EECS 16B
Designing Information Devices and Systems II
Department of Electrical Engineering and Computer Sciences

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A little bit about myself ... 

• Live in Orinda with my wife and two children. 
• Grew up in India and received PhD from EECS, UC Berkeley. 
• Primary research area is networks. 
• Divide my time between academia and industries. 
• Currently focusing on performance issues in 5G networks. 
• Fond of hiking/trekking, photography, cricket, tennis, ... 
• Like to understand things starting from the basics.
Important Quiz ...
System-Level Issues

• System-Level Model
• Stability
• Impact of Feedback
• Controllability & Reachability
• Minimum Energy Control
• Linearization of a Non-Linear Model
• ...

...
Discrete-Time System Model

\[ \ddot{x}_d[i+1] = A_d \ddot{x}_d[i] + B_d \dot{u}_d[i] + \ddot{w}_d[i] \]

(Going forward, we’ll drop the subscript d for simplicity.)
Scalar Discrete-Time System Model

\[ x[i+1] = ax[i] + bu[i] + w[i] \]

System Identification Problem: Find the model parameters a & b.

There are 2 unknowns.

Collect data by choosing \( u[k], k: 0, \ldots, T-1 \)

measuring \( x[k], k: 0, \ldots, T \)

\[
\begin{align*}
    x[1] &\approx ax[0] + bu[0] \\
    \vdots \\
    x[T] &\approx ax[T-1] + bu[T-1]
\end{align*}
\]
Let's write the eigs as

\[ \begin{bmatrix} X[0] & u[0] \\ \vdots & \vdots \\ X[T-1] & u[T-1] \end{bmatrix} \rightarrow \begin{bmatrix} q_0 \\ b \end{bmatrix} \rightarrow \begin{bmatrix} X[0] \\ \vdots \\ X[T] \end{bmatrix} \]

\[ T \times 2 \quad 2 \times 1 \quad T \times 1 \]

Data Parameters State

\[ D \rightarrow P \rightarrow S \]

Least Squares soln:

\[ P = (D^T D)^{-1} D^T S \]

\[ \rightarrow \text{Need } D^T D \text{ to be invertible.} \]

\[ S = S + e \]

\[ S = D P \rightarrow \text{Column space } A \sim D \]

\[ D = [v_1 \ldots v_n] \begin{bmatrix} \lambda_1^* \\ \vdots \\ \lambda_n^{*1} \end{bmatrix} = \lambda_1^* v_1 + \ldots + \lambda_n^{*1} v_n \]
\[D^T e^2 = 0 \]
\[D^T (S - D \hat{P}) = 0 \]
\[D^T S - D^T D \hat{P} = 0 \]
\[D^T S \hat{P} = \underbrace{D^T D \hat{P}}_{\hat{S}} \]
\[\hat{P} = (D^T D)^{-1} D^T S \]
Vector Discrete-Time System Model

\[ \tilde{x}[i+1] = A\tilde{x}[i] + B\tilde{u}[i] + \tilde{w}[i] \]

System Identification Problem: Find the model parameters A & B.

\[ \tilde{x}[i] \text{ is } n \text{-dim vector} \]
\[ \tilde{u}[i] \text{ is } m \text{-dim vector} \]

\[ A = \begin{bmatrix} a_1 & \cdots & a_n \end{bmatrix}^{\top} \quad n \times n \]
\[ B = \begin{bmatrix} b_1 & \cdots & b_m \end{bmatrix}^{\top} \quad n \times m \]

Note: A is unknown.

\[ A = \begin{bmatrix} a_1 & \cdots & a_n \end{bmatrix}^{\top} \quad n \times n \]
Let's present \( E_{ij} \) for L-S formulation:

Let's focus on row \( r \):

\[
x_{r[i+1]} = \overrightarrow{U_j^T} \overrightarrow{X_{[i]}} + \overrightarrow{b_{r^i}} \overrightarrow{U_{[i]}}
\]

Scalar \( 1 \times n \) \( n \times 1 \) \( 1 \times m \) \( m \times 1 \)

\[
\begin{bmatrix}
\overrightarrow{x_{[0]}} & \overrightarrow{u_{[0]}} \\
\overrightarrow{x_{[r-1]}} & \overrightarrow{u_{[r-1]}}
\end{bmatrix}
\begin{bmatrix}
\overrightarrow{a_r} \\
\overrightarrow{b_{r^i}}
\end{bmatrix}
\overset{\text{vertically stacked}}{\rightarrow}
\begin{bmatrix}
\overrightarrow{x_{r[i]}} \\
\overrightarrow{x_{r[i+1]}}
\end{bmatrix}
\]

\( T \times (n+m) \) \( (n+m) \times 1 \) \( T \times 1 \)

\[
D \overset{\rightarrow}{Pr} \overset{\rightarrow}{S_0}
\]
Least squares estimation:

\[ \hat{P}_r = (D^T D)^{-1} D^T S_r \Rightarrow \hat{P}_r = M \cdot S_r \]

- \( D^T D \) is assumed to be invertible
- \( D^T D \) is \((n+m) \times (n+m)\)
- \( M \) does not depend on \( r \)
- \( M \) is \((n+m) \times T\)

\[ S_r = D \hat{P}_r \]

\[ e = S_r - D \hat{P}_r \]

\[ \hat{P}_r = M \cdot S_r \]

\[ P = M \cdot S^T \]

\[ \begin{bmatrix} x_1(0) & \ldots & x_n(0) \\ x_1(1) & \ldots & x_n(1) \end{bmatrix} \]
Least Squares: \[ A = M S \]

\[
\begin{bmatrix}
\hat{A} \\
\hat{B}
\end{bmatrix} = \begin{bmatrix}
X_0 \quad [T] \\
X_C \quad [T]
\end{bmatrix}
\]

\[
A = (D^TD)^{-1} D^TS
\]

\[
(n+m)n = n^2 + mn
\]

Parameters

Note: \(T \geq n+m\) Why?

(Since the Least Squares formulation for \(a^r \& b^r\) has \(n+m\) unknowns)
Validation

• How good is $\hat{P}$?
  – Gold Standard: Use it in the discrete-time model with to see if specific $\tilde{u}[i]'s$ result in the desired $\tilde{x}[i]'s$.
  – Silver Standard: Test $\hat{P}$ against some test data.

\[ \begin{align*}
  \tilde{x}_{\text{test}}[0] & \cdots \\
  \tilde{u}_{\text{test}}[0] & \cdots \\
\end{align*} \]
Is $D^T D$ invertible?

• $\text{Null}(D^T D) = \text{Null}(D)$.
  – Hence, if $D$ has linearly independent columns, $D^T D$ will also have linearly independent columns.
  – Hence, if $D$ has linearly independent columns, $D^T D$ is invertible.

• Highly likely that $D$ has linearly independent columns.
  – Also, we have some control over $D$.

• We’ll deal with the case of non-invertible $D^T D$ when we discuss SVD (and Moore-Penrose Pseudoinverse).
$$\text{Null}(D^TD) = \text{Null}(D)$$

Proof: Suppose $$\bar{v} \in \text{Null}(D^TD)$$

$$D^TD\bar{v} = 0$$

$$\Rightarrow \quad \bar{v}^T D^T D \bar{v} = 0$$

$$\Rightarrow (DV)^T DV = 0$$

$$\Rightarrow \quad \|DV\|^2 = 0$$

$$\Rightarrow \quad DV = 0 \Rightarrow \quad \bar{v} \in \text{Null}(D)$$

Suppose $$\bar{v} \in \text{Null}(D)$$

$$\Rightarrow \quad D\bar{v} = 0$$

$$\Rightarrow \quad D^T D\bar{v} = 0 \Rightarrow \quad \bar{v} \in \text{Null}(D^TD)$$
If \( D \) has linearly ind. columns

\[ = \quad D^TD \quad \text{has linearly ind. columns.} \]

**Proof:**

\[ D = \begin{bmatrix} v_1 & \cdots & v_m \end{bmatrix} \]

\( D \) is full column rank

\[
\text{rank}(D) + \text{nullity}(D) = m + n \leq m + n
\]

\[ \Rightarrow \quad \text{nullity}(D) = 0 \]

\[ \Rightarrow \quad \text{nullity}(D^TD) = 0 \]

\[ \Rightarrow \quad D^TD \quad \text{is full rank} \]