Note 5: Second-Order Differential Equations with RLC Circuits

1 Second Order Differential Equations

Definition 1 (Second Order, Linear Differential Equation)

A second order, linear differential equation can be put into the form

$$\frac{\mathrm{d}^2 x(t)}{\mathrm{d}t^2} + 2\alpha \frac{\mathrm{d}x(t)}{\mathrm{d}t} + \omega_0^2 x(t) = f(t) \tag{1}$$

for some constants $\alpha, \omega_0 \in \mathbb{R}$ (often referred to as the *damping coefficient* and *undamped resonant frequency* respectively) and some function of time $f(t)$ (this is sometimes called a *forcing function*). The solution to this differential equation can be separated into homogeneous and particular solutions of the form

$$x(t) = x_h(t) + x_p(t) \tag{2}$$

where $x_h(t)$ represents the homogeneous solution and $x_p(t)$ represents the particular solution.

We typically solve separately for the homogeneous and particular solutions. The homogeneous solution is the solution to

$$\frac{\mathrm{d}^2 x(t)}{\mathrm{d}t^2} + 2\alpha \frac{\mathrm{d}x(t)}{\mathrm{d}t} + \omega_0^2 x(t) = 0 \tag{3}$$

Theorem 2 (Homogeneous Solution to Second Order Differential Equations)

Define $s_1 := -\alpha + \sqrt{\alpha^2 - \omega_0^2}$ and $s_2 := -\alpha - \sqrt{\alpha^2 - \omega_0^2}$. The homogeneous solution will take on one of the following forms, depending on the value of $\frac{\alpha}{\omega_0}$, called the *damping ratio*.

1. *Overdamped case: ($\frac{\alpha}{\omega_0} > 1$)*

$$x_h(t) = K_1 e^{s_1 t} + K_2 e^{s_2 t} \tag{4}$$

2. *Critically damped case: ($\frac{\alpha}{\omega_0} = 1$)*

$$x_h(t) = K_1 e^{s_1 t} + K_2 t e^{s_1 t} \tag{5}$$

Note that $s_1 = s_2$ in this case.

3. *Underdamped case: ($\frac{\alpha}{\omega_0} < 1$)*
Note that s_1 and s_2 will be complex, so we can rewrite them as $s_1 = -\alpha + j\omega_n$ and $s_2 = -\alpha - j\omega_n$ where $\omega_n := \sqrt{\omega_0^2 - \alpha^2}$ is defined as the natural frequency. The solution is of the form

$$x_h(t) = K_1 e^{-\alpha t}\cos(\omega_n t) + K_2 e^{-\alpha t}\sin(\omega_n t) \tag{6}$$

In all of the cases above, K_1 and K_2 are arbitrary constants that are determined by initial conditions. Note that you will need two initial conditions to completely solve a second order differential equation.

Concept Check: This note will not prove the solutions from first principles as that is out of scope, but as an exercise, you are encouraged to verify that the solutions satisfy eq. (3).

In general, finding the particular solution is not easy, but we can consider the specific case for a DC forcing function as we started with when looking at first-order differential equations. In other words, we can consider the case where $f(t) = C$ for some constant $C \in \mathbb{R}$. To solve for the particular solution in this case, we can replace circuit components by their DC steady-state equivalents (so a capacitor becomes an open circuit and an inductor becomes a wire) and then solve for $x_p(t)$ using circuit analysis.

1.1 Example: LC Tank

Consider the following circuit.

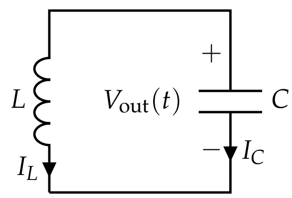

Figure 1: An LC Tank.

We can model $V_{\text{out}}(t)$ using differential equations. Suppose that $V_{\text{out}}(0) = 0$ and $I_L(0) = 1\,\text{A}$. From KVL, we have

$$V_C(t) = V_L(t) \tag{7}$$

$$V_{\text{out}}(t) = L\frac{\mathrm{d}I_L(t)}{\mathrm{d}t} \tag{8}$$

Further, we have from KCL that $I_L(t) = -I_C(t)$. Plugging this in above, we get

$$-L\frac{\mathrm{d}}{\mathrm{d}t}(I_C(t)) = V_{\text{out}}(t) \tag{9}$$

For a capacitor, we have $I_C(t) = C\frac{\mathrm{d}V_C(t)}{\mathrm{d}t} = C\frac{\mathrm{d}V_{\text{out}}(t)}{\mathrm{d}t}$. Plugging this in above, we get

$$-L\frac{\mathrm{d}}{\mathrm{d}t}\left(C\frac{\mathrm{d}V_{\text{out}}(t)}{\mathrm{d}t}\right) = V_{\text{out}}(t) \tag{10}$$

$$-LC\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\mathrm{d}V_{\text{out}}(t)}{\mathrm{d}t}\right) = V_{\text{out}}(t) \tag{11}$$

$$-LC\frac{\mathrm{d}^2V_{\text{out}}(t)}{\mathrm{d}t^2} = V_{\text{out}}(t) \tag{12}$$

$$-\frac{\mathrm{d}^2V_{\text{out}}(t)}{\mathrm{d}t^2} = \frac{1}{LC}V_{\text{out}}(t) \tag{13}$$

$$\frac{\mathrm{d}^2V_{\text{out}}(t)}{\mathrm{d}t^2} + \frac{1}{LC}V_{\text{out}}(t) = 0 \tag{14}$$

Pattern matching to eq. (1), we have $\omega_0^2 = \frac{1}{LC} \implies \omega_0 = \frac{1}{\sqrt{LC}}$ (we only consider the positive ω_0 since it represents the undamped resonant frequency). This means that $\frac{\alpha}{\omega_0} = 0$, and $f(t) = 0$. Hence, we are dealing with the underdamped case. Since $f(t) = 0$, we only need to solve for $x_h(t)$ (i.e., $x(t) = x_h(t)$).

Following Theorem 2, we have $\omega_n = \omega_0 = \sqrt{\frac{1}{LC}}$. This means that

$$V_{\text{out}}(t) = K_1 \cos\left(\sqrt{\frac{1}{LC}}t\right) + K_2 \sin\left(\sqrt{\frac{1}{LC}}t\right) \tag{15}$$

Now, we can apply the initial conditions to solve for K_1 and K_2. We are told that $V_{\text{out}}(0) = 0$. Plugging in $t = 0$ to eq. (15), we have

$$V_{\text{out}}(0) = K_1 \cos\left(0 \cdot \sqrt{\frac{1}{LC}}\right) + K_2 \sin\left(0 \cdot \sqrt{\frac{1}{LC}}\right) = K_1 \tag{16}$$

so we have $K_1 = V_{\text{out}}(0) = 0$. Now, we can rewrite eq. (15) as

$$V_{\text{out}}(t) = K_2 \sin\left(\sqrt{\frac{1}{LC}}t\right) \tag{17}$$

We can incorporate the fact that $I_L(0) = 1\,\text{A}$. We know that $I_L(t) = -I_C(t) = -C\frac{\mathrm{d}V_{\text{out}}(t)}{\mathrm{d}t}$. Plugging in eq. (17), we have

$$I_L(t) = -C\frac{\mathrm{d}}{\mathrm{d}t}\left(K_2 \sin\left(\sqrt{\frac{1}{LC}}t\right)\right) = -K_2\frac{C}{\sqrt{LC}} \cos\left(\sqrt{\frac{1}{LC}}t\right) = -K_2\sqrt{\frac{C}{L}} \cos\left(\sqrt{\frac{1}{LC}}t\right) \tag{18}$$

So, plugging in $t = 0$ above, we get

$$I_L(0) = -K_2\sqrt{\frac{C}{L}} \cos\left(0 \cdot \sqrt{\frac{1}{LC}}\right) = -K_2\sqrt{\frac{C}{L}} \tag{19}$$

Using the fact that $I_L(0) = 1$, we can solve for K_2 to obtain $K_2 = -\sqrt{\frac{L}{C}}$. Thus, plugging in for K_2 into eq. (17), we have

$$V_{\text{out}}(t) = -\sqrt{\frac{L}{C}} \sin\left(\sqrt{\frac{1}{LC}}t\right) \tag{20}$$

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