EECS 16B
Designing Information Devices and Systems II

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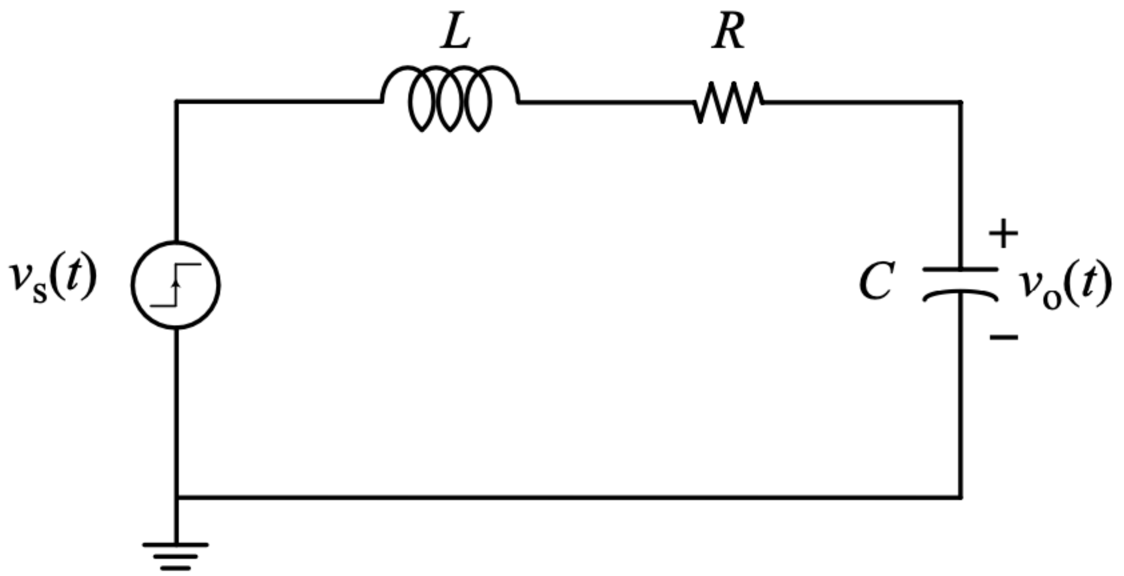
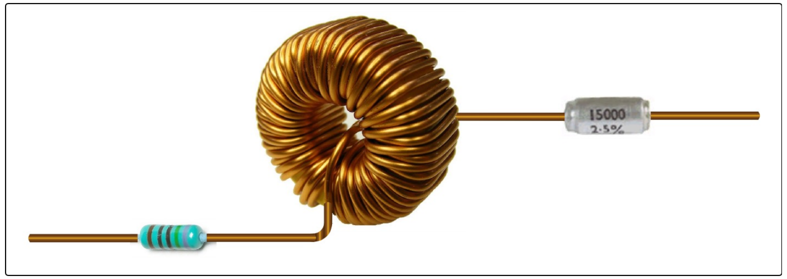
Lecture 5: RLC Circuits

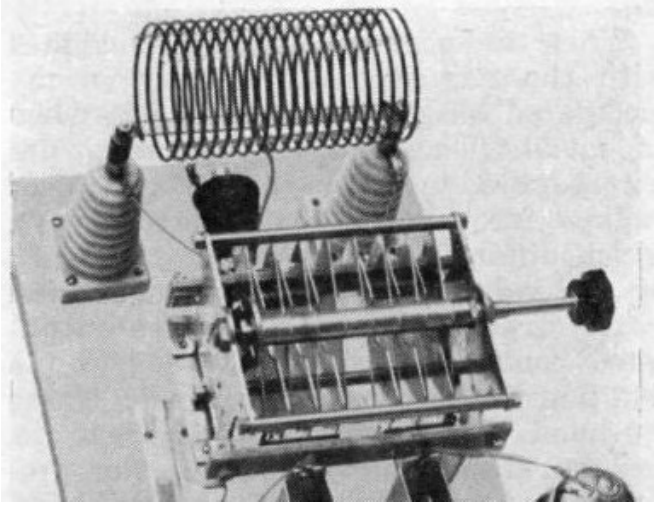
EECS 16B

Series RLC Circuit



- This is a very important circuit, and we'll spend some time understanding the behavior of the circuit.





Radio Tuner

Images from Wikipedia

KVL For Series RLC Circuit

$$v_s = iR + v_c + v_L$$

$$i = C\frac{dv_c}{dt}$$

$$v_s = RC\frac{dv_c}{dt} + v_c + L\frac{di}{dt}$$

$$v_o(0) = v_C(0) = 0\ V$$

$$i(0) = i_L(0) = 0\ A$$

- Due to interaction between current in inductor and voltage on capacitor, we end up with a 2^{nd} order differential equation
- We must specify two initial conditions, the voltage (or charge) on the capacitor and the current (or flux) in the inductor

Solution for Constant Inputs

- We’ll solve the situation when we apply a constant input to the circuit at some time.
- Note the final value of the state of the circuit is predictable based on DC steady-state:

$$V_{dd} = v_c(t) + RC\frac{dv_c}{dt} + LC\frac{d^2v_C}{dt^2}$$

$$V_{dd} = v_C(\infty)$$

Steady-State Solution

- Let's simply plug in the steady-state solution and solve for the unknown *transient* solution, which is the solution to the homogeneous differential equation:

$$v_c(t) = V_{dd} + v(t)$$

$$V_{dd} = V_{dd} + v(t) + RC\frac{dv}{dt} + LC\frac{d^2v}{dt^2}$$

Homogeneous Solution

- Try an exponential solution as before to satisfy the homogeneous equation:

$$LC\frac{d^2v}{dt^2} + RC\frac{dv}{dt} + v = 0$$

$$\frac{d^2v}{dt^2} + \frac{R}{L}\frac{dv}{dt} + \frac{1}{LC}v = 0$$

$$\frac{d^2v}{dt^2} + 2\alpha\frac{dv}{dt} + \omega_0^2\ v = 0$$

where

$\alpha = \frac{R}{2L}$: damping Coefficient

$\omega_0 = \frac{1}{\sqrt{LC}}$: resonant frequency

Try solutions: $v(t) = A\,e^{st}$

$$s^2 + 2\alpha\,s + \omega_0^2 = 0$$

Solution:

$$s_1 = -\alpha + \sqrt{\alpha^2 - \omega_0^2}$$

$$s_2 = -\alpha - \sqrt{\alpha^2 - \omega_0^2}$$

(1) Overdamped: $\alpha > \omega_0$

- s_1 and s_2 are both real number with distinctive negative values
- Solution: $v(t) = Ae^{s_1 t} + Be^{s_2 t}$
- Exponential decay functions

(2) Critically Damped: $\alpha = \omega_0$

- s_1 and s_2 are both real and equal: $s_1 = s_2 = -\alpha$
- However, the original general solution $v(t) = Ae^{s_1 t} + Be^{s_1 t} = Ce^{-\alpha t}$
- Only one constant, cannot satisfy both initial conditions
- Need to modify the form of general solution:

- $v(t) = Ae^{-\alpha t} + Bte^{-\alpha t}$

- For explanation and proof, see https://youtu.be/NW9JfMvIsxw

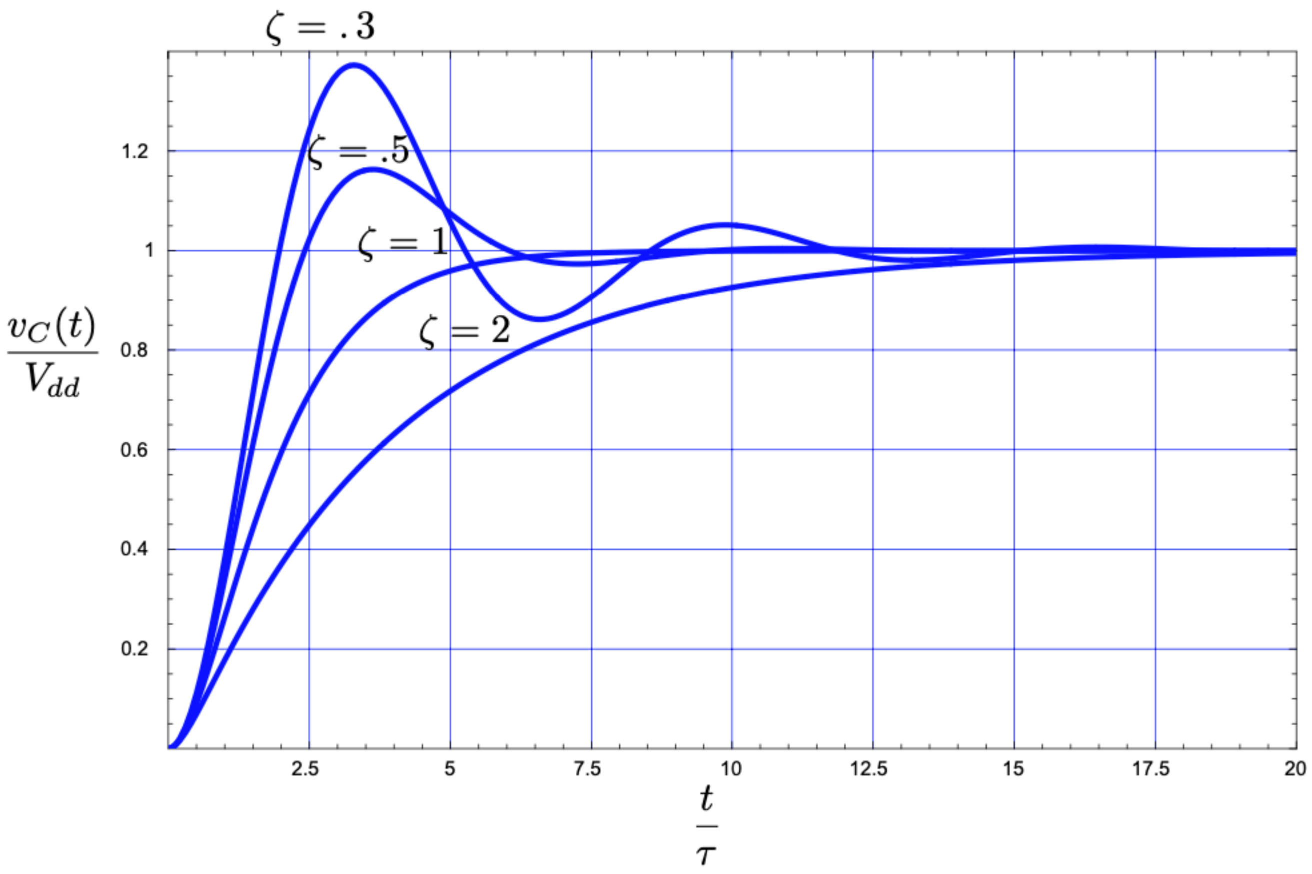
(3) Under Damped: $\alpha < \omega_0$

- s_1 and s_2 are are complex numbers
- $s_1 = -\alpha + j\omega_n$
- $s_2 = -\alpha - j\omega_n$

where $j = \sqrt{-1}$ and $\omega_n = \sqrt{\omega_0^2 - \alpha^2}$

- Solution: $v(t) = Ae^{-\alpha t}\cos(\omega_n) + Be^{-\alpha t}\sin(\omega_n)$
- Also exponential decay, but it may overshoot before damped out

Summary of Under, Critical, and Over Damped Solutions



$$\zeta = \frac{\alpha}{\omega_0} : \text{damping ratio}$$

$\zeta < 1$ Underdamped

$\zeta = 1$ Critically Damped

$\zeta > 1$ Overdamped

General Solution: Over Damped

- If the roots are distinct, the form of the general solution is as follows. We can find constants A and B from initial conditions.
- If both roots are real, we have two decaying exponentials:

$$v_C(t) = V_{dd} + A\exp(s_1 t) + B\exp(s_2 t)$$

$$s_1 = -\alpha + \sqrt{\alpha^2 - \omega_0^2} \qquad s_2 = -\alpha - \sqrt{\alpha^2 - \omega_0^2}$$

$$v_C(0) = V_{dd} + A + B = 0$$

$$i(0) = C\frac{dv_C(t)}{dt}|_{t=0} = 0$$

$$As_1 + Bs_2 = 0$$

$$A + B = -V_{dd}$$

$$A = \frac{-V_{dd}}{1-\sigma}$$

$$B = \frac{\sigma V_{dd}}{1-\sigma}$$

$$\sigma = \frac{s_1}{s_2}$$

$$v_C(t) = V_{dd}\left(1 - \frac{1}{1-\sigma}\left(e^{s_1 t} - \sigma e^{s_2 t}\right)\right)$$

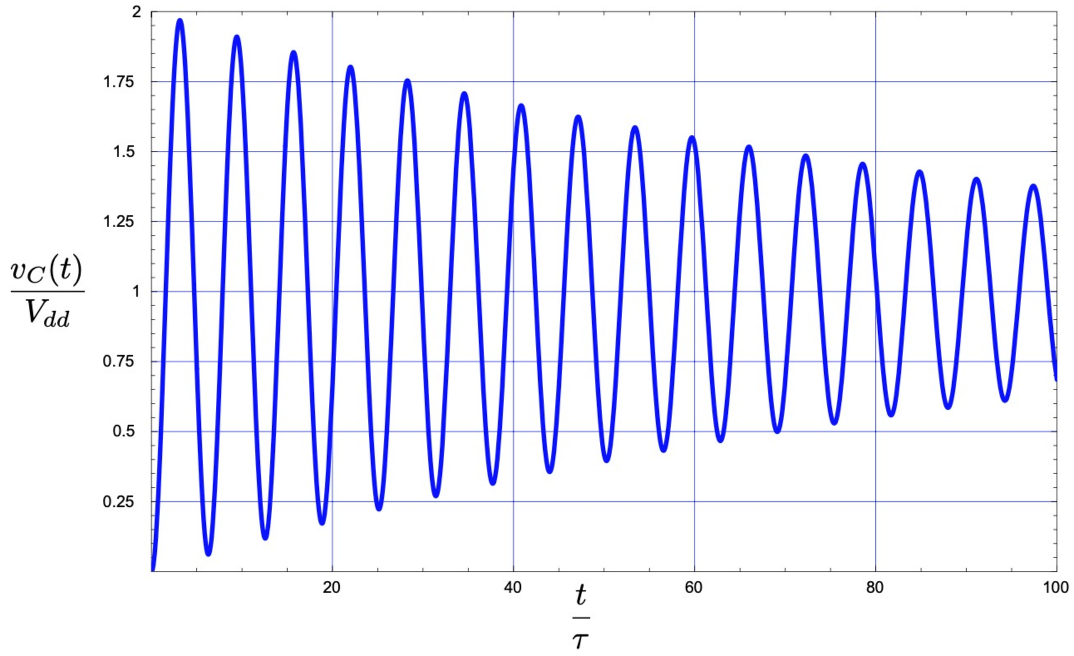
General Solution: Critically Damped

- $v_c(t) = V_{dd} + Ae^{-\alpha t} + Bte^{-\alpha t}$
- Initial conditions:
- $v_c(0) = 0, \quad V_{dd} + A = 0, \; A = -V_{dd}$
- $i_C(0) = C\frac{dv_C}{dt} = 0, \quad -\alpha A + B = 0, \quad B = \alpha A = -\alpha V_{dd}$
- $v_c(t) = V_{dd}(1 - e^{-\alpha t} - \alpha te^{-\alpha t}) = V_{dd}[1 - (1 + \alpha t)\, e^{-\alpha t}]$

- Fastest charging without overshoot (oscillation)!

Underdamped

- If the roots of the equation are underdamped, they are complex and lead to oscillatory behavior



$$v_C(t) = Vdd + Ae^{-\alpha t}\cos(\omega_n t) + Be^{-\alpha t}\sin(\omega_n t)$$

General Solution: Underdamped

- Initial conditions:
- $v_C(t) = 0$
- $\left.\frac{dv_C}{dt}\right|_{t=0} = 0$
- $v_C(t) = V_{dd}\left[1 - e^{-\alpha t}\left(\cos(\omega_n t) + \frac{\alpha}{\omega_n}\sin(\omega_n t)\right)\right]$

How much energy is stored in the “tank”?

- An “LCR” circuit is often referred to as a “tank”
- Let’s assume the tank is lossless. Then the energy stored in the inductor and capacitor is given by:

$$w_L = \tfrac{1}{2}Li^2(t) = \tfrac{1}{2}LI_M^2 \cos^2 \omega_0 t$$

$$w_C = \tfrac{1}{2}Cv_C^2(t) = \tfrac{1}{2}C\left(\frac{1}{C}\int i(\tau)d\tau\right)^2$$

$$w_C = \tfrac{1}{2}\frac{I_M^2}{\omega_0^2 C}\sin^2 \omega_0 t$$

Total Tank Energy

- If we sum the energy stored in the inductor and capacitor at any given time, we find that the sum is constant.
- Since the tank is lossless, this is logical and a statement of the conservation of energy.
- We observe that the maximum energy of the inductor or capacitor occurs when other is storing zero energy:

$$w_s = w_L + w_C = \tfrac{1}{2} I_M^2 \left(L \cos^2 \omega_0 t + \frac{1}{\omega_0^2 C} \sin^2 \omega_0 t \right) = \tfrac{1}{2} I_M^2 L$$

$$w_{L,\max} = w_s = \tfrac{1}{2} I_M^2 L \qquad w_{C,\max} = w_s = \tfrac{1}{2} V_M^2 C$$

Lossy Case

- Now let's introduce loss. The energy dissipated by the resistor per cycle is given by:

$$w_d = P \cdot T = \tfrac{1}{2} I_M^2 R \cdot \frac{2\pi}{\omega_0}$$

- Comparing the energy lost to the energy stored in the inductor, we have:

$$\frac{w_s}{w_d} = \frac{\frac{1}{2} L I_M^2}{\frac{1}{2} I_M^2 R \frac{2\pi}{\omega_0}} = \frac{\omega_0 L}{R} \frac{1}{2\pi} = \frac{Q}{2\pi}$$

$$Q = 2\pi \frac{w_s}{w_d}$$

Parallel LCR

- Using the concept of "duality", we expect the equations to take on the exam same form as before: