EECS 16B
Designing Information Devices and Systems II

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Lecture 6: Impedances, Frequency Domain Theory, Complex Exponentials

EECS 16B

Outline

- Phasers
- Concept of Impedance
- AC Circuits
- Examples

AC Voltage, Current of Resistive Load

- $v(t) = V_m \cos(\omega t)$
  - Amplitude:
  - Frequency:

- Resistive load:

$$i(t) = \frac{v(t)}{R} = \frac{V_m}{R}\cos(\omega t)$$

- Seeing signals on Oscilloscope

AC Power of Resistive Load

- $p(t) = v(t)i(t) = \frac{V_m^2}{R} cos^2(\omega t)$
- Note: power is consumed in both positive and negative cycles
- $P_{avg} = \frac{1}{T}\int_0^T p(t)dt = \frac{1}{2}\frac{V_m^2}{R}$

AC Voltage, Current of Capacitive Load

- $v(t) = V_m \cos(\omega t)$
- Capacitive load:
  $$i(t) = C\frac{dv(t)}{dt} = -CV_m\omega\sin(\omega t)$$
- What’s the difference between $\cos(\omega t)$ and $-\sin(\omega t)$?
- Seeing signals on Oscilloscope

AC Power of Capacitive Load

- $p(t) = v(t)i(t)$

  $= -\omega C V_m^2 \cos(\omega t) \sin(\omega t)$

  $= -\omega C V_m^2 \frac{1}{2} \sin(2\omega t)$
- Note: power oscillates between positive and negative values
- $P_{avg} = \frac{1}{T}\int_0^T p(t)dt = 0$

AC Voltage, Current of Inductive Load

- $v(t) = V_m \cos(\omega t)$
- Capacitive load:
  $$i(t) = \frac{1}{L}\int v(t)dt = \frac{V_m}{\omega L}\sin(\omega t)$$
- What’s the difference between $\sin(\omega t)$, and $\cos(\omega t)$?

- Seeing signals on Oscilloscope

AC Power of Inductive Load

- $p(t) = v(t)i(t)$

$$= \frac{V_m^2}{\omega L}\cos(\omega t)\sin(\omega t)$$

$$= \frac{V_m^2}{\omega L}\frac{1}{2}\sin(2\omega t)$$

- Note: power oscillates between positive and negative values
- $P_{avg} = \frac{1}{T}\int_0^T p(t)dt = 0$

Complex Exponential

- $e^{j\omega t} = \cos(\omega t) + j\,sin(\omega t)$
- Complex plane:
  - X-axis = real part
  - Y-axis = imaginary part
- $e^{j\omega t}$ is a rotating vector on a unit circle
- $\cos(\omega t)$ is the projection the vector on X-axis

Complex Exponential: Phase

- 2 vectors, $e^{j\omega t}$ and $e^{j(\omega t+\theta)}$
- Rotating at the same frequency ω but with a phase difference
- It is sufficient to keep track of the phases

"Phasors"

- Sinusoidal signals
- $v(t) = V_m \cos(\omega t + \theta) = \text{Re}\,[\,V_m e^{j(\omega t+\theta)}\,]$
- For linear circuit, all signals have the same frequency, but their phases may be different
- It is sufficient to keep track of amplitude and phase
- $\tilde{V} = V_m e^{j\theta}$ or $V_m \angle \theta$ is called **Phasor**

Phasor along Imaginary Axis

- $1\angle 90° = j$
- $1\angle -90° = -j$
- $1\angle 180° =$
- $1\angle 270° =$

Phasor Algebra - Addition

- Same as any complex number
- $V_1\angle\theta_1 + V_2\angle\theta_2$

Phasor Algebra - Multiplication

- $H_1\angle\theta_1 \times H_2\angle\theta_2$

Phasor Algebra - Division

$$\frac{H_1 \angle \theta_1}{H_2 \angle \theta_2} =$$

Concept of Impedance

- Voltages and currents phasors (sinusoidal signals)
- $\tilde{V} = V_m \angle \theta_1$
- $\tilde{I} = I_m \angle \theta_2$
- Complex impedance

$$Z = \frac{\tilde{V}}{\tilde{I}} = \frac{V_m \angle \theta_1}{I_m \angle \theta_2} = \frac{V_m}{I_m} \angle (\theta_1 - \theta_2)$$

  – Real part is called resistance
  – Imaginary part is called reactance

Impedance of a Capacitor

$$v(t) = V_m \cos(\omega t)$$
$$\tilde{V} = V_m \angle 0$$

$$i(t) = C\frac{dv(t)}{dt} = -CV_m\omega \sin(\omega t)$$
$$= CV_m\omega \ \cos(\omega t + 90°)$$
$$\tilde{I} = CV_m\omega \ \angle 90°$$

$$Z = \frac{V_m\angle 0}{CV_m\omega\angle 90°} = \frac{1}{j\omega C}$$

- Alternative method:

$$v(t) = V_m e^{j\omega t}$$

$$i(t) = C\frac{dv}{dt} = j\omega C V_m \ e^{j\omega t}$$

$$Z = \frac{v(t)}{i(t)} = \frac{V_m}{j\omega C V_m} = \frac{1}{j\omega C}$$

Impedance of an Inductor

$$i(t) = I_m \cos(\omega t)$$
$$\tilde{I} = I_0 \angle 0$$
$$v(t) = L\frac{di}{dt} = -\omega L I_m \sin(\omega t)$$
$$= \omega L I_m \cos(\omega t + 90°)$$

$$\tilde{V} = \omega L I_m \angle 90°$$

$$Z = \frac{\omega L I_m \angle 90°}{I_m \angle 0} = j\omega L$$

- Alternative method:

$$i(t) = I_m e^{j\omega t}$$

$$v(t) = L\frac{di}{dt} = j\omega L I_m \; e^{j\omega t}$$

$$Z = \frac{v(t)}{i(t)} = \frac{j\omega L I_m}{I_m} = j\omega L$$

AC Power Consumption

- Power consumption of a circuit element with complex impedance
- $\tilde{V} = V_m \angle \theta_1$, $\tilde{I} = I_m \angle \theta_2$

$$P_{avg} = \frac{1}{T}\int_0^T p(t)dt = \frac{1}{T}\int_0^T V_m I_m \cos(\omega t + \theta_1)\cos(\omega t + \theta_2)\, dt$$

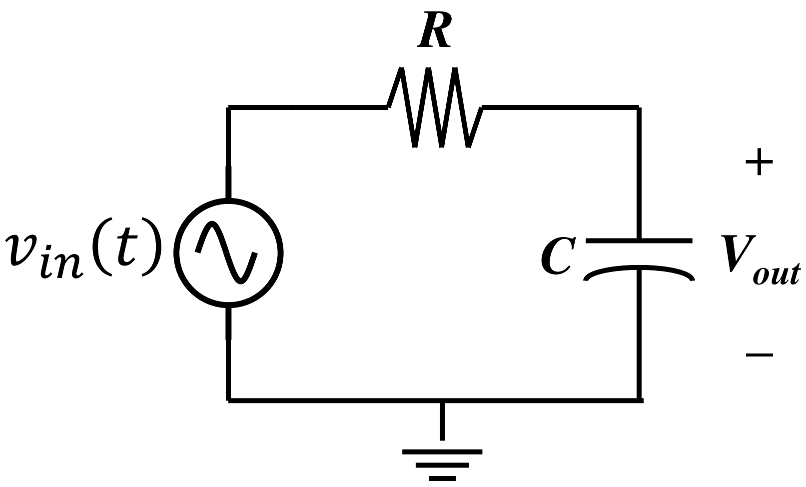
$$= \frac{V_m I_m}{2T}\int_0^T [\,\cos(\theta_1 - \theta_2) + \cos(2\omega t + \theta_1 + \theta_2)\,]\, dt$$

$$= \frac{V_m I_m}{2}\cos(\theta_1 - \theta_2)$$

Analyzing Circuit Using Impedance

- Use impedance like how we use resistance
  - Voltage divider, current divider, etc.
- KCL and KVL
- Thevenin and Norton equivalent circuits
- Output voltage or current is complex number
  - Phase delays or advances w.r.t. input signals
  - **Steady state solution** with **sinusoidal input**

RC Low-Pass Filter



Temporal Response

- Low frequency

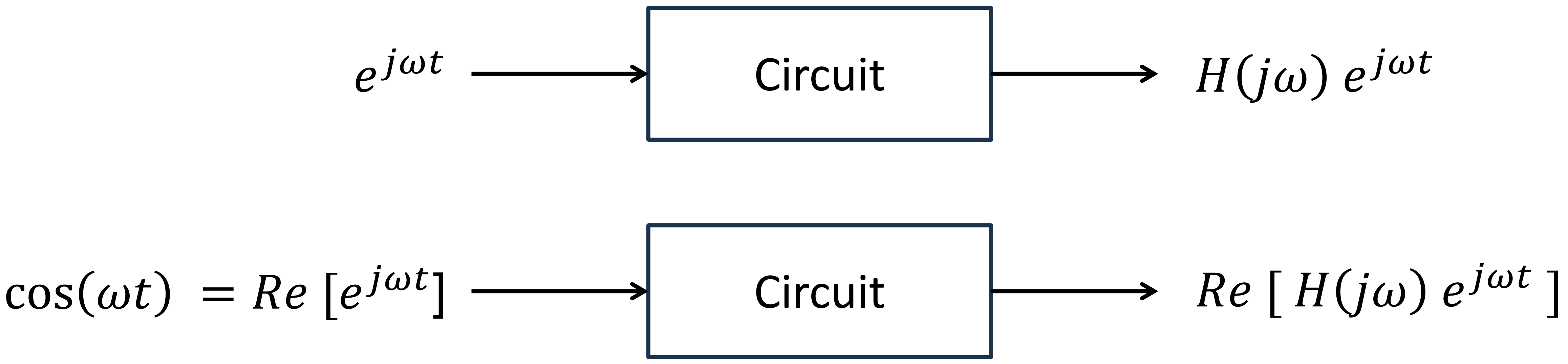
- High frequency

- At corner frequency ($\omega = 1/RC$)

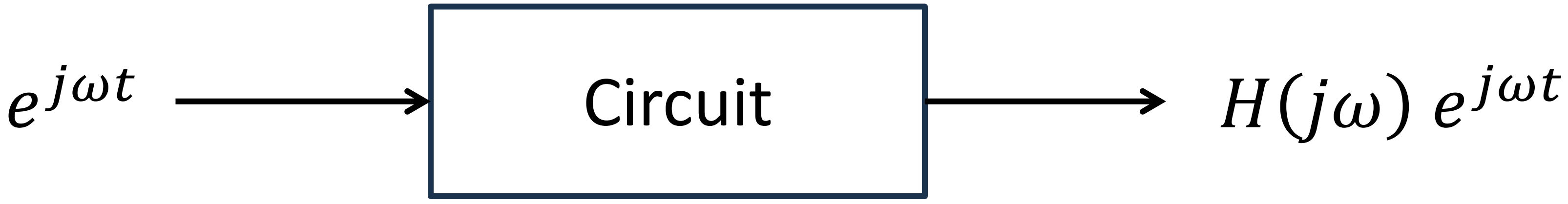
Frequency Response

Frequency Response in Log-Log Scale

Frequency Domain Theory



Frequency Response



- Transfer Function: $H(j\omega) = \frac{V_{out}(j\omega)}{V_{in}(j\omega)}$
- Magnitude: $|H(j\omega)|$
- Phase: $\angle H(j\omega)$