Lecture 6: Impedances, Frequency Domain Theory, Complex Exponentials

EECS 16B Phasor
Outline

• Phasers
• Concept of Impedance
• AC Circuits
• Examples
AC Voltage, Current of Resistive Load

- \( v(t) = V_m \cos(\omega t) \)
  - Amplitude: \( V_m \)
  - Frequency: \( \omega = \frac{2\pi}{T} \)

- Resistive load:
  \( i(t) = \frac{v(t)}{R} = \frac{V_m}{R} \cos(\omega t) \)

- Seeing signals on Oscilloscope

\[ \frac{V_m}{R} \]

\[ f = \frac{1}{T}, \quad \omega = 2\pi f = \frac{2\pi}{T} \]
AC Power of Resistive Load

- $p(t) = v(t)i(t) = \left(\frac{V_m^2}{R}\right)\cos^2(\omega t)$

- Note: power is consumed in both positive and negative cycles

- $P_{avg} = \frac{1}{T} \int_0^T p(t) dt = \frac{1}{2} \frac{V_m^2}{R}$
AC Voltage, Current of Capacitive Load

- \( v(t) = V_m \cos(\omega t) \)

- Capacitive load:
  \[
  i(t) = C \frac{dv(t)}{dt} = -CV_m \omega \sin(\omega t)
  \]

- What’s the difference between \( \cos(\omega t) \) and \( -\sin(\omega t) \)?
  \[
  -\sin(\omega t) = \cos(\omega t + 90^\circ)
  \]

- Seeing signals on Oscilloscope

\[
\cos(\alpha + \beta) = \cos \alpha \cos \beta - \sin \alpha \sin \beta
\]

\[
\cos(\alpha - \beta) = \cos \alpha \cos \beta + \sin \alpha \sin \beta
\]
AC Power of Capacitive Load

- \( p(t) = v(t)i(t) \)
  \[ = -\omega C V_m^2 \cos(\omega t) \sin(\omega t) \]
  \[ = -\omega C V_m^2 \frac{1}{2} \sin(2\omega t) \]

- Note: power oscillates between positive and negative values

- \( P_{avg} = \frac{1}{T} \int_0^T p(t)dt = 0 \)

\( \Rightarrow \) Capacitive load does not consume energy (power)

Power is "slushed" back and forth
AC Voltage, Current of Inductive Load

- \[ v(t) = V_m \cos(\omega t) \]
- \[ i(t) = \frac{1}{L} \int v(t) dt = \frac{V_m}{\omega L} \sin(\omega t) \]
- Capacitive load:
  \[ v = L \frac{di}{dt} \]

- What’s the difference between \( \sin(\omega t) \), and \( \cos(\omega t) \)?

- Seeing signals on Oscilloscope

\[
\cos(\alpha + \beta) = \cos \alpha \cos \beta - \sin \alpha \sin \beta \\
\cos(\alpha - \beta) = \cos \alpha \cos \beta + \sin \alpha \sin \beta
\]
AC Power of Inductive Load

- \( p(t) = v(t)i(t) \)
  \[
  = \frac{V_m^2}{\omega L} \cos(\omega t) \sin(\omega t) \\
  = \frac{V_m^2}{\omega L} \frac{1}{2} \sin(2\omega t)
  \]

- Note: power oscillates between positive and negative values

- \( P_{avg} = \frac{1}{T} \int_0^T p(t) dt = 0 \)

  Inductive load does not consume power
Complex Exponential

\( e^{j\omega t} = \cos(\omega t) + j \sin(\omega t) \)

- Complex plane:
  - X-axis = real part
  - Y-axis = imaginary part

- \( e^{j\omega t} \) is a rotating vector on a unit circle
- \( \cos(\omega t) \) is the projection the vector on X-axis
Complex Exponential: Phase

- 2 vectors, $e^{j\omega t}$ and $e^{j(\omega t + \theta)}$
- Rotating at the same frequency $\omega$ but with a phase difference
- It is sufficient to keep track of the phases
“Phasors”

• Sinusoidal signals

\[ \nu(t) = V_m \cos(\omega t + \theta) = \text{Re} \left[ V_m e^{j(\omega t + \theta)} \right] \]

• For linear circuit, all signals have the same frequency, but their phases may be different

• It is sufficient to keep track of amplitude and phase

• \( \tilde{V} = V_m e^{j\theta} \) or \( V_m \angle \theta \) is called **Phasor**
Phasor along Imaginary Axis

- $1 \angle 90^\circ = j$
- $1 \angle -90^\circ = -j$
- $1 \angle 180^\circ =$
- $1 \angle 270^\circ =$
Phasor Algebra - Addition

• Same as any complex number
• $V_1 \angle \theta_1 + V_2 \angle \theta_2$
Phasor Algebra - Multiplication

• $H_1 \angle \theta_1 \times H_2 \angle \theta_2$
Phasor Algebra - Division

\[
\frac{H_1 \angle \theta_1}{H_2 \angle \theta_2} =
\]
Concept of Impedance

- Voltages and currents phasors (sinusoidal signals)
  \[ \tilde{V} = V_m \angle \theta_1 \]
  \[ \tilde{I} = I_m \angle \theta_2 \]
- Complex impedance
  \[ Z = \frac{\tilde{V}}{\tilde{I}} = \frac{V_m \angle \theta_1}{I_m \angle \theta_2} = \frac{V_m}{I_m} \angle (\theta_1 - \theta_2) \]
  - Real part is called resistance
  - Imaginary part is called reactance
Impedance of a Capacitor

\[ v(t) = V_m \cos(\omega t) \]
\[ \tilde{V} = V_m \angle 0 \]

\[ i(t) = C \frac{dv(t)}{dt} = -CV_m \omega \sin(\omega t) \]
\[ = CV_m \omega \cos(\omega t + 90^\circ) \]
\[ \tilde{I} = CV_m \omega \angle 90^\circ \]

\[ Z = \frac{V_m \angle 0}{CV_m \omega \angle 90^\circ} = \frac{1}{j\omega C} \]

\[ v(t) = V_m e^{j\omega t} \]

\[ i(t) = C \frac{dv}{dt} = j\omega CV_m e^{j\omega t} \]

\[ Z = \frac{v(t)}{i(t)} = \frac{V_m}{j\omega CV_m} = \frac{1}{j\omega C} \]
Impedance of an Inductor

\[ i(t) = I_m \cos(\omega t) \]
\[ \tilde{I} = I_0 \angle 0 \]
\[ v(t) = L \frac{di}{dt} = -\omega LI_m \sin(\omega t) \]
\[ = \omega LI_m \cos(\omega t + 90^\circ) \]
\[ \tilde{V} = \omega LI_m \angle 90^\circ \]
\[ Z = \frac{\omega LI_m \angle 90^\circ}{I_m \angle 0} = j\omega L \]

• Alternative method:
\[ i(t) = I_m e^{j\omega t} \]
\[ v(t) = L \frac{di}{dt} = j\omega LI_m \ e^{j\omega t} \]
\[ Z = \frac{v(t)}{i(t)} = \frac{j\omega LI_m}{I_m} = j\omega L \]
AC Power Consumption

• Power consumption of a circuit element with complex impedance

\[ \tilde{V} = V_m \angle \theta_1, \quad \tilde{I} = I_m \angle \theta_2 \]

\[ P_{avg} = \frac{1}{T} \int_0^T p(t) dt = \frac{1}{T} \int_0^T V_m I_m \cos(\omega t + \theta_1) \cos(\omega t + \theta_2) dt \]

\[ = \frac{V_m I_m}{2T} \int_0^T \left[ \cos(\theta_1 - \theta_2) + \cos(2\omega t + \theta_1 + \theta_2) \right] dt \]

\[ = \frac{V_m I_m}{2} \cos(\theta_1 - \theta_2) \]

\[
\begin{align*}
\cos(\alpha + \beta) &= \cos \alpha \cos \beta - \sin \alpha \sin \beta \\
\cos(\alpha - \beta) &= \cos \alpha \cos \beta + \sin \alpha \sin \beta
\end{align*}
\]
Analyzing Circuit Using Impedance

• Use impedance like how we use resistance
  – Voltage divider, current divider, etc.
• KCL and KVL
• Thevenin and Norton equivalent circuits
• Output voltage or current is complex number
  – Phase delays or advances w.r.t. input signals
  – Steady state solution with sinusoidal input
RC Low-Pass Filter

\[ v_{in}(t) \quad \]  
\[ R \quad C \quad v_{out} \]

Recall (from Lecture 1) that electronic building blocks referred to as "logic gates" are used to implement logical functions (NAND, NOR, NOT) in digital ICs.

- Any logical function can be implemented using these gates.
- A logic gate can be modeled as a simple R-C circuit:

\[
\begin{align*}
V_{out}(t) &= V_{in}(t) + RC \frac{d}{dt} \left( V_{in}(t) \right) \\
&= V_{in}(t) + RC \frac{d}{dt} \left( V_{in}(t) \right)
\end{align*}
\]

When the input voltage pulse width is large enough, the output pulse is not distorted. We need to wait for the output to reach a recognizable logic level, before changing the input again.

\[ V_{in} \quad R \quad C \quad V_{out} \]

Suppose a voltage pulse of width 5 µs and height 4 V is applied to the input of this circuit beginning at \( t = 0 \):

\[ R = 2.5 \text{ k}\Omega \]
\[ C = 1 \text{ nF} \]

- First, \( V_{out} \) will increase exponentially toward 4 V.
- When \( V_{in} \) goes back down, \( V_{out} \) will decrease exponentially back down to 0 V.

What is the peak value of \( V_{out} \)?

The output increases for 5 µs, or 2 time constants.

\[ \tau = RC \]

\[ \tau = 2.5 \mu \text{s} \]

It reaches 1-e^{-2} or 86% of the final value.

\[ 0.86 \times 4 \text{ V} = 3.44 \text{ V} \]

Example

\[ \tau = RC \]

First Order Circuits: Forced Response
Temporal Response

- Low frequency

- High frequency

- At corner frequency ($\omega = 1/RC$)
Frequency Response
Frequency Response in Log-Log Scale
Frequency Domain Theory

\[ e^{j\omega t} \rightarrow \text{Circuit} \rightarrow H(j\omega) e^{j\omega t} \]

\[ \cos(\omega t) = \text{Re} \left[ e^{j\omega t} \right] \rightarrow \text{Circuit} \rightarrow \text{Re} \left[ H(j\omega) e^{j\omega t} \right] \]
Frequency Response

- Transfer Function: \( H(j\omega) = \frac{V_{out}(j\omega)}{V_{in}(j\omega)} \)
- Magnitude: \(|H(j\omega)|\)
- Phase: \(\angle H(j\omega)\)