EECS 16B
Designing Information Devices and Systems II

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Lecture 5: RLC Circuits

EECS 16B
• This is a very important circuit, and we’ll spend some time understanding the behavior of the circuit.
KVL For Series RLC Circuit

Due to interaction between current in inductor and voltage on capacitor, we end up with a 2\textsuperscript{nd} order differential equation.

We must specify two initial conditions, the voltage (or charge) on the capacitor and the current (or flux) in the inductor.

\[ v_s = iR + v_c + v_L \]

\[ i = C \frac{dv_c}{dt} \]

\[ v_s = RC \frac{dv_c}{dt} + v_c + L \frac{di}{dt} \]
Solution for Constant Inputs

- We’ll solve the situation when we apply a constant input to the circuit at some time.
- Note the final value of the state of the circuit is predictable based on DC steady-state:

\[ V_{dd} = v_c(t) + RC \frac{dv_c}{dt} + LC \frac{d^2v_c}{dt^2} \]

\[ i = C \frac{dv_c}{dt} \]

Steady State Solution

\[ V_{dd} = v_c(\infty) \]

\[ t \to \infty \]

\[ v_c(t=\infty) = V_{dd} \]
Steady-State Solution

• Let’s simply plug in the steady-state solution and solve for the unknown transient solution, which is the solution to the homogeneous differential equation:

\[ v_c(t) = V_{dd} + v(t) \]

\[ V_{dd} = (V_{dd} + v(t)) + RC \frac{dv}{dt} + LC \frac{d^2v}{dt^2} \]

\[ LC \frac{d^2v}{dt^2} + RC \frac{dv}{dt} + V = 0 \rightarrow \text{Homogeneous D.E.} \]
Homogeneous Solution

- Try an exponential solution as before to satisfy the homogeneous equation:

\[
L \frac{d^2 v}{dt^2} + R \frac{dv}{dt} + \frac{1}{LC} v(t) = 0
\]

\[
\frac{d^2 v}{dt^2} + (2\alpha) \frac{dv}{dt} + \omega_0^2 v(t) = 0
\]

\[
\frac{d^2 v}{dt^2} + 2\alpha \frac{dv}{dt} + \omega_0^2 v = 0
\]

Try solutions: \( v(t) = A e^{st} \)

\[
s^2 + 2\alpha s + \omega_0^2 = 0
\]

Solution:

\[
\begin{align*}
  s_1 &= -\alpha + \sqrt{\alpha^2 - \omega_0^2} \\
  s_2 &= -\alpha - \sqrt{\alpha^2 - \omega_0^2}
\end{align*}
\]

\( \alpha > \omega_0 \Rightarrow 2 \text{ real numbers} \)

\( \alpha = \omega_0 \Rightarrow s_1 = s_2 = -\alpha \)

\( \alpha < \omega_0 \Rightarrow \sqrt{\alpha^2 - \omega_0^2} = \omega_n \)

\( \omega_n = \sqrt{\omega_0^2 - \alpha^2} \)
(1) Overdamped: $\alpha > \omega_0$

- $s_1$ and $s_2$ are both real number with distinctive negative values
- Solution: $v(t) = Ae^{s_1 t} + Be^{s_2 t}$
- Exponential decay functions

Initial condition determined by $A, B$.
(2) Critically Damped: $\alpha = \omega_0$

- $s_1$ and $s_2$ are both real and equal: $s_1 = s_2 = -\alpha$
- However, the original general solution $v(t) = Ae^{s_1t} + Be^{s_1t} = Ce^{-\alpha t}$
- Only one constant, cannot satisfy both initial conditions
- Need to modify the form of general solution:
  - $v(t) = Ae^{-\alpha t} + Bte^{-\alpha t}$
- For explanation and proof, see https://youtu.be/NW9JfMvIsxw
(3) Under Damped: $\alpha < \omega_0$

- $s_1$ and $s_2$ are complex numbers
  
- \[
  \begin{align*}
  s_1 &= -\alpha + j\omega_n \\
  s_2 &= -\alpha - j\omega_n
  \end{align*}
  \]
  where $j = \sqrt{-1}$ and $\omega_n = \sqrt{\omega_0^2 - \alpha^2}$

- Solution: \[
  v(t) = Ae^{-\alpha t} \cos(\omega_n t) + Be^{-\alpha t} \sin(\omega_n t)
  \]
  Also exponential decay, but it may overshoot before damped out

- Under damped resonance $\omega_0$ is the undamped resonance freq.
  $\omega_n$ is the oscillation freq. $\omega_n = 2\pi f_n$

- $\omega_0 = \omega_n$ when $\alpha = 0$, i.e. $R = 0$

- Euler's Formula
  
  \[
  e^{j\omega} = \cos\omega + j\sin\omega
  \]
Summary of Under, Critical, and Over Damped Solutions

\[ \zeta = \frac{\alpha}{\omega_0} : \text{damping ratio} \]

- \( \zeta < 1 \) Underdamped
- \( \zeta = 1 \) Critically Damped
- \( \zeta > 1 \) Overdamped

Critical damping \( \Rightarrow \) Fastest solution to reach steady state without oscillation
General Solution: Over Damped

- If the roots are distinct, the form of the general solution is as follows. We can find constants A and B from initial conditions.
- If both roots are real, we have two decaying exponentials:

\[ v_C(t) = V_{dd} + A \exp(s_1 t) + B \exp(s_2 t) \]

\[ s_1 = -\alpha + \sqrt{\alpha^2 - \omega_0^2} \]
\[ s_2 = -\alpha - \sqrt{\alpha^2 - \omega_0^2} \]

\[ v_C(0) = V_{dd} + A + B = 0 \]
\[ A s_1 + B s_2 = 0 \]
\[ A + B = -V_{dd} \]
\[ \sigma = \frac{s_1}{s_2} \]

\[ i(0) = C \frac{dv_C(t)}{dt} \bigg|_{t=0} = 0 \]
\[ A = -\frac{V_{dd}}{1 - \sigma} \]
\[ B = \frac{\sigma V_{dd}}{1 - \sigma} \]

\[ v_C(t) = V_{dd} \left( 1 - \frac{1}{1 - \sigma} (e^{s_1 t} - \sigma e^{s_2 t}) \right) \]
General Solution: Critically Damped

- \( v_c(t) = V_{dd} + Ae^{-\alpha t} + Bte^{-\alpha t} \)
- Initial conditions:
  - \( v_c(0) = 0, \quad V_{dd} + A = 0, \quad A = -V_{dd} \)
  - \( i_c(0) = C \frac{dv_c}{dt} = 0, \quad -\alpha A + B = 0, \quad B = \alpha A = -\alpha V_{dd} \)
  - \( v_c(t) = V_{dd} (1 - e^{-\alpha t} - \alpha te^{-\alpha t}) = V_{dd} [1 - (1 + \alpha t) e^{-\alpha t}] \)
- Fastest charging without overshoot (oscillation)!
Underdamped

- If the roots of the equation are underdamped, they are complex and lead to oscillatory behavior.

\[ v_C(t) = V_{dd} + A e^{-\alpha t} \cos(\omega_n t) + B e^{-\alpha t} \sin(\omega_n t) \]
General Solution: Underdamped

- Initial conditions:
  - $v_C(t) = 0$
  - $\frac{dv_C}{dt} \bigg|_{t=0} = 0$

- $v_C(t) = V_{dd} \left[ 1 - e^{-\alpha t} \left( \cos(\omega_n t) + \frac{\alpha}{\omega_n} \sin(\omega_n t) \right) \right]$
How much energy is stored in the “tank”?  

- An “LCR” circuit is often referred to as a “tank”
- Let’s assume the tank is lossless. Then the energy stored in the inductor and capacitor is given by:

\[ w_L = \frac{1}{2} L i^2(t) = \frac{1}{2} LI_M^2 \cos^2 \omega_0 t \]

\[ w_C = \frac{1}{2} C v_C^2(t) = \frac{1}{2} C \left( \frac{1}{C} \int i(\tau)d\tau \right)^2 \]

\[ w_C = \frac{1}{2} \frac{I_M^2}{\omega_0^2 C} \sin^2 \omega_0 t \]
Total Tank Energy

• If we sum the energy stored in the inductor and capacitor at any given time, we find that the sum is constant.

• Since the tank is lossless, this is logical and a statement of the conservation of energy.

• We observe that the maximum energy of the inductor or capacitor occurs when other is storing zero energy:

\[ w_s = w_L + w_C = \frac{1}{2} I_M^2 \left( L \cos^2 \omega_0 t + \frac{1}{\omega_0^2 C} \sin^2 \omega_0 t \right) = \frac{1}{2} I_M^2 L \]

\[ w_{L,\text{max}} = w_s = \frac{1}{2} I_M^2 L \quad w_{C,\text{max}} = w_s = \frac{1}{2} V_M^2 C \]
Lossy Case

- Now let’s introduce loss. The energy dissipated by the resistor per cycle is given by:

\[ w_d = P \cdot T = \frac{1}{2} I_M^2 R \cdot \frac{2\pi}{\omega_0} \]

- Comparing the energy lost to the energy stored in the inductor, we have:

\[ \frac{w_s}{w_d} = \frac{\frac{1}{2} L I_M^2}{\frac{1}{2} I_M^2 R \cdot \frac{2\pi}{\omega_0}} = \frac{\omega_0 L}{R} \cdot \frac{1}{2\pi} = \frac{Q}{2\pi} \]

\[ Q = 2\pi \frac{w_s}{w_d} \]
Parallel LCR

- Using the concept of “duality”, we expect the equations to take on the same form as before: