Note 1: Capacitors, RC Circuits, and Differential Equations

1 Mathematical Approach to RC Circuits

We know from EECS 16A that $q = Cv$ describes the charge in a capacitor as a function of the voltage across the capacitor and capacitance. From EECS16A, we know that the voltage across the capacitor will gradually change over time. So, we may write charge as a function of time, namely

$$q(t) = Cv(t) \tag{1}$$

We can assume that capacitance is a constant with respect to time, since this is a quantity inherent to the physical nature of the component. This will allow us to come up with a *differential equation*.

Definition 1 (Differential Equation)
A differential equation is an equation which includes any kind of derivative (ordinary derivative or partial derivative) of any order (e.g. first order, second order, etc.).

We can derive a differential equation for capacitors based on eq. (1).

Theorem 2 (Capacitor Differential Equation)
A differential equation relating the time evolution of current through and voltage across a capacitor is given by

$$I(t) = C\frac{\mathrm{d}v(t)}{\mathrm{d}t} \tag{2}$$

Proof. Current is the rate of flow of charge over time, so we may write $\frac{\mathrm{d}q(t)}{\mathrm{d}t} = I(t)$. Taking time derivatives on both sides of eq. (1) yields

$$I(t) = C\frac{\mathrm{d}v(t)}{\mathrm{d}t} \tag{3}$$

□

1.1 RC Circuit Example

Consider the following circuit.

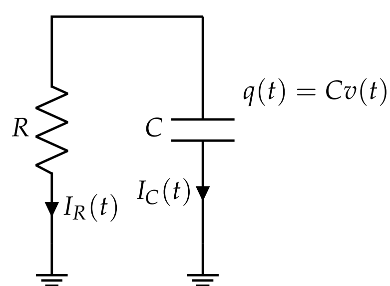

Figure 1: Capacitor discharging through circuit

Using the results from the previous section and KVL, we have the following differential equation governing the behavior of this circuit

$$C\frac{\mathrm{d}v(t)}{\mathrm{d}t} = -\frac{v(t)}{R} \tag{4}$$

$$\frac{\mathrm{d}v(t)}{\mathrm{d}t} = -\frac{v(t)}{RC} \tag{5}$$

2 Differential Equations

We now will generalize what we've seen to solve some types of differential equations.

Definition 3 (Scalar Constant Differential Equation)
A scalar constant differential equation is defined as

$$\frac{\mathrm{d}}{\mathrm{d}t}x(t) = b \tag{6}$$

To solve this differential equation and find $x(t)$, we need a few key components.

Key Idea 4 (Components of Differential Equations)
When solving differential equations, we need two main components:

1. The differential equation itself. An example that we have already seen is $\frac{\mathrm{d}v(t)}{\mathrm{d}t} = -\frac{1}{RC}v(t)$.
2. An initial condition. This will tell us what the solution to our differential equation is at a specific time. For example the example above, we would need to know a concrete value for $v(t_0)$, for some time t_0.

Theorem 5 (Scalar Constant Differential Equation Solution)
The scalar constant differential equation defined in Definition 3 admits a solution of the form

$$x(t) = k + b(t - t_0) \tag{7}$$

with the initial condition being $x(t_0) = k$.

Proof. We can integrate both sides of eq. (6). To solve this integral, we can introduce a dummy variable τ and integrate with respect to it as follows:

$$\int_{t_0}^{t} \frac{\mathrm{d}}{\mathrm{d}\tau}x(\tau)\,\mathrm{d}\tau = \int_{t_0}^{t} b\,\mathrm{d}\tau \tag{8}$$

Applying the fundamental theorem of calculus,

$$\int_{t_0}^{t} \frac{\mathrm{d}}{\mathrm{d}\tau}x(\tau)\,\mathrm{d}\tau = \int_{t_0}^{t} b\,\mathrm{d}\tau \tag{9}$$

$$x(t) - x(t_0) = b(t - t_0) \tag{10}$$

$$x(t) = k + b(t - t_0) \tag{11}$$

□

2.1 "Homogeneous" Differential Equations

Next, we work to extend this reasoning beyond $\frac{\mathrm{d}}{\mathrm{d}t}x(t) = b$ to more general differential equations of the form $\frac{\mathrm{d}}{\mathrm{d}t}x(t) = ax(t)$ where $a \in \mathbb{R}$ is a constant. This is known as a *homogeneous* differential equation.

Definition 6 (Homogeneous Differential Equation)
A homogeneous differential equation can be written as:

$$\frac{\mathrm{d}}{\mathrm{d}t}x(t) = \lambda x(t) \tag{12}$$

for some $\lambda \in \mathbb{R}$.

We will only consider the case where $\lambda \neq 0$ for this subsection, since if $\lambda = 0$, then the differential equation is exactly as in eq. (6) with $b = 0$. To solve this equation, we employ a method of guessing the solution.

Key Idea 7 (Homogeneous Differential Equation Solution Form)
We guess that the solution to the homogeneous differential equation is

$$x(t) = Ae^{bt} \tag{13}$$

for some constants $A, b \in \mathbb{R}$.

We can typically use the initial condition to find A. Given an initial condition for $x(t_0)$, we can apply this in eq. (13) as follows:

$$x(t_0) = Ae^{bt_0} \tag{14}$$

We typically use the differential equation itself to find b.

Theorem 8 (Homogeneous Differential Equation Solution)
If the initial condition is $x(t_0) = k \neq 0$, we obtain a solution of the form

$$x(t) = ke^{\lambda(t-t_0)} \tag{15}$$

for the same λ defined in Definition 6.
If the initial condition is $x(t_0) = k \neq 0$, the solution will be $x(t) = 0$ for all $t \geq 0$.

Proof. Case 1: Suppose $x(t_0) = k \neq 0$. We can start by solving for A as follows:

$$x(t_0) = Ae^{bt_0} \tag{16}$$

$$k = Ae^{bt_0} \tag{17}$$

$$\implies A = ke^{-bt_0} \tag{18}$$

Plugging this back into the differential equation, we see

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(ke^{b(t-t_0)}\right) = \lambda\left(ke^{b(t-t_0)}\right) \tag{19}$$

$$b\left(ke^{b(t-t_0)}\right) = \lambda\left(ke^{b(t-t_0)}\right) \tag{20}$$

$$\implies b = \lambda \tag{21}$$

which concludes that $x(t) = ke^{\lambda(t-t_0)}$ when $k \neq 0$. Crucially, we used the fact that $k \neq 0$ in eq. (20), along with the fact that e^{anything} is nonzero.

Case 2: Suppose $k = 0$. Hence,

$$x(t_0) = Ae^{bt_0} \tag{22}$$

$$\underbrace{k}_{0} = Ae^{bt_0} \tag{23}$$

$$\implies A = 0 \tag{24}$$

Thus, $x(t) = 0e^{bt} = 0$ for all $t \geq 0$. □

2.1.1 Example

Given the differential equation for the circuit in fig. 1, the solution to the differential equation for $v(t)$ would be

$$v(t) = Ae^{-\frac{t}{RC}} \tag{25}$$

Note here that $\lambda = -\frac{1}{RC}$. At $t = 0$, we can assume the capacitor is fully charged, i.e. $v(0) = v_S$ for some initial nonzero voltage value v_S. Hence, we can use this information to find A:

$$\underbrace{v(0)}_{v_S} = Ae^{-\frac{0}{RC}} \tag{26}$$

$$v_S = A \tag{27}$$

so, altogether, we have

$$v(t) = v_S e^{-\frac{t}{RC}} \tag{28}$$

We will see the voltage across the capacitor follow the shape of the graph in fig. 2.

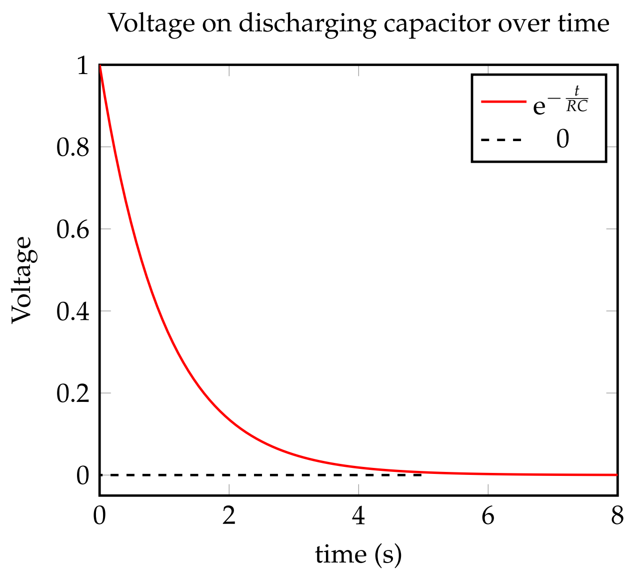

Figure 2

2.2 Uniqueness

Now that we have found a set of potential solutions, the other question that arises is whether there is a unique solution to the differential equation that we are solving.

Theorem 9 (Uniqueness of Homogeneous Differential Equations)
Given a differential equation of the form in Definition 6 and given an initial condition, the solution of the form

$$x(t) = Ae^{bt} \tag{29}$$

satisfying the differential equation and initial condition is unique.

Concept Check: Prove Theorem 9 as a (optional) homework exercise.

2.3 Nonhomogeneous Differential Equations

So far, we have learned to solve homogeneous differential equations, let us learn to solve a specific kind of nonhomogeneous differential equations.

Definition 10 (Nonhomogeneous Differential Equations)
A nonhomogeneous differential equation is defined as

$$\frac{\mathrm{d}}{\mathrm{d}t}x(t) = \lambda x(t) + u(t) \tag{30}$$

for some function $u(t) : \mathbb{R}_+ \to \mathbb{R}$.

Key Idea 11 (Homogeneous And Particular Solution:)
We have already shown that $x_h(t) = Ae^{bt}$ is the form of the solution for the homogeneous differential equation $\frac{\mathrm{d}}{\mathrm{d}t}x_h(t) = \lambda x_h(t)$. Thus, a reasonable (and indeed correct) intuition to have is that the solution to the non-homogeneous equation

$$\frac{\mathrm{d}}{\mathrm{d}t}x(t) = \lambda x(t) + u(t) \tag{31}$$

would be comprised of the same *homogeneous* solution, $x_h(t)$, along with an additional *particular* solution, $x_p(t)$ to account for the $u(t)$ term. Thus, the entire solution to the above equation is of the form:

$$x(t) = x_h(t) + x_p(t) \tag{32}$$

We can analyze the solution for this differential equation.

What does this particular solution represent though? One way to see this is by putting $x(t) = x_h(t) + x_p(t)$ into the differential equation that it is defined as a solution for.

$$\frac{\mathrm{d}}{\mathrm{d}t}x(t) = \frac{\mathrm{d}}{\mathrm{d}t}\left(x_h(t) + x_p(t)\right) \tag{33}$$

$$= \frac{\mathrm{d}}{\mathrm{d}t}x_h(t) + \frac{\mathrm{d}}{\mathrm{d}t}x_p(t) \tag{34}$$

$$= \lambda x_h(t) + \frac{\mathrm{d}}{\mathrm{d}t} x_p(t) \tag{35}$$

For the last step, we use the fact that $x_h(t)$ is, by definition, a solution to the homogeneous differential equation $\frac{\mathrm{d}}{\mathrm{d}t}x(t) = \lambda x(t)$.

The differential equation itself states that

$$\frac{\mathrm{d}}{\mathrm{d}t} x(t) = \lambda x(t) + u(t) = \lambda x_h(t) + \lambda x_p(t) + u(t) \tag{36}$$

Thus, this implies that

$$\frac{\mathrm{d}}{\mathrm{d}t} x_p(t) = \lambda x_p(t) + u(t) \tag{37}$$

which is simply the original differential equation we wanted to solve, with $x_p(t)$ instead of $x(t)$!

Essentially, the particular solution $x_p(t)$ can be any function that satisfies the original differential equation; any variation between solutions (which comes due to different initial conditions) is handled by the arbitrary constant present in the homogeneous solution. There exist many different particular solutions, though we usually try to find the simplest one when solving differential equations.

We state here (without proof) that the particular solution for a differential equation is often related to the input function $u(t)$. For example, if $u(t)$ is constant, a simple particular solution will usually be constant as well, and if $u(t)$ is sinusoidal, a simple particular solution will usually be sinusoidal as well. For now, we will mostly focus on the case where $u(t)$ is constant, which corresponds to circuits in the case where the input voltage or current source is constant.

Key Idea 12 (Particular Solutions with Constant Input:)
Suppose we have the differential equation

$$\frac{\mathrm{d}}{\mathrm{d}t} x(t) = \lambda x(t) + u \tag{38}$$

where u is a constant input term.
The particular solution we will use in this case is a constant value, $x_p(t) = A$.

For this general form of the differential equation, we can show how to solve for the value of A in terms of the given variables.

$$\frac{\mathrm{d}}{\mathrm{d}t} x_p(t) = \lambda x_p(t) + u \tag{39}$$

$$\frac{\mathrm{d}}{\mathrm{d}t}(A) = \lambda(A) + u \tag{40}$$

$$0 = \lambda A + u \tag{41}$$

Thus, $A = -\frac{u}{\lambda}$ would be a solution for the differential equation, and thus a particular solution $x_p(t) = -\frac{u}{\lambda}$.

Mathematically, this would be a way to solve for the particular solution with a constant input, but what does this represent for a circuit, a system that we can represent with such a differential equation? It turns out that a convenient particular solution for a circuit such as an RC circuit is the **steady-state solution** of the circuit, which is the solution as $t \to \infty$!

Key Idea 13 (Particular and Steady-State Solutions:)
For a circuit with constant input, the circuit's DC steady-state solution (a solution as $t \to \infty$) can be used as a particular solution for the differential equation represented by the circuit.

The reason the steady-state solution is a convenient solution for a circuit's differential equation is because a steady-state solution describes a stable solution that does not change anymore; this will often correspond to when derivative terms in the differential equation are 0, such as when we calculated the particular solution with constant input, and reduces a differential equations problem into a basic algebra problem (at least for the particular solution). We can calculate the steady-state solution for a circuit even without having the differential equation as the idea of steady-state in a circuit can be represented by simplifying each element of the circuit to its steady state equivalent.

Key Idea 14 (Capacitor DC Steady-State:)
In DC steady-state (as $t \to \infty$ with only DC/constant inputs), a capacitor behaves like an open circuit.

We can see why this makes sense by using the idea that at DC steady state, all state variables (currents and voltages) are constant with time.

If we assume this to be true, using the capacitor equation $I(t) = C\frac{\mathrm{d}}{\mathrm{d}t}v(t) = C(0) = 0$ since $\frac{\mathrm{d}}{\mathrm{d}t}v(t) = 0$ for steady-state.

Thus, in steady-state, the current through a capacitor is 0, which is equivalent to an open circuit behavior.

Once this substitution is made in the circuit diagram (specifically for the purpose of calculating the steady-state solution, not permanently), the steady-state solution simply becomes the solution of the remaining resistor network with source voltage(s)/current(s), and this constant value can be used as the particular solution for the differential equation.

2.3.1 Example:

Consider the circuit in fig. 4.

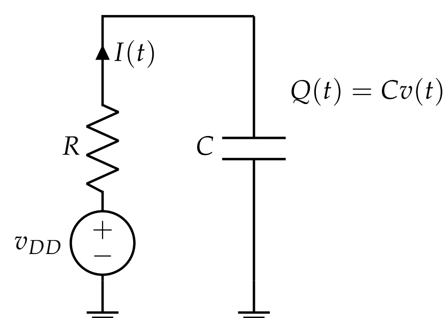

Figure 3: Capacitor charging through resistor circuit

Using Kirchhoff's voltage Law (KvL), we can see that

$$v_{DD} = RI(t) + v(t) \tag{42}$$

where $v(t)$ is the voltage across the capacitor. Using the fact that $I(t) = C\frac{\mathrm{d}}{\mathrm{d}t}v(t)$, our resulting differential equation is

$$RC\frac{\mathrm{d}}{\mathrm{d}t}v(t) + v(t) = v_{DD} \tag{43}$$

$$\frac{\mathrm{d}}{\mathrm{d}t}v(t) = -\frac{1}{RC}v(t) + \frac{v_{DD}}{RC} \tag{44}$$

Suppose the capacitor is initially uncharged at time $t = 0$, i.e. $v(0) = 0$.

First, let's find the homogeneous solution, which is the solution to the following differential equation:

$$\frac{\mathrm{d}}{\mathrm{d}t}v_h(t) = -\frac{1}{RC}v_h(t) \tag{45}$$

From the homogeneous solutions section, we know that the solution will be

$$v_h(t) = A\mathrm{e}^{-\frac{1}{RC}t} \tag{46}$$

with A as a constant we will solve for later with our initial condition.

To find the particular solution, we will use the idea of steady-state solutions. To do so, we can redraw our circuit, with the capacitor replaced with an open circuit:

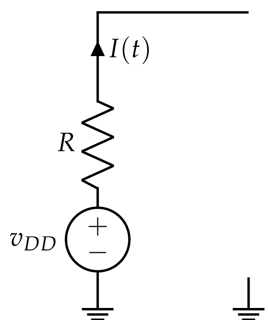

Figure 4: Capacitor charging through resistor circuit

Due to the open circuit, there can be no current through resistor, and thus no voltage drop across it. Thus, the steady-state voltage across the capacitor (which is an open circuit in the current diagram) is $v_p(t) = v_{DD}$. With both the homogeneous and particular solutions, we can construct the full solution:

$$v(t) = v_h(t) + v_p(t) = A\mathrm{e}^{-\frac{1}{RC}t} + v_{DD} \tag{47}$$

With the initial condition $v(0) = 0$, we can find that $A = -v_{DD}$ and the solution is:

$$v(t) = v_h(t) + v_p(t) = v_{DD}(1 - \mathrm{e}^{-\frac{1}{RC}t}) \tag{48}$$

which is a relatively common solution structure for an RC circuit charging a capacitor (and one to become familiar with over time). A plot of $v(t)$ will follow the shape of the graph in fig. 5.

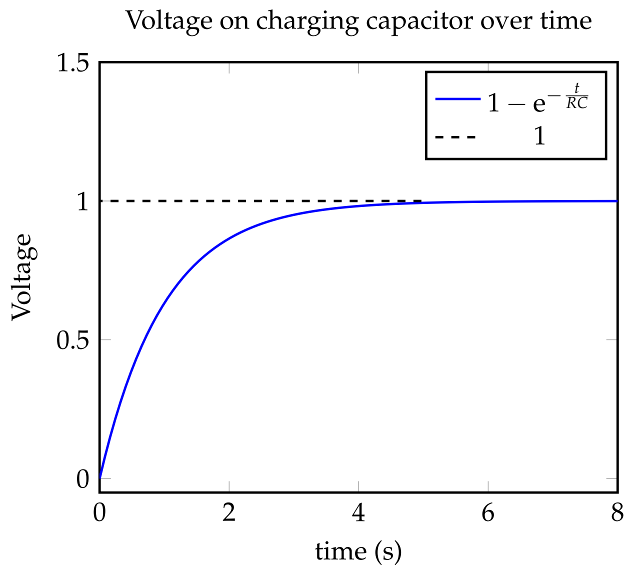

Figure 5

2.4 Generalized Solution to Linear Differential Equation with Constant Input:

Key idea 12 shows us how to solve for the particular solution of a linear differential equation with a constant input. We have also seen how to solve for the overall solution to the differential equation of a circuit. Here, we will combine what we have learned to derive a general solution for a differential equation with constant input. (Note: You should not have to memorize the general solution; it will simply be a tool to help with future derivations where we already have the differential equation and need to solve it. It is more important to understand the process of solving a differential equation using the homogeneous and particular solution method as established in the previous sections.)

2.4.1 Derivation:

We are given two key pieces of information:

1. Our system is defined by the differential equation $\frac{\mathrm{d}}{\mathrm{d}t}x(t) = \lambda x(t) + u$.
2. This system has some known initial condition (or state) $x(t_0) = k$.

Based on what we have already established, the solution is of the form $x(t) = x_h(t) + x_p(t)$. The homogeneous solution will be the solution to the differential equation without input:

$$\frac{\mathrm{d}}{\mathrm{d}t}x_h(t) = \lambda x_h(t) \tag{49}$$

From Key idea 7, we can determine that the solution to this homogeneous differential equation will be:

$$x_h(t) = Ae^{\lambda t} \tag{50}$$

with A as an arbitrary constant that we will solve for later.

We have already shown from Key idea 12 that the particular solution will be $x_p(t) = -\frac{u}{\lambda}$.

Thus, the combined solution (prior to applying the initial condition), will be:

$$x(t) = Ae^{\lambda t} - \frac{u}{\lambda} \tag{51}$$

Now, we can use our initial condition to solve for A:

$$x(t_0) = Ae^{\lambda(t_0)} - \frac{u}{\lambda} \tag{52}$$

$$k = Ae^{\lambda t_0} - \frac{u}{\lambda} \tag{53}$$

$$A = (k + \frac{u}{\lambda})e^{\lambda t_0} \tag{54}$$

With this, we can establish our full solution in the following theorem.

Theorem 15 (Solution to Differential Equations with Constant Nonhomogeneous Term)
Consider a differential equation

$$\frac{\mathrm{d}}{\mathrm{d}t}x(t) = \lambda x(t) + u \tag{55}$$

for real constants $\lambda, u \in \mathbb{R}$, where $\lambda \neq 0$ and $u \neq 0$[a]. This differential equation admits a solution of the form

$$x(t) = \left(k + \frac{u}{\lambda}\right)e^{\lambda(t-t_0)} - \frac{u}{\lambda} \tag{56}$$

where the initial condition is $x(t_0) = k$.

[a] If $u = 0$, then the differential equation is of the form in Definition 6, and if $\lambda = 0$, then the differential equation is of the form in eq. (6).

3 OPTIONAL: Change of Variables Method for Solving Differential Equations:

Key Idea 16 (Change of Variables)
A change of variables is the technique of defining a new $\widetilde{x}(t)$ such that we are able to transform a new type of differential equation into a differential equation for $\widetilde{x}(t)$ that we already know how to solve.

We can analyze the solution for this differential equation.

Theorem 17 (Solution to Differential Equations with Constant Nonhomogeneous Term)
Consider a differential equation

$$\frac{\mathrm{d}}{\mathrm{d}t}x(t) = \lambda x(t) + u \tag{57}$$

for real constants $\lambda, u \in \mathbb{R}$, where $\lambda \neq 0$ and $u \neq 0$[a]. This differential equation admits a solution of the form

$$x(t) = \left(k + \frac{u}{\lambda}\right)e^{\lambda(t-t_0)} - \frac{u}{\lambda} \tag{58}$$

where the initial condition is $x(t_0) = k$.

[a] If $u = 0$, then the differential equation is of the form in Definition 6, and if $\lambda = 0$, then the differential equation is of the form in eq. (6).

Proof. Define

$$\widetilde{x}(t) = x(t) + \frac{u}{\lambda} \iff x(t) = \widetilde{x}(t) - \frac{u}{\lambda} \tag{59}$$

We can use this change of variables to define a new differential equation:

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(\widetilde{x}(t) - \frac{u}{\lambda}\right) = \lambda\left(\widetilde{x}(t) - \frac{u}{\lambda}\right) + u \tag{60}$$

$$\frac{\mathrm{d}}{\mathrm{d}t}\widetilde{x}(t) = \lambda\widetilde{x}(t) \tag{61}$$

We have to define the initial condition. We are given that $x(t_0) = k$ for some constant $k \neq 0$. This means that $\widetilde{x}(t_0) = x(t_0) + \frac{u}{\lambda} = k + \frac{u}{\lambda}$.

Case 1: Suppose $k + \frac{u}{\lambda} \neq 0$. Hence, the solution for $\widetilde{x}(t)$ will follow the form

$$\widetilde{x}(t) = A\mathrm{e}^{\lambda t} \tag{62}$$

for some A. We can find A by plugging in the initial condition:

$$\widetilde{x}(t_0) = A\mathrm{e}^{\lambda t_0} \tag{63}$$

$$k + \frac{u}{\lambda} = A\mathrm{e}^{\lambda t_0} \tag{64}$$

$$\left(k + \frac{u}{\lambda}\right)\mathrm{e}^{-\lambda t_0} = A \tag{65}$$

So, the solution for $\widetilde{x}(t)$ is

$$\widetilde{x}(t) = \left(k + \frac{u}{\lambda}\right)\mathrm{e}^{\lambda(t-t_0)} \tag{66}$$

Plugging this back in to the change of variables defined in eq. (59), we can find the solution for $x(t)$:

$$x(t) = \left(k + \frac{u}{\lambda}\right)\mathrm{e}^{\lambda(t-t_0)} - \frac{u}{\lambda} \tag{67}$$

Case 2: Suppose $k + \frac{u}{\lambda} = 0$. This means the value of the solution for $\widetilde{x}(t)$ is 0 at t_0, which means that $\widetilde{x}(t) = 0$ for all $t \geq 0$. This is because $\mathrm{e}^{\lambda t}$ will always be positive. Plugging in $\widetilde{x}(t) = 0$ into eq. (59), we can find $x(t)$:

$$x(t) = -\frac{u}{\lambda} \tag{68}$$

□

4 OPTIONAL: Nonlinear Differential Equations

In the above sections, we only talked about linear differential equations where $\frac{\mathrm{d}}{\mathrm{d}t}x(t) = ax(t) + b$. However, you may encounter differential equations like $\frac{\mathrm{d}}{\mathrm{d}t}x(t) = x(t)^2$ and other such nonlinear functions of $x(t)$. In general, there are various "techniques" that can be used to attempt to guess potential solutions for such equations. At the end of the day, all of these guesses need to be checked and the appropriate uniqueness theorems proved to make sure that we have got the single true solution. Only then can this solution used for any predictive purposes.

Without a uniqueness theorem, such solutions cannot be trusted for prediction. In the homework, you will see an example that illustrates how a seemingly innocuous differential equation can have non-unique solutions. In that homework, we will also share another technique that can be used to guess solutions to nonlinear differential equations — a technique known as "separation of variables." There are many such techniques out there, and different ones tend to work for different types of equations. You will encounter these techniques in later courses alongside the kinds of differential equations for which they tend to work.

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