Minimum Energy Control & Singular Value Decomposition (SVD)

EECS 16B
Designing Information Devices and Systems II

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Minimum Energy Control

- stability (Is system going to blow up?)
- controllability (Can I reach where I want to?)
- Efficiency (How efficiently can I reach the desired destination)

$$\vec{x}[i+1] = A\vec{x}[i] + \vec{b}u[i]$$

$$\vec{x}[k] = A^{k}x[0] + A^{k-1}\vec{b}\vec{u}[0] + \cdots + \vec{b}u[k-1]$$

$$k = 100, \quad \vec{x}[0] = 0$$

$$\vec{x}[100] = A^{99}\vec{b}u[0] + \cdots + \vec{b}u[99]$$

$$= \underbrace{\left[A^{99}\vec{b} \;\middle|\; \cdots \;\middle|\; A\vec{b} \;\middle|\; \vec{b} \right]}_{C} \underbrace{\begin{bmatrix} u[0] \\ \vdots \\ u[99] \end{bmatrix}}_{\vec{u}}$$

$\vec{d}$

$$\vec{x}[100] = C\vec{u}$$

Min. Energy Control

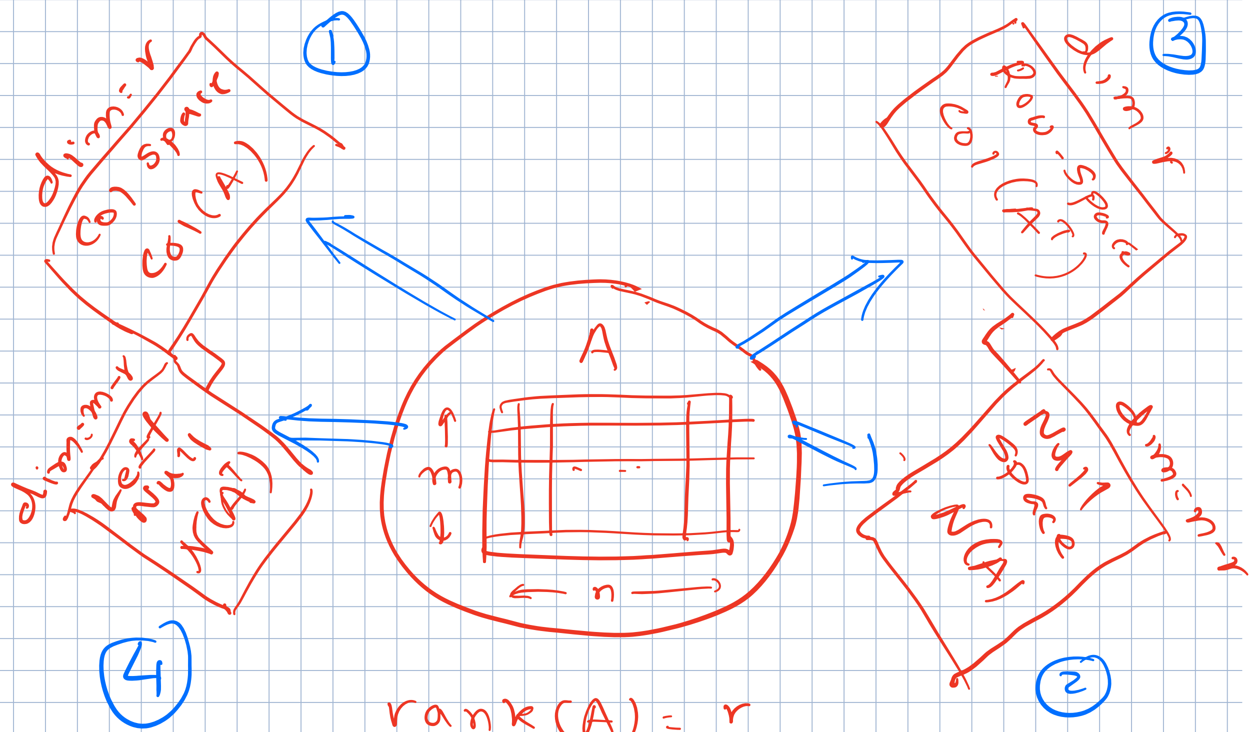
(Min norm solution)

$$\min_{\vec{u}} \|\vec{u}\|^2$$
$$\text{s.t. } \vec{x}[100] = C\vec{u}$$

More formally,

$$\vec{u}^* = \underset{\vec{u}}{\operatorname{argmin}} \|\vec{u}\|^2 \quad \text{s.t. } C\vec{u} = \vec{d}$$

4 Fundamental Spaces (Review)



$$\text{rank}(A) = r$$

$$r \le \min(m, n)$$

① $\{\vec{x} = A\vec{w},\ \forall \vec{w} \in R^n\} = \text{col}(A)$

② $\{\vec{w} \in R^n : A\vec{w} = 0\} = \text{Null}(A)$

③ $\{\vec{x} : A^T\vec{w},\ \forall \vec{w} \in R^m\} = \text{Row-space}(A) = \text{col}(A^T)$

④ $\{\vec{w} \in R^m : A^T w = 0\} = \text{Null}(A^T)$

Fact: $N(A) \perp$ Row-Space(A)

Proof: $\vec{w} \in N(A)$

$\vec{A}_i^T$ is i^{th} row of A

$\iff A\vec{w} = 0$

$$\iff \begin{bmatrix} \vec{A}_1^T \\ \vdots \\ \vec{A}_m^T \end{bmatrix} \vec{w} = \begin{bmatrix} \langle \vec{A}_1, \vec{w} \rangle \\ \vdots \\ \langle \vec{A}_m, \vec{w} \rangle \end{bmatrix} = \vec{0}_m$$

$\iff \vec{w}$ is $\perp$ to each row of A.

$\iff \vec{w}$ is $\perp$ to any lin. comb. of rows of A.

Ex: $x[k+1] = x[k] + u[k]$

Scalar eqn. with $A=1$, $b=1$

Suppose $x[0]=0$, $d = x[2] = 2$.

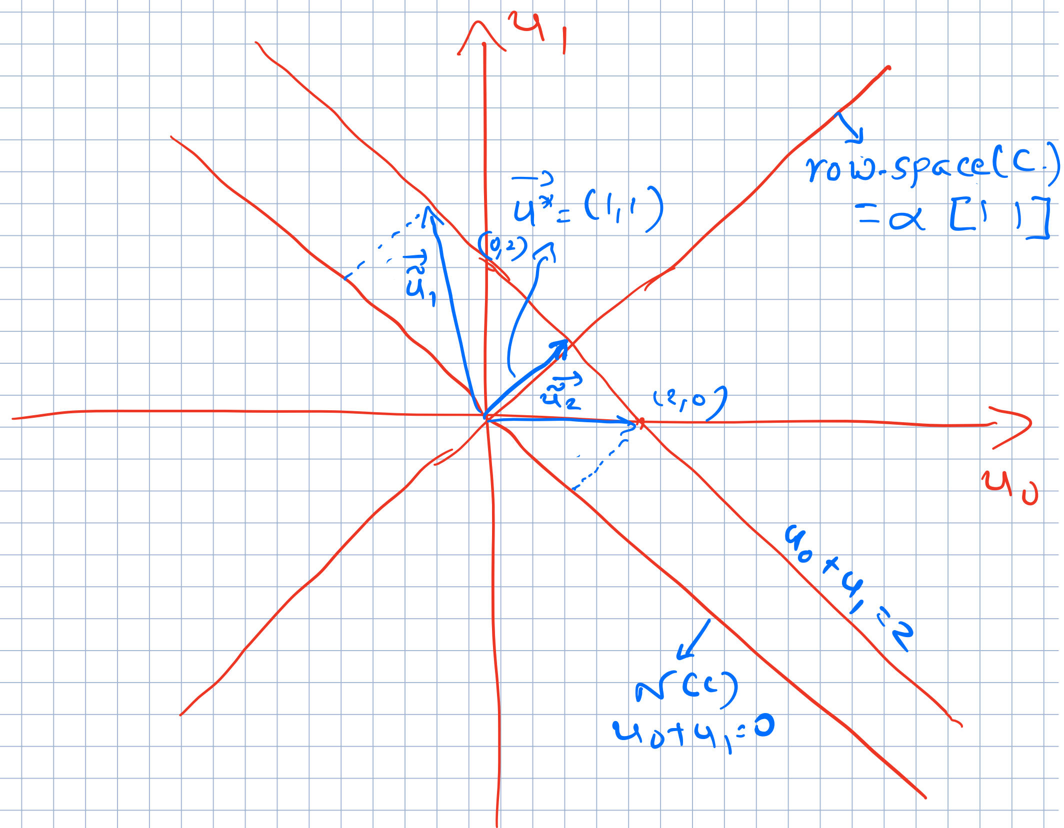
$$x[2] = x[1] + u[1]$$
$$= x[0] + u[0] + u[1]$$

we want to find $u[0]$ & $u[1]$ s.t.

$|u[0]|^2 + |u[1]|^2$ is minimum

There are infinitely many soln.

$(u_0, u_1) = (2,0), (1.5, 0.5) \ldots (0,2)$



u_1
u_0
$\vec{u}^* = (1,1)$
row-space(C) $= \alpha\,[1\ 1]$
(0,2)
(2,0)
$\vec{u}_1$
$\vec{u}_2$
$u_0 + u_1 = 2$
$N(C)$
$u_0 + u_1 = 0$

Warm-up for SVD