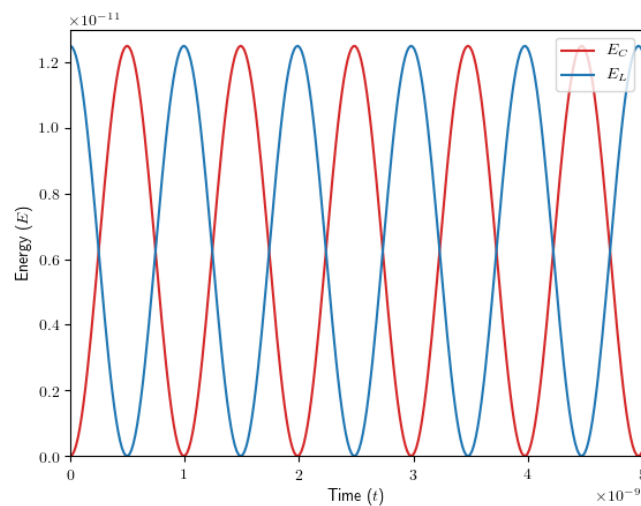
**Figure 5:** Voltage and Current response of LC Tank

We can find the energy in the capacitor and inductor respectively:

$$E_C = \frac{1}{2} C V_{\text{out}}^2 = \frac{L}{2} \sin^2\left(\frac{t}{\sqrt{LC}}\right) \quad (80)$$

$$E_L = \frac{1}{2} L I_L^2 = \frac{L}{2} \cos^2\left(\frac{t}{\sqrt{LC}}\right) \quad (81)$$

Notice that  $E_C + E_L = \frac{L}{2}$ , so the energy is constant over time. This is expected, since physics tells us that energy in this closed system should be conserved. A plot of  $E_C$  and  $E_L$  will resemble Figure 6.



**Figure 6:** Energy stored in Inductor and Capacitor. Notice the sum is constant.

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