EECS 16B
Designing Information Devices and Systems II

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Admin:

1. Midterm is Monday of next week (Oct 16, 7-9pm)

2. Homework - the big picture…
   ○ Unlikely to determine your grade
   ○ Purpose is to solidify the material (so that you do well on exams, future classes, and in your career post-UCB)
   ○ Very important to do them on your own

3. Plan for today
   ○ Review/MT
   ○ Discrete Time
Signals and Systems

The systems we are studying are generally:
- Linear
- Time-Invariant
- Multiple Input / Multiple Output
Systems

Process:
1. Define inputs, outputs, state variables.
2. Model system using differential equations to capture system behavior.
3. Test, refine model as necessary.
State Space Representation

\[ \frac{d\vec{x}}{dt} = A\vec{x} + B\vec{u} \]
\[ \vec{y} = C\vec{x} \]

Notes:

- Input, output, and state vectors are signals. (These are \textit{time-varying}.)
- A, B, and C matrices capture system behavior. (These are \textit{time-invariant}.)
- We often ignore the output equation and treat the states as the outputs.
- Higher order derivatives (Newton’s Laws, etc) are handled by defining additional states.
Analysis, Path Planning, and Control

**Analysis:** Given inputs and initial state, find state trajectory.

**Path Planning:** Given initial state and desired final state, find feasible (or optimal) state trajectory.

**Control:** Given desired state trajectory, find inputs.
Change of Basis and Matrix Diagonalization

A basis of particular interest is the “eigenbasis” of a given square matrix.

\[ S^{-1}AS = \Lambda = \begin{bmatrix} \lambda_1 & & \\ & \lambda_2 & \\ & & \ddots \\ & & & \lambda_n \end{bmatrix} \]
Solving VDEs

Difficult due to coupling.

Strategy 1 (starting vector decomposition method):

1. Write starting state vector as a linear combination of eigenvectors:

\[ \bar{x}(0) = c_1 \bar{v}_1 + c_2 \bar{v}_2 + \cdots + c_n \bar{v}_n \]

2. For each eigenvector, \( A \) can be replaced by the corresponding \( \lambda \). State trajectory remains on the eigenvector.

3. Solution is the linear combination of the solutions for each eigenvector:

\[ \bar{x}(t) = c_1 e^{\lambda_1 t} \bar{v}_1 + c_2 e^{\lambda_2 t} \bar{v}_2 + \cdots + c_n e^{\lambda_n t} \bar{v}_n \]

\( \bar{x} \) and \( \frac{d\bar{x}}{dt} \) are co-linear.
Solving VDEs

Strategy 2 (change of basis method):

1. Change basis to eigenbasis.
2. System will be diagonal $\Rightarrow$ solve using initial conditions in new basis.
3. Change back to standard basis.
Midterm

For module 2 (systems and controls), be sure that you can:

● Given a circuit, write the differential equations.
● Convert differential equations to a VDE in state space form.
● Solve the system using change of basis method. (Homogeneous case only.)
Solving Homogeneous 2x2 VDE

Given: \( \frac{d\mathbf{x}}{dt} = A\mathbf{x} \)

\[ A = \begin{bmatrix} -4 & -3 \\ 1 & 0 \end{bmatrix} \]

\[ \mathbf{x}(0) = \begin{bmatrix} 1 \\ 1 \end{bmatrix} \]

1. Find \( \lambda, \mathbf{v}, \mathbf{v}', \mathbf{V}, \mathbf{V}^{-1} \)

\[ \begin{vmatrix} -4 - \lambda & -3 \\ 1 & -\lambda \end{vmatrix} = 0 \]

\[ \lambda^2 + 4\lambda + 3 = 0 \]

\[ (\lambda + 3)(\lambda + 1) = 0 \]

\[ \lambda_1 = -3 \]

\[ \begin{bmatrix} -1 & -3 \\ 1 & 3 \end{bmatrix} \mathbf{v}_1 = \mathbf{0} \]

\[ \mathbf{v}_1 = \begin{bmatrix} 3 \\ -1 \end{bmatrix} \]
\[ \lambda_2 = -1 \quad \begin{bmatrix} -3 & -3 \\ 1 & 1 \end{bmatrix} \overline{v_2} = 0 \quad \overline{v_2} = \begin{bmatrix} 1 \\ -1 \end{bmatrix} \]

\[ V = \begin{bmatrix} 3 & 1 \\ -1 & -1 \end{bmatrix} \quad V^{-1} = \frac{1}{2} \begin{bmatrix} -1 & -1 \\ 1 & 3 \end{bmatrix} = \begin{bmatrix} -\frac{1}{2} & \frac{1}{2} \\ -\frac{1}{2} & -\frac{3}{2} \end{bmatrix} \]

(2) Change to eigenvector basis

\[ \frac{d\overline{x}}{dt} = A \overline{x} \quad \overline{x} = V \overline{y} \quad \frac{d\overline{x}}{dt} = V \frac{d\overline{y}}{dt} \]

\[ vV \frac{d\overline{y}}{dt} = v \Lambda \overline{v} \overline{y} \]

\[ \overline{y} = V^{-1} \overline{x} \quad \Rightarrow \overline{y}(0) = V^{-1} x(0) \]

\[ \frac{d\overline{y}}{dt} = \Lambda \overline{y} = \begin{bmatrix} -3 & 0 \\ 0 & -1 \end{bmatrix} \overline{y} \]
\[ \mathbf{Y} = \begin{bmatrix} y_1(0) e^{-3t} \\ y_2(0) e^{t} \end{bmatrix} = \begin{bmatrix} e^{-3t} \\ -2 e^{t} \end{bmatrix} \]

\[ \mathbf{y}(0) = \begin{bmatrix} \frac{1}{2} \\ \frac{1}{2} \\ \frac{1}{2} \\ \frac{3}{2} \end{bmatrix} \]

Change back to std basis

\[ \mathbf{x} = \mathbf{V} \mathbf{Y} = \begin{bmatrix} 3 & 1 \\ -1 & -1 \end{bmatrix} \mathbf{Y} \]

\[ \mathbf{x} = \begin{bmatrix} 3 e^{-3t} - 2 e^{t} \\ -e^{-3t} + 2 e^{t} \end{bmatrix} \]

(4) check \( x(0) \)

\[ x(0) = \begin{bmatrix} 1 \end{bmatrix} \]

\[ \checkmark \]
Continuous Time vs Discrete Time

Discrete time signals may come from sampling CT signals.

There are also signals that are inherently DT, for example:
  - end-of-day stock prices
  - periodic population counts (census data, etc)
  - daily weather data
  - etc

Is CT vs DT the same as analog vs digital?...
Analog vs Digital

Not exactly…

Analog means CT \textit{and} values can be any real number.

Digital means DT \textit{and} a finite (although perhaps large) set of possible values.

We are going to focus on DT but not digital, i.e. we will assume values can be any real number.
CT vs DT Duality

**Continuous Time**

\[ y''(i) + 3y'(i) + 2y(t) = u(t) \]

differential equation

\[ \frac{d\bar{x}(t)}{dt} = A\bar{x}(t) + B\bar{u}(t) \]

state space form in CT

\[ x(t) = \sum_i C_i e^{r_i t} \]

form of homogeneous solution

**Discrete Time**

\[ y[n] + 3y[n - 1] + 2y[n - 2] = u[n] \]

difference equation

\[ \bar{x}[n + 1] = A_d\bar{x}[n] + B_d\bar{u}[n] \]

state space form in DT

\[ x[n] = \sum_i C_i r^n_i \]

form of homogeneous solution
Solving - DT Nonhomogeneous Case

For discrete time, we can solve directly by “unrolling” the difference equation, i.e. going back in time, step-by-step, all the way to $i=0$:

\[
\begin{align*}
\bar{x}[i] &= A\bar{x}[i - 1] + Bu[i - 1] \\
&= A(A\bar{x}[i - 2] + Bu[i - 2]) + Bu[i - 1] \\
&= A^2\bar{x}[i - 2] + ABu[i - 2] + Bu[i - 1] \\
&= A^2(A\bar{x}[i - 3] + Bu[i - 3]) + ABu[i - 2] + Bu[i - 1] \\
&= A^3\bar{x}[i - 3] + A^2Bu[i - 3] + ABu[i - 2] + Bu[i - 1] \\
&= \ldots \ldots \ldots \ldots
\end{align*}
\]

\[
\begin{align*}
&= A^{i-1}(A\bar{x}[0] + Bu[0]) + \sum_{k=1}^{i-1} A^{i-k-1}Bu[k] \\
&= A^i\bar{x}[0] + \sum_{k=0}^{i-1} A^{i-k-1}Bu[k] \\
&= A^i\bar{x}_0 + \sum_{k=0}^{i-1} A^{i-k-1}Bu[k].
\end{align*}
\]
Solving - CT Nonhomogeneous Case

\[ \frac{d\vec{x}}{dt} = A\vec{x} - \beta\vec{u} \]

\[ \nu^{-1} \frac{d\vec{y}}{dt} = \nu^T A \nu \vec{y} + \nu^T \beta \vec{u} \]

\[ \frac{d\vec{y}}{dt} = \Lambda \vec{y} + \nu^{-1} \beta \vec{u} \Rightarrow \text{Solve } n \text{ scalar equations} \]

Then to get solution in standard basis, use

\[ \vec{x} = \nu \vec{y} \]
Still need to solve scalar case.

\[ \frac{dx}{dt} = ax + bu \]

**Option 1** Solve for \( x_n \), guess for \( x_p \)

**Option 2** Use general solution:

\[ x(t) = x(0)e^{at} + \int_{0}^{t} be^{a(t-\tau)} u(\tau) d\tau \]
Discretization of a System

Want to describe this

\[ ZOH = \text{zero order hold} \]

\[ \Rightarrow \text{"stair step" signal based on } u[n] \]
Problem Statement

Given $A$, $B$, timestep $\Delta t$,
Find $A_d$ and $B_d$ s.t.

\[ x[n+1] = A_d x[n] + B_d u[n] \]
Strategy

1. Consider scalar case first. What happens during each timestep?

2. Diagonalize system.

3. Solve (using 1 above)

4. Convert back to std basis.

This is what you did in discussion 7A to get Ad and Bd.