EE16B Designing Information Devices and Systems II

Lecture 12A
Sampling
Aliasing
Discrete Signals
Intro

• Last time:
  – Interpolation
  – Started the sampling theorem

• Today:
  – Sampling theorem
  – Aliasing
  – Discrete signals
Sampling and Recovery

- Can we perfectly recover an analog signal from its samples?

Analog signal:
\[ y(x) = f(x) \]

Sample:
\[ y[n] = f(n\Delta) \]

Interpolate:
\[ \hat{f}(x) = \sum_{n=-\infty}^{\infty} y[n] \Phi(x - n\Delta) = f(x) \]
Sampling a sinusoid

- What rate should you be sampling a sinusoid?
Bandlimitedness

• The sinc function does not contain frequencies beyond a certain bandwidth

\[
\text{sinc}(x) = \frac{1}{\pi} \int_{0}^{\pi} \cos(\omega x) d\omega
\]

\[
= \frac{\sin(\pi x)}{\pi x} \bigg|_{0}^{\pi} = \frac{\sin \pi x}{\pi x} \quad x \neq 0
\]

\[
\text{sinc} \left( \frac{x}{\Delta} \right) = \frac{1}{\pi} \int_{0}^{\pi} \cos \left( \frac{\omega}{\Delta} x \right) d\omega \quad \Rightarrow \omega_{\text{max}} = \frac{\pi}{\Delta}
\]
Sampling Theorem

- If \( f(x) \) is bandlimited by frequency \( \omega_{\text{max}} \), then

\[
f(x) = \hat{f}(x) = \sum_{n=-\infty}^{\infty} y[n] \Phi(x - n\Delta)
\]

\[\Phi(x) = \text{sinc} \left( \frac{x}{\Delta} \right)\]

As long as,

\[
\omega_{\text{max}} < \frac{\pi}{\Delta}
\]

\[
\frac{\omega_{\text{max}}}{\pi} < \frac{1}{\Delta}
\]

\[
\frac{\omega_{\text{max}}}{2\pi} < \frac{1}{\Delta}
\]

\[
2f_{\text{max}} < f_s
\]

\[
\omega_s > 2\omega_{\text{max}}
\]

Proof: EE120, EE123
Examples

• Audio Signals:
  – Can hear up to 18-20KHz
  – Sampling 44.1KHz, or 48KHz

• Speech: 500Hz – 3.5KHz
  – MSP430 samples at about 2.8KHz. Is that enough?
Example 1

\[ f(x) = \cos\left(\frac{2\pi}{3} x\right) \quad \Delta = 1 \quad \omega_{\text{max}} < \frac{\pi}{\Delta} \]
Example 1

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Example 1

$$f(x) = \cos\left(\frac{2\pi}{3} x\right) \quad \Delta = 1 \quad \omega_{\text{max}} < \frac{\pi}{\Delta}$$
Example 1

\[ f(x) = \cos\left(\frac{2\pi}{3} x \right) \quad \Delta = 1 \quad \omega_{\text{max}} < \frac{\pi}{\Delta} \]
Example 1

\[ f(x) = \cos\left(\frac{2\pi}{3} x\right) \quad \Delta = 1 \quad \omega_{\text{max}} < ? \frac{\pi}{\Delta} \]
Example 1

\[ f(x) = \cos\left(\frac{2\pi}{3} x\right) \quad \Delta = 1 \quad \omega_{\max} \leq ? \frac{\pi}{\Delta} \]
Example 1

\[ f(x) = \cos\left(\frac{2\pi}{3} x\right) \quad \Delta = 1 \quad \omega_{\text{max}} < ? \frac{\pi}{\Delta} \]
Example 2

\[ f(x) = \cos\left(\frac{4\pi}{3} x\right) \quad \Delta = 1 \quad \omega_{\text{max}} < \frac{\pi}{\Delta} \]
Example 2

\[ f(x) = \cos \left( \frac{4\pi}{3} x \right) \quad \Delta = 1 \quad \omega_{\text{max}} < ? \frac{\pi}{\Delta} \]
Example 2

\[ f(x) = \cos\left(\frac{4\pi}{3}x\right) \quad \Delta = 1 \quad \omega_{\text{max}} < ? \frac{\pi}{\Delta} \]

\[ \hat{f}(x) = \cos\left(\frac{2\pi}{3}x\right) \]

• Sinc interpolation gives:  

Aliasing of high frequencies into lower ones!
Aliasing and Phase Reversal

\[ f(x) = \cos(\omega x + \phi) \quad \Delta = 1 \]

\[ y[n] = \cos(\omega n + \phi) \]

- Highest interpolated frequency will not be higher than \( \pi \)

\[ y[n] = \cos(\omega n + \phi) = \cos(2\pi n - (\omega n + \phi)) = \cos((2\pi - \omega)n - \phi) \]

If \( \pi < w < 2\pi \) and \( \Delta = 1 \), there’s an equivalent lower frequency signal with the same samples!

\[ \hat{f}(x) = \cos((2\pi - \omega)x - \phi) \]
Example 2

\[ f(x) = \cos(\omega x + \phi) \quad \Delta = 1 \quad \omega = \frac{4\pi}{3} \quad \phi = 0 \]

\[ \hat{f}(x) = \cos((2\pi - \omega)x - \phi) \]

\[ = \cos\left(\frac{2\pi}{3}x\right) \]
Example 3

\[ f(x) = \sin(1.9\pi x) \quad \Delta = 1 \]

\[ = \cos(1.9\pi x - \frac{\pi}{2}) \]

\[ \hat{f}(x) = \cos(0.1\pi x + \frac{\pi}{2}) \]

\[ = -\sin(0.1\pi x) \]
Aliasing Demo
Aliasing in video

https://www.youtube.com/watch?v=cxddi8m_mzk
Aliasing in Images
Aliasing in MRI

Of parallel imaging methods, those that work in the raw data or k-space domain and those that work in the image domain. The most common k-space-based techniques are Generalized Autocalibrating Partially Parallel Acquisitions (GRAPPA) and Autocalibrating Reconstruction for Cartesian (ARC) imaging, whereas SENSitivity Encoding (SENSE) is the most common image-based method. Both techniques derive their speed advantage by subsampling k-space such that a reduced number of phase encode steps are acquired. If the overall extent of k-space is unchanged but there is a greater spacing between phase encoding steps, image resolution is maintained but the phase FOV is reduced. As described above, reducing the phase FOV can result in phase aliasing or wrap. In k-space-based parallel imaging techniques, the coil sensitivity information is used to synthesize the missing lines of k-space so that the resultant images are free from aliasing, whereas in the image-based techniques, the coil sensitivity information is used to unwrap the images after reconstruction. One typical artifact that is often seen with SENSE type reconstruction is shown in Figure 12a, where phase aliasing was present in the image before parallel imaging was applied. In a standard, i.e., nonparallel, imaging acquisition a small amount of wrap-around at the edges of an image, due to a reduced phase FOV, can often be tolerated. However, in the case of a SENSE-based acquisition and reconstruction such phase aliasing appears in the center of the image and may often cause problems in diagnosis, particularly if the artifact is subtle. For this reason, it is especially important to prescribe a large enough phase FOV, to accommodate the anatomy being imaged, whenever SENSE is used. As described above SENSE parallel imaging implementations require knowledge of the individual coil spatial sensitivities to unwrap the image. These sensitivity maps are generally obtained before image acquisition as part of a calibration acquisition. There is, therefore, an assumption that the coil sensitivity profile does not change between acquisition of the calibration data and the image. However, if the patient or the coils move, the calibration may be suboptimal and incomplete.
Discrete Time Signals

• Samples of a CT signal:

\[ y[n] = f(n\Delta) \]

• Or, inherently discrete (Examples?)
Discrete Time Signals

• At their core are “just samples”!
Basic Sequences

- Unit Impulse
  \[ \delta[n] = \begin{cases} 
  1 & n = 0 \\
  0 & n \neq 0 
\end{cases} \]

- Unit Step
  \[ U[n] = \begin{cases} 
  1 & n \geq 0 \\
  0 & n < 0 
\end{cases} \]
Basic Sequences

- Exponential

\[ y[n] = \begin{cases} 
  A\alpha^n & n \geq 0 \\
  0 & n < 0 
\end{cases} \]

- **Bounded**
  - \( 0 < \alpha < 1 \) \( \ldots \)
  - \( -1 < \alpha < 0 \) \( \ldots \)

- **unBounded**
  - \( \alpha > 1 \) \( \ldots \)
  - \( \alpha < -1 \) \( \ldots \)
Discrete Sinusoids

\[ y[n] = A \cos(\omega_0 n + \phi) \]

or, \[ y[n] = Ae^{j\omega_0 n + j\phi} \]

Q: Is \( y[n] \) periodic? \[ y[n + N] = y[n] \quad |N \in \text{Integer} \]

Q: Only if \( \omega_0 / \pi \) is rational

• To find fundamental period, \( N \)
  – Find the smallest integers \( N, K \):

\[ \omega_0 N = 2\pi K \]
Discrete Sinusoids

- Examples:

\[ \cos\left(\frac{\pi}{5n}\right) \quad N = 10 \quad K = 1 \]
\[ \cos\left(\frac{5\pi}{7}n\right) \quad N = 14 \quad K = 5 \]

\[ \cos\left(\frac{5\pi}{7}n\right) + \cos\left(\frac{\pi}{5n}\right) \quad \Rightarrow N = \text{S.C.M}\{10, 14\} = 70 \]

\[ \omega_0 N = 2\pi K \]
Discrete Sinusoids

Q: Which signal has a higher frequency?

\[ \cos(\pi n) \]

\[ \cos\left(\frac{3\pi}{2}n\right) = \cos\left(\frac{\pi}{2}n\right) \]
Discrete Sinusoids

• What’s the lowest discrete frequency?

\[ y[n] = \cos(0n) = 1 \]

• What’s the highest discrete frequency?

\[ y[n] = \cos(\pi n) = (-1)^n \]
Discrete Sinusoids

\[ \cos(w_0 n) \]

- \( \omega_0 = 0 \)
- \( \omega_0 = \pi/8 \)
- \( \omega_0 = \pi/4 \)
- \( \omega_0 = \pi \)
Discrete Sinusoids

$\cos(w_0 n)$

$\omega_0 = 2\pi$

$\omega_0 = \frac{15}{8} \pi$

$\omega_0 = \frac{7}{4} \pi$

$\omega_0 = \pi$
Complex Frequencies

- Sinusoids are sums of left and right rotating complex exponentials

\[ 2 \cos(\omega t) = e^{j\omega t} + e^{-j\omega t} \]

“Positive” and “Negative” frequencies

Discrete frequencies with period N:

\[ y[n] = e^{j2\pi n/N} \]

\[ W_N \overset{\Delta}{=} e^{j2\pi/N} \Rightarrow y[n] = W_N^n \]
Complex Frequencies

\[ W_N \triangleq e^{\frac{j2\pi}{N}} \Rightarrow y[n] = W_N^n \]

- **N = 4** \[ y[n] = W_4^n \]

- **N = 6, neg. freq.** \[ y[n] = W_6^{-n} \]