Note 3: Inductors and RL Circuits

1 Inductors

1.1 Introduction to Inductors

Here, we introduce a new passive component, the inductor. This new component will help us design more interesting circuits and introduce oscillations within our circuits.

Definition 1 (Inductor)

An inductor is denoted as in Figure 1.

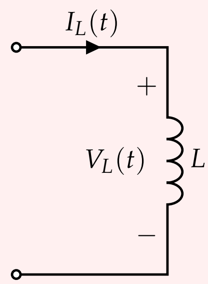


Figure 1: Example Inductor Circuit

The voltage across the inductor is related to its current as follows:

$$V_L(t) = L\frac{\mathrm{d}I_L(t)}{\mathrm{d}t} \tag{1}$$

where L is the *inductance* of the inductor. The SI unit of inductance is the Henry (H).

The following are important facts about inductors:

1. The current through an inductor cannot change instantaneously.
2. Immediately after a current is passed through the inductor, the inductor acts as an open circuit, but as $t \to \infty$, the inductor acts like a short.
   (a) Why is this the case? Well, note that steady-state (i.e. when $t \to \infty$) is when the circuit achieves an equilibrium state. In other words, the current across the inductor will be constant. Given the inductor definition equation $V_L(t) = L\frac{\mathrm{d}I_L(t)}{\mathrm{d}t} = (L)(0)$, we can confirm that an inductor is indeed a short in steady-state.

Notice that the voltage-current relationship written in eq. (1) is similar to that of a capacitor, but with voltage and current swapped. The short term and long term behavior of inductors and capacitors are also opposites of each other.

Theorem 2 (Series Equivalence)
Consider the two inductors in series configuration in Figure 2, and suppose we wish to find the series equivalent as in Figure 3.

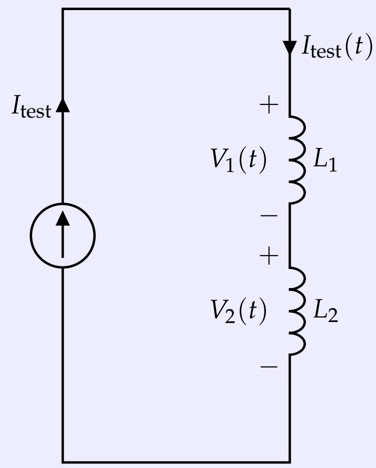


Figure 2: Series Inductor Circuit

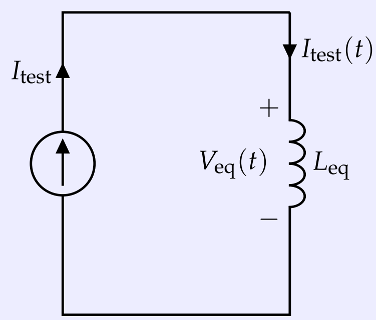


Figure 3: Equivalent Series Inductor Circuit

The equivalent series inductance is $L_{eq} = L_1 + L_2$.

Proof. We use the test current source, $I_{test}(t)$, depicted in Figure 2 and Figure 3 to find the equivalent voltage across both inductors, i.e., $V_{eq}(t)$. Using KVL, we have

$$V_1(t) + V_2(t) = V_{eq}(t) \tag{2}$$

$$L_1 \frac{dI_L(t)}{dt} + L_2 \frac{dI_L(t)}{dt} = V_{eq}(t) \tag{3}$$

$$\underbrace{(L_1 + L_2)}_{L_{eq}} \frac{dI_L(t)}{dt} = V_{eq}(t) \tag{4}$$

as desired. □

Theorem 3 (Parallel Equivalence)
Consider the two inductors in parallel configuration in Figure 4, and suppose we wish to find the parallel equivalent as in Figure 5.

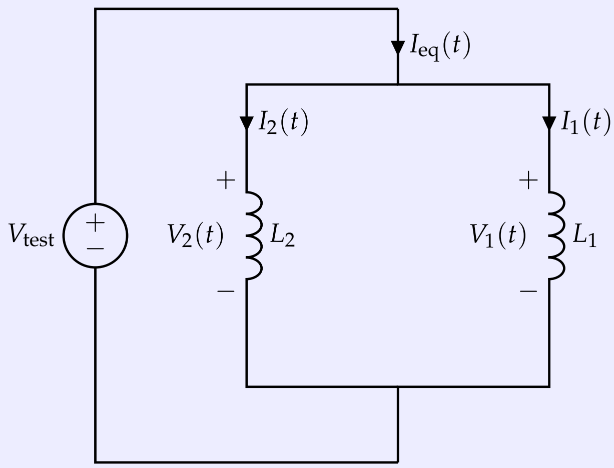


Figure 4: Parallel Inductor Circuit

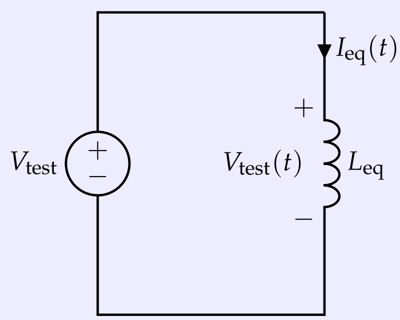


Figure 5: Equivalent Parallel Inductor Circuit

The equivalent inductance is given by $L_{eq} = \left(\frac{1}{L_1} + \frac{1}{L_2}\right)^{-1}$.

Proof. We can apply the test voltage V_{test} as depicted in Figure 4 and Figure 5 to find the equivalent current through both inductors, i.e., $I_{eq}(t)$. By NVA, we have that

$$V_1(t) = V_2(t) = V_{test}(t) \tag{5}$$

$$L_1 \frac{dI_1}{dt} = L_2 \frac{dI_2}{dt} = L_{eq} \frac{dI_{eq}}{dt} \tag{6}$$

and from KCL we have

$$I_{eq}(t) = I_1(t) + I_2(t) \tag{7}$$

$$\frac{dI_{eq}}{dt} = \frac{dI_1}{dt} + \frac{dI_2}{dt} \tag{8}$$

$$\frac{dI_{eq}}{dt} = \frac{L_{eq}}{L_1}\frac{dI_{eq}}{dt} + \frac{L_{eq}}{L_2}\frac{dI_{eq}}{dt} \tag{9}$$

$$\frac{1}{L_{eq}} = \frac{1}{L_1} + \frac{1}{L_2} \tag{10}$$

$$L_{\text{eq}} = \left(\frac{1}{L_1} + \frac{1}{L_2}\right)^{-1} \tag{11}$$

as desired. □

Theorem 4 (Stored Energy)
The stored energy in an inductor can be written as

$$E = \frac{1}{2} L i_L^2 \tag{12}$$

where i is the current through the inductor.

Proof. The formula for power can be manipulated as follows:

$$P_L = v_L i_L \tag{13}$$
$$P_L = \left(L \frac{\mathrm{d}i_L}{\mathrm{d}t}\right) i_L \tag{14}$$
$$P_L \, \mathrm{d}t = L i_L \, \mathrm{d}i_L \tag{15}$$

Integrating both sides to find stored energy, we have

$$\int P_L \, \mathrm{d}t = \int L i_L \, \mathrm{d}i_L \tag{16}$$
$$E = \frac{1}{2} L i_L^2 \tag{17}$$

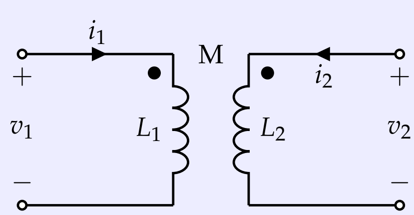
□

Definition 5 (Mutual Inductance)
The mutual inductance between two inductors L_1 and L_2 is given by

$$M = \frac{N_2 \Phi_{21}}{i_1} = \frac{N_1 \Phi_{12}}{i_2} \tag{18}$$

where N_1 and N_2 are the number of windings in the coils for inductors L_1 and L_2 respectively, and i_1 and i_2 are the current through the respective inductors. Φ_{12} is the flux passing through coil 1 from the magnetic field induced by coil 2, and Φ_{21} is the flux passing through coil 2 from the magnetic field induced by coil 1.

Theorem 6 (Induced Voltage from Mutual Inductance)
Consider the circuit below, with two inductors L_1 and L_2, with mutual inductance M.

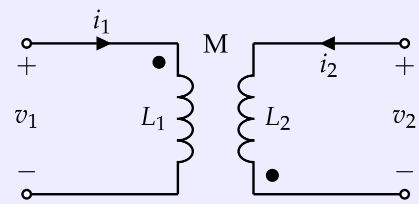


The dots in the circuit indicate the orientation of the inductors. For the given orientation, the following equations hold:

$$v_1 = L_1 \frac{\mathrm{d}i_1}{\mathrm{d}t} + M \frac{\mathrm{d}i_2}{\mathrm{d}t} \tag{19}$$

$$v_2 = M \frac{\mathrm{d}i_1}{\mathrm{d}t} + L_2 \frac{\mathrm{d}i_2}{\mathrm{d}t} \tag{20}$$

If the orientation of L_2 is flipped, as shown in the circuit below



then the following equations hold:

$$v_1 = L_1 \frac{\mathrm{d}i_1}{\mathrm{d}t} - M \frac{\mathrm{d}i_2}{\mathrm{d}t} \tag{21}$$

$$v_2 = -M \frac{\mathrm{d}i_1}{\mathrm{d}t} + L_2 \frac{\mathrm{d}i_2}{\mathrm{d}t} \tag{22}$$

Proof. Understanding of this proof is **optional**, but knowing/understanding mutual inductance is still in scope. Here, we will only prove the first part of the theorem since the second part follows by a symmetry argument, with a negated value of EMF to account for the flipped orientation. First, we can find the induced EMF in L_2 due to mutual inductance only. We can apply Faraday's law,

$$\mathcal{E}_{2,\text{mutual}} = -N_2 \frac{\mathrm{d}\Phi_{21}}{\mathrm{d}t} \tag{23}$$

$$= -N_2 \frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{Mi_1}{N_2}\right) \tag{24}$$

$$= -M \frac{\mathrm{d}i_1}{\mathrm{d}t} \tag{25}$$

where in eq. (24) we apply Definition 5. Now, notice that there is also current flowing through the second inductor, so we have an induced EMF from that. We can compute that, using Definition 1, as follows

$$\mathcal{E}_{2,\text{current}} = L_2 \frac{\mathrm{d}i_2}{\mathrm{d}t} \tag{26}$$

Combining these two EMFs using superposition and taking care to note the orientation of L_2, we obtain

$$v_2 = -\mathcal{E}_{2,\text{mutual}} + \mathcal{E}_{2,\text{current}} = M \frac{\mathrm{d}i_1}{\mathrm{d}t} + L_2 \frac{\mathrm{d}i_2}{\mathrm{d}t} \tag{27}$$

We can apply the exact same argument symmetrically to L_1 to obtain

$$v_1 = L_1 \frac{\mathrm{d}i_1}{\mathrm{d}t} + M \frac{\mathrm{d}i_2}{\mathrm{d}t} \tag{28}$$

□

1.2 Physics behind Inductors

Inductors store energy in a magnetic field. In the same way that a capacitor separates charge (Q) and this leads to an electric field ($\vec{E}$), anytime current flows down a conductor, it creates a magnetic field ($\vec{B}$), and this magnetic field can store energy. Inductors' behavior can be described using **Faraday's Law of Induction**.

The magnitude of magnetic field created by a straight wire is pretty small, so we usually use other geometries to create useful inductances. A **solenoid** is a good example, where we wind a wire around a conductor like a copper rod:

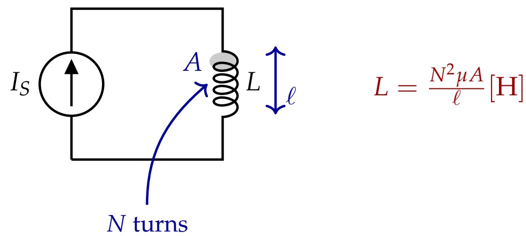


Figure 6: The Inductance of a Solenoid: a wire coiled around something.

Note that the inductance (L) depends on the **geometry** and a material property called **magnetic permeability** (μ) of the solenoid core material. In the case of the solenoid in fig. 6, the inductance depends on the number of turns (N), the length of the solenoid (l) and the area (A) of the loops. Inductors are useful in many applications such as wireless communications, chargers, DC-DC converters, key card locks, transformers in the power grid, etc. But in many high speed applications, their presence might be undesirable as they create delays in the time response of the circuit (analogous to capacitors).

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