This homework is due on Thursday, November 5, 2020, at 10:59PM.

Self-grades are due on Thursday, November 12, 2020, at 10:59PM.

1 SVD I

Find the singular value decomposition of the following matrix (leave all work in exact form, not decimal):

$$A = \begin{bmatrix} 2 & 2 \\ 3 & -3 \end{bmatrix}$$

a) Find the eigenvalues of $AA^\top$ and order them from largest to smallest, $\lambda_1 > \lambda_2$.

Solution

$$\lambda_1 = 18 \quad , \quad \lambda_2 = 8$$

b) Find orthonormal eigenvectors $\vec{u}_i$ of $AA^\top$ (all eigenvectors are mutually orthogonal and unit length).

Solution

$$\vec{u}_1 = \begin{bmatrix} 0 \\ 1 \end{bmatrix} \quad , \quad \vec{u}_2 = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$$

c) Find the singular values $\sigma_i = \sqrt{\lambda_i}$. Find the $\vec{v}_i$ vectors from:

$$A^\top \vec{u}_i = \sigma_i \vec{v}_i$$

Solution

$$\sigma_1 = 3\sqrt{2} \quad , \quad \sigma_2 = 2\sqrt{2}$$

$$\vec{v}_1 = \begin{bmatrix} \frac{1}{\sqrt{2}} \\ -\frac{1}{\sqrt{2}} \end{bmatrix} \quad , \quad \vec{v}_2 = \begin{bmatrix} \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} \end{bmatrix}$$

d) Write out A as a weighted sum of rank-1 matrices:

$$A = \sigma_1 \vec{u}_1 \vec{v}_1^\top + \sigma_2 \vec{u}_2 \vec{v}_2^\top$$

Solution

$$A = 3\sqrt{2} \begin{bmatrix} 0 & 0 \\ \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} \end{bmatrix} + 2\sqrt{2} \begin{bmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ 0 & 0 \end{bmatrix}$$

2 Rank 1 Decomposition

In this problem, we will decompose a few images into linear combinations of rank 1 matrices. Remember that outer product of two vectors $\vec{s}\vec{g}^T$ gives a rank 1 matrix. It has rank 1 because clearly, the column span is one-dimensional — multiples of $\vec{s}$ only — and the row span is also one dimensional — multiples of $\vec{g}^T$ only.

For example, if $\vec{s}$ and $\vec{g}$ are two vectors of dimension 5, then $\vec{s}\vec{g}^T$ is given as follows.

$$\vec{s} = \begin{bmatrix} s_1 \\ s_2 \\ s_3 \\ s_4 \\ s_5 \end{bmatrix} \qquad \vec{g} = \begin{bmatrix} g_1 \\ g_2 \\ g_3 \\ g_4 \\ g_5 \end{bmatrix}$$

$$\vec{s}\vec{g}^T = \begin{bmatrix} s_1 \\ s_2 \\ s_3 \\ s_4 \\ s_5 \end{bmatrix} \begin{bmatrix} g_1 & g_2 & g_3 & g_4 & g_5 \end{bmatrix} = \begin{bmatrix} s_1g_1 & s_1g_2 & s_1g_3 & s_1g_4 & s_1g_5 \\ s_2g_1 & s_2g_2 & s_2g_3 & s_2g_4 & s_2g_5 \\ s_3g_1 & s_3g_2 & s_3g_3 & s_3g_4 & s_3g_5 \\ s_4g_1 & s_4g_2 & s_4g_3 & s_4g_4 & s_4g_5 \\ s_5g_1 & s_5g_2 & s_5g_3 & s_5g_4 & s_5g_5 \end{bmatrix}$$

a) Consider a standard 8×8 chessboard shown in Figure 1. Assume that black colors represent -1 and that white colors represent 1.

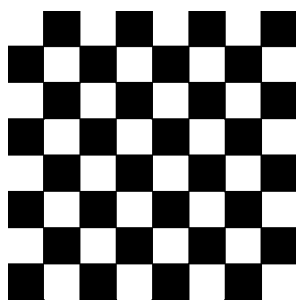

Figure 1: 8×8 chessboard.

Hence, that the chessboard is given by the following 8×8 matrix C_1:

$$C_1 = \begin{bmatrix} 1 & -1 & 1 & -1 & 1 & -1 & 1 & -1 \\ -1 & 1 & -1 & 1 & -1 & 1 & -1 & 1 \\ 1 & -1 & 1 & -1 & 1 & -1 & 1 & -1 \\ -1 & 1 & -1 & 1 & -1 & 1 & -1 & 1 \\ 1 & -1 & 1 & -1 & 1 & -1 & 1 & -1 \\ -1 & 1 & -1 & 1 & -1 & 1 & -1 & 1 \\ 1 & -1 & 1 & -1 & 1 & -1 & 1 & -1 \\ -1 & 1 & -1 & 1 & -1 & 1 & -1 & 1 \end{bmatrix}$$

Express C_1 as a linear combination of outer products. *Hint: In order to determine how many rank 1 matrices you need to combine to represent the matrix, find the rank of the matrix you are trying to represent.*

Solution

The matrix C_1 only has rank 1, since column vectors 1, 3, 5, and 7 are the same, and column vectors 2, 4, 6, and 8 are multiples of the other columns. This means that we can express C_1 by multiplying the first column vector $\begin{bmatrix} 1 & -1 & 1 & -1 & 1 & -1 & 1 & -1 \end{bmatrix}^T$ by the multiples required to generate the other columns, which are $1, -1, 1, \ldots, -1$. As a result, we get the following outer product form:

$$C_1 = \begin{bmatrix} 1 \\ -1 \\ 1 \\ -1 \\ 1 \\ -1 \\ 1 \\ -1 \end{bmatrix} \begin{bmatrix} 1 \\ -1 \\ 1 \\ -1 \\ 1 \\ -1 \\ 1 \\ -1 \end{bmatrix}^T$$

b) For the same chessboard shown in Figure 1, now assume that black colors represent 0 and that white colors represent 1.

Hence, the chessboard is given by the following 8×8 matrix C_2:

$$C_2 = \begin{bmatrix} 1 & 0 & 1 & 0 & 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 & 0 & 1 & 0 & 1 \\ 1 & 0 & 1 & 0 & 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 & 0 & 1 & 0 & 1 \\ 1 & 0 & 1 & 0 & 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 & 0 & 1 & 0 & 1 \\ 1 & 0 & 1 & 0 & 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 & 0 & 1 & 0 & 1 \end{bmatrix}$$

Express C_2 as a linear combination of outer products.

Solution

The chessboard is now a rank 2 image, so we need to decompose it.

There are multiple valid solutions. This is just one of them. Give yourself full credit for any valid solution.

We can look at the new matrix as a linear combination of the matrix from part (a) and a new matrix. Hence, we can write it as:

$$C_2 = \begin{bmatrix} 1 & 0 & 1 & 0 & 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 & 0 & 1 & 0 & 1 \\ 1 & 0 & 1 & 0 & 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 & 0 & 1 & 0 & 1 \\ 1 & 0 & 1 & 0 & 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 & 0 & 1 & 0 & 1 \\ 1 & 0 & 1 & 0 & 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 & 0 & 1 & 0 & 1 \end{bmatrix}$$

$$= \frac{1}{2}\left(\begin{bmatrix} 1 & -1 & 1 & -1 & 1 & -1 & 1 & -1 \\ -1 & 1 & -1 & 1 & -1 & 1 & -1 & 1 \\ 1 & -1 & 1 & -1 & 1 & -1 & 1 & -1 \\ -1 & 1 & -1 & 1 & -1 & 1 & -1 & 1 \\ 1 & -1 & 1 & -1 & 1 & -1 & 1 & -1 \\ -1 & 1 & -1 & 1 & -1 & 1 & -1 & 1 \\ 1 & -1 & 1 & -1 & 1 & -1 & 1 & -1 \\ -1 & 1 & -1 & 1 & -1 & 1 & -1 & 1 \end{bmatrix} + \begin{bmatrix} 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \end{bmatrix}\right)$$

$$= \frac{1}{2}\left(\begin{bmatrix} 1 \\ -1 \\ 1 \\ -1 \\ 1 \\ -1 \\ 1 \\ -1 \end{bmatrix}\begin{bmatrix} 1 \\ -1 \\ 1 \\ -1 \\ 1 \\ -1 \\ 1 \\ -1 \end{bmatrix}^T + \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \\ 1 \\ 1 \\ 1 \\ 1 \end{bmatrix}\begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \\ 1 \\ 1 \\ 1 \\ 1 \end{bmatrix}\right)$$

c) Now consider the Swiss flag shown in Figure 2. Assume that red colors represent 0 and that white colors represent 1.

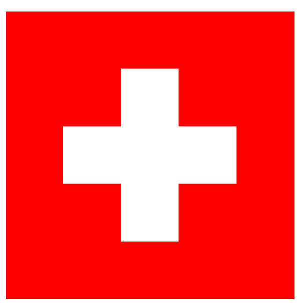

Figure 2: Swiss flag.

Assume that the Swiss flag is given by the following 5×5 matrix S:

$$S = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 1 & 1 & 1 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

Furthermore, we know that the Swiss flag can be viewed as a superposition of the following pairs of images:

Figure 3: Pairs of images - Option 1

Figure 4: Pairs of images - Option 2

Express the S in two different ways: i) as a linear combination of the outer products inspired by the Option 1 images and ii) as a linear combination of outer products inspired by the Option 2 images.

Solution

Based on the given images, we can decompose the Swiss flag into the following rank-1 matrices.

Option 1:

$$S = \begin{bmatrix} 0 \\ 1 \\ 1 \\ 1 \\ 0 \end{bmatrix} \begin{bmatrix} 0 \\ 1 \\ 1 \\ 1 \\ 0 \end{bmatrix}^T - \begin{bmatrix} 0 \\ 1 \\ 0 \\ 1 \\ 0 \end{bmatrix} \begin{bmatrix} 0 \\ 1 \\ 0 \\ 1 \\ 0 \end{bmatrix}^T$$

Option 2:

$$S = \begin{bmatrix}0\\1\\1\\1\\0\end{bmatrix}\begin{bmatrix}0\\0\\1\\0\\0\end{bmatrix}^T + \begin{bmatrix}0\\0\\1\\0\\0\end{bmatrix}\begin{bmatrix}0\\1\\0\\1\\0\end{bmatrix}^T$$

Note here that the there does not necessarily exist a unique decomposition for an image.

3 SVD properties

In this question, we look at some properties of SVD. Below we consider a m by n matrix whose SVD writes $A = \sum_{i=1}^{r} \sigma_i \vec{u}_i \vec{v}_i^T$ where r is the rank of the matrix.

a) We know that A can also be represented in matrix form as $A = \sum_{i=1}^{r} \sigma_i \vec{u}_i \vec{v}_i^T = U_1 S {V_1}^T$ where $U_1 = [\vec{u}_1, \vec{u}_2, \ldots, \vec{u}_r]$, $S = \text{diag}(\sigma_1, \sigma_2, \ldots, \sigma_r)$, and $V_1 = [\vec{v}_1, \vec{v}_2, \ldots, \vec{v}_r]$. Show that $\sum_{i=1}^{r} \sigma_i \vec{u}_i \vec{v}_i^T = U_1 S {V_1}^T$. Note that the math does not assume any further property for U_1 and V_1 than that they are of compatible shape as long as S is diagonal.

Solution

$$U_1 S {V_1}^T = \begin{bmatrix} | & | & & | \\ \vec{u}_1 & \vec{u}_2 & \ldots & \vec{u}_r \\ | & | & & | \end{bmatrix} \begin{bmatrix} \sigma_1 & & & \\ & \sigma_2 & & \\ & & \ddots & \\ & & & \sigma_r \end{bmatrix} \begin{bmatrix} - & \vec{v}_1^T & - \\ - & \vec{v}_2^T & - \\ & \vdots & \\ - & \vec{v}_r^T & - \end{bmatrix}$$

$$= \sum_{i=1}^{r} \begin{bmatrix} | & | & & | \\ \vec{u}_1 & \vec{u}_2 & \ldots & \vec{u}_r \\ | & | & & | \end{bmatrix} \begin{bmatrix} \ddots & & & \\ & 0 & & \\ & & \sigma_i & \\ & & & 0 & \\ & & & & \ddots \end{bmatrix} \begin{bmatrix} - & \vec{v}_1^T & - \\ - & \vec{v}_2^T & - \\ & \vdots & \\ - & \vec{v}_r^T & - \end{bmatrix}$$

$$= \sum_{i=1}^{r} \begin{bmatrix} \ldots & | & | & | & \ldots \\ \ldots & 0 & \sigma_i \vec{u}_i & 0 & \ldots \\ & | & | & | & \end{bmatrix} \begin{bmatrix} - & \vec{v}_1^T & - \\ - & \vec{v}_2^T & - \\ & \vdots & \\ - & \vec{v}_r^T & - \end{bmatrix}$$

$$= \sum_{i=1}^{r} \sigma_i \vec{u}_i \vec{v}_i^T$$

The full SVD of A writes $A = U\Sigma V^T$ where $U = [\vec{u}_1, \vec{u}_2, \ldots, \vec{u}_r, \vec{u}_{r+1}, \ldots, \vec{u}_m]$ and $V = [\vec{v}_1, \vec{v}_2, \ldots, \vec{v}_r, \vec{v}_{r+1}, \ldots, \vec{v}_n]$ are orthornormal matrices with the first r columns being the same as those of U' and V'.

$$\Sigma = \begin{bmatrix} S_{r\times r} & 0_{r\times(n-r)} \\ 0_{(m-r)\times r} & 0_{(m-r)\times(n-r)} \end{bmatrix}.$$

b) Suppose $r < \min(m, n)$, what is a basis of the null space of A? Prove your answer.

Solution

A basis for the null space of A is $\{\vec{v}_{r+1}, \ldots, \vec{v}_n\}$.

(<=) For some vector $\vec{x}$ in the span of $\{\vec{v}_{r+1}, \ldots, \vec{v}_n\}$, $\vec{x}$ can be written as $\vec{x} = \sum_{i=r+1}^{n} a_i \vec{v}_i$ and $V_1^T \vec{x}$ will be 0 because $\vec{v}_i \vec{v}_j^T = 0$ when $i \neq j$ and $\vec{v}_i \vec{v}_j^T = 1$ when $i = j$. Now $A\vec{x} = U_1 S V_1^T \vec{x} = 0$. $\vec{x}$ is in the null space of A.

(=>) By contrapositive. Suppose $\vec{x}$ is not in the span of $\{\vec{v}_{r+1}, \ldots, \vec{v}_n\}$, $\exists i <= r, \vec{x}^T \vec{v}_i \neq 0$, therefore both $V_1^T \vec{x}$ and $S V_1^T \vec{x}$ are non zero. Then $A\vec{x} = U_1 \vec{b} = \sum_{i=1}^{r} b_i \vec{u}_i$ where $\vec{b} = S V_1^T \vec{x}$ is non zero. Because $\vec{u}_1, \vec{u}_2, \ldots, \vec{u}_m$ are linearly independent, $A\vec{x}$ cannot be 0. Such $\vec{x}$ is not in the null space of A.

c) What is the basis of the range of A? Prove your answer.

Solution

A basis for the range of A is $\{\vec{u}_1, \ldots, \vec{u}_r\}$.

(<=) For some vector $\vec{y}$ in the span of $\{\vec{u}_1, \ldots, \vec{u}_r\}$, $\vec{y}$ can be written as $\vec{y} = \sum_{i=r+1}^{n} b_i \vec{u}_i$, we can construct $\vec{x} = V_1 S^{-1} \vec{b}$ such that $A\vec{x} = U_1 \vec{b} = \sum_{i=1}^{r} b_i \vec{u}_i = \vec{y}$, where $\vec{b} = [b_1, b_2, ..., b_r]^T$. Therefore $\vec{y}$ is in the range of A.

(=>) For some vector $\vec{y}$ in the range of A, $\vec{y} = A\vec{x}$ for some x. We have $\vec{y} = U_1 S V_1^T \vec{x} = U_1 \vec{b} = \sum_{i=1}^{r} b_i \vec{u}_i$ where $\vec{b} = S V_1^T \vec{x}$. Therefore such $\vec{y}$ is in the span of $\{\vec{u}_1, \ldots, \vec{u}_r\}$.

d) Suppose $n = m = r$, that is, A is square and full rank. Find the inverse of A in terms of U, Σ, and V.

Solution

Since A is full rank, we know Σ is a diagonal matrix with positive diagonal elements. Therefore we can construct its inverse $\Sigma^{-1} = \text{diag}(\sigma_1^{-1}, \sigma_2^{-1}, \ldots, \sigma_n^{-1})$. Now the inverse writes,

$$A^{-1} = V \Sigma^{-1} U^T.$$

Because $A V \Sigma^{-1} U^T = U \Sigma V^T V \Sigma^{-1} U^T = U \Sigma \Sigma^{-1} U^T = U U^T = I$.

4 Induced Matrix Norms

Often, the general effect of matrices on their inputs is really hard to predict. To overcome this, we usually try to "bound" the effect a matrix has on input vectors. We will work through a simple case of bounding the output of an $n \times n$ matrix T given that T is diagonalizable and symmetric. We will consider the system

$$\vec{y} = T\vec{x},$$

where $\vec{x}$ is the input vector and $\vec{y}$ is the output vector.

a) Let $U = [\vec{u}_1, \vec{u}_2, \ldots, \vec{u}_n]$ be a set of orthonormal eigenvectors of the matrix T. **Decompose the generic input vector $\vec{x}$ into a linear combination of these eigenvectors.**

Solution

We want the coordinates for $\vec{x}$ in U basis. From 16A, we know this means that we multiply $\vec{x}$ by U^{-1} since $UU^{-1}\vec{x} = \vec{x}$. Using the fact that eigenvectors of symmetric matrices are orthonormal, so $U^T U = I$ and so $U^{-1} = U^T$. Writing this out as a sum we have

$$\vec{x} = \sum_{i=1}^{n} (\vec{u}_i^T \vec{x})\vec{u}_i$$

b) Let the eigenvalues of T be $\{\lambda_1, \lambda_2, \ldots, \lambda_n\}$ with $|\lambda_1| \geq |\lambda_2| \geq \ldots |\lambda_n|$. If we represent $\vec{x}$ as a linear combination of the eigenvectors of T, **what is the Euclidean norm of the output vector $\vec{y}$, $\|\vec{y}\|$?**
Hint: You can use the fact that Euclidean norms are preserved with orthonormal transforms.

Solution

First,

$$\vec{y} = T\vec{x} \tag{1}$$
$$= T\sum_{i=1}^{n} (\vec{u}_i^T \vec{x})\vec{u}_i \tag{2}$$
$$= \sum_{i=1}^{n} (\vec{u}_i^T \vec{x})T\vec{u}_i \tag{3}$$
$$= \sum_{i=1}^{n} (\vec{u}_i^T \vec{x})\lambda_i\vec{u}_i \tag{4}$$

and since the $\vec{u}_i$ are orthonormal,

$$||\vec{y}|| = \sqrt{\sum_{i=1}^{n} (\lambda_i \vec{u}_i^T \vec{x})^2}$$

c) Say you do not know all the eigenvalues of T, but you know the largest eigenvalue λ_1. If the norm $||\vec{x}|| = \alpha$, **how big could $\|\vec{y}\|$ be**?

Solution

To see how big it could be, it is easier to first ask how big $\|\vec{y}\|^2$ could be:

$$||T\vec{x}||^2 = \sum_{i=1}^{n} (\lambda_i \vec{u}_i^T \vec{x})^2 \tag{5}$$
$$\leq \sum_{i=1}^{n} \lambda_1^2 (\vec{u}_i^T \vec{x})^2 \tag{6}$$
$$= \lambda_1^2 \sum_{i=1}^{n} (\vec{u}_i^T \vec{x})^2 \tag{7}$$
$$= \lambda_1^2 ||\vec{x}||^2 \tag{8}$$
$$= \lambda_1^2 \alpha^2 \tag{9}$$

Where (6) is because the first eigenvalue is the biggest and they're all non-negative and (8) is because the $\vec{u}_i$ are orthonormal and so these are the squared coordinates of $\vec{x}$ in that basis. Therefore,

$$||\vec{y}|| \le |\lambda_1|\alpha$$

The maximum factor by which a square matrix can grow the norm of a vector is called the induced norm for that matrix. Although we had you do the derivation above for a symmetric matrix T, the fact that $\|A\vec{x}\| = \sqrt{\vec{x}^T A^T A\vec{x}}$ can be used to show how to generalize this concept of induced norm for general matrices.

5 Closed-Loop Control of SIXT33N

To make our control more robust, we introduce feedback, turning our open-loop controller into a closed-loop controller. In this problem, we derive the closed-loop control scheme you will use to make SIXT33N reliably drive straight.

We introduce $\delta(k) = d_L(k) - d_R(k)$ as the difference in positions between the two wheels. We will consider a proportional control scheme, which introduces a feedback term into our input equation in which we apply gains k_L and k_R to $\delta(k)$ to modify our input at each timestep in an effort to prevent $|\delta(k)|$ from growing without bound. To do this, we will modify our inputs $u_L(k)$ and $u_R(k)$ to be:

$$\begin{aligned} u_L(k) &= \frac{v^* + \beta_L}{\theta_L} - k_L\frac{\delta(k)}{\theta_L} \\ u_R(k) &= \frac{v^* + \beta_R}{\theta_R} + k_R\frac{\delta(k)}{\theta_R} \end{aligned}$$

Substituting into the open-loop equations

$$\begin{aligned} d_L(k+1) - d_L(k) &= \theta_L u_L(k) - \beta_L \\ d_R(k+1) - d_R(k) &= \theta_R u_R(k) - \beta_R \end{aligned} \tag{1}$$

we obtain:

$$\begin{aligned} d_L(k+1) - d_L(k) &= v^* - k_L\delta(k) \\ d_R(k+1) - d_R(k) &= v^* + k_R\delta(k) \end{aligned} \tag{2}$$

a) Let's look a bit more closely at picking k_L and k_R. First, we need to figure out what happens to $\delta(k)$ over time. Find $\delta(k+1)$ in terms of $\delta(k)$.

Solution

$$\begin{aligned} \delta(k+1) &= d_L(k+1) - d_R(k+1) \\ &= v^* - k_L\delta(k) + d_L(k) - (v^* + k_R\delta(k) + d_R(k)) \\ &= v^* - k_L\delta(k) + d_L(k) - v^* - k_R\delta(k) - d_R(k) \\ &= -k_L\delta(k) - k_R\delta(k) + (d_L(k) - d_R(k)) \\ &= -k_L\delta(k) - k_R\delta(k) + \delta(k) \\ &= \delta(k)(1 - k_L - k_R) \end{aligned}$$

b) Given your work above, what is the eigenvalue of the system defined by $\delta(k)$? For discrete-time systems like our system, $\lambda \in (-1, 1)$ is considered stable. Are $\lambda \in [0, 1)$ and $\lambda \in (-1, 0]$ identical in function for our system? Which one is "better"? (*Hint:* Preventing oscillation is a desired benefit.)

Based on your choice for the range of λ above, how should we set k_L and k_R in the end?

Solution

The eigenvalue is $\lambda = 1 - k_L - k_R$.

As a discrete system, both are stable, but $\lambda \in (-1, 0]$ will cause the car to oscillate due to overly high gain. Therefore, we should choose $\lambda \in [0, 1)$.

As a result, $1 - k_L - k_R \in [0, 1) \implies (k_L + k_R) \in (0, 1]$ means that we should set the gains such that $(k_L + k_R) \in [0, 1)$.

c) Let's re-introduce the model mismatch in order to model environmental discrepancies, disturbances, etc. How does closed-loop control fare under model mismatch? Find $\delta_{ss} = \delta[k \to \infty]$, assuming that $\delta[0] = \delta_0$. What is δ_{ss}? (To make this easier, you may leave your answer in terms of appropriately defined c and λ obtained from an equation in the form of $\delta(k + 1) = \delta(k)\lambda + c$.)

Check your work by verifying that you reproduce the equation in part (c) if all model mismatch terms are zero. Is it better than the open-loop model mismatch?

$$
\begin{aligned}
d_L(k+1) - d_L(k) &= (\theta_L + \Delta\theta_L)u_L(k) - (\beta_L + \Delta\beta_L) \\
d_R(k+1) - d_R(k) &= (\theta_R + \Delta\theta_R)u_R(k) - (\beta_R + \Delta\beta_R)
\end{aligned}
$$

$$
\begin{aligned}
u_L(k) &= \frac{v^* + \beta_L}{\theta_L} - k_L \frac{\delta(k)}{\theta_L} \\
u_R(k) &= \frac{v^* + \beta_R}{\theta_R} + k_R \frac{\delta(k)}{\theta_R}
\end{aligned}
$$

Solution

$$\begin{aligned}
\delta(k+1) &= d_L(k+1) - d_R(k+1) \\
&= (\theta_L + \Delta\theta_L)u_L(k) - (\beta_L + \Delta\beta_L) + d_L(k) - ((\theta_R + \Delta\theta_R)u_R(k) - (\beta_R + \Delta\beta_R) + d_R(k)) \\
&= \theta_L u_L(k) - \beta_L + \Delta\theta_L u_L(k) - \Delta\beta_L + d_L(k) - (\theta_R u_R(k) - \beta_R + \Delta\theta_R u_R(k) - \Delta\beta_R + d_R(k)) \\
&= v^* - k_L\delta(k) + \Delta\theta_L u_L(k) - \Delta\beta_L + d_L(k) - (v^* + k_R\delta(k) + \Delta\theta_R u_R(k) - \Delta\beta_R + d_R(k)) \\
&= v^* - k_L\delta(k) + \Delta\theta_L u_L(k) - \Delta\beta_L + d_L(k) - v^* - k_R\delta(k) - \Delta\theta_R u_R(k) + \Delta\beta_R - d_R(k) \\
&= v^* - v^* + (d_L(k) - d_R(k)) - k_L\delta(k) - k_R\delta(k) + \Delta\theta_L u_L(k) - \Delta\beta_L - \Delta\theta_R u_R(k) + \Delta\beta_R \\
&= \delta(k)(1 - k_L - k_R) + \Delta\theta_L u_L(k) - \Delta\beta_L - \Delta\theta_R u_R(k) + \Delta\beta_R \\
&= \delta(k)(1 - k_L - k_R) + \Delta\theta_L\left(\frac{v^* + \beta_L}{\theta_L} - k_L\frac{\delta(k)}{\theta_L}\right) - \Delta\theta_R\left(\frac{v^* + \beta_R}{\theta_R} + k_R\frac{\delta(k)}{\theta_R}\right) - \\
&\Delta\beta_L + \Delta\beta_R \\
&= \delta(k)(1 - k_L - k_R) + \frac{\Delta\theta_L}{\theta_L}(v^* + \beta_L) - \delta(k)k_L\frac{\Delta\theta_L}{\theta_L} - \frac{\Delta\theta_R}{\theta_R}(v^* + \beta_R) - \\
&\delta(k)k_R\frac{\Delta\theta_R}{\theta_R} - \Delta\beta_L + \Delta\beta_R \\
&= \delta(k)\left(1 - k_L - k_R - k_L\frac{\Delta\theta_L}{\theta_L} - k_R\frac{\Delta\theta_R}{\theta_R}\right) + \frac{\Delta\theta_L}{\theta_L}(v^* + \beta_L) - \frac{\Delta\theta_R}{\theta_R}(v^* + \beta_R) - \\
&\Delta\beta_L + \Delta\beta_R \\
&= \delta(k)\left(1 - k_L - k_R - k_L\frac{\Delta\theta_L}{\theta_L} - k_R\frac{\Delta\theta_R}{\theta_R}\right) + \left(\frac{\Delta\theta_L}{\theta_L}(v^* + \beta_L) - \Delta\beta_L\right) - \\
&\left(\frac{\Delta\theta_R}{\theta_R}(v^* + \beta_R) - \Delta\beta_R\right)
\end{aligned}$$

Let us define $c = \left(\frac{\Delta\theta_L}{\theta_L}(v^* + \beta_L) - \Delta\beta_L\right) - \left(\frac{\Delta\theta_R}{\theta_R}(v^* + \beta_R) - \Delta\beta_R\right)$, and our new eigenvalue $\lambda = 1 - k_L - k_R - k_L\frac{\Delta\theta_L}{\theta_L} - k_R\frac{\Delta\theta_R}{\theta_R}$. In this case,

$$\begin{aligned}
\delta[1] &= \delta_0\lambda + c \\
\delta[2] &= \lambda(\delta_0\lambda + c) + c & &= \delta_0\lambda^2 + c\lambda + c \\
\delta[3] &= \lambda(\delta_0\lambda^2 + c\lambda + c) + c & &= \delta_0\lambda^3 + c\lambda^2 + c\lambda + c \\
\delta[4] &= \lambda(\delta_0\lambda^3 + c\lambda^2 + c\lambda + c) + c & &= \delta_0\lambda^4 + c\lambda^3 + c\lambda^2 + c\lambda + c \\
\delta[5] &= \delta_0\lambda^5 + c(\lambda^4 + \lambda^3 + \lambda^2 + \lambda^1 + 1) \\
\delta[n] &= \delta_0\lambda^n + c(1 + \lambda^1 + \lambda^2 + \lambda^3 + \lambda^4 + \ldots + \lambda^n) \\
\delta[n] &= \delta_0\lambda^n + c\left(\sum_{k=0}^{n}\lambda^k\right) \text{ (rewriting in sum notation)} \\
\delta[n] &= \delta_0\lambda^n + c\left(\frac{1 - \lambda^n}{1 - \lambda}\right) \text{ (sum of a geometric series)}
\end{aligned}$$

If $\lambda < 1$, then $\lambda^\infty = 0$, so those terms drop out:

$$\delta[n = t \to \infty] = \delta_0 \lambda^\infty + c\left(\frac{1-\lambda^\infty}{1-\lambda}\right)$$
$$\delta[n = t \to \infty] = c\frac{1}{1-\lambda}$$

$$\delta_{ss} = c\frac{1}{1-\lambda}$$

For your entertainment only: δ_{ss} is fully-expanded form (not required) is

$$\frac{\left(\frac{\Delta\theta_L}{\theta_L}(v^* + \beta_L) - \Delta\beta_L\right) - \left(\frac{\Delta\theta_R}{\theta_R}(v^* + \beta_R) - \Delta\beta_R\right)}{k_L + k_R + k_L\frac{\Delta\theta_L}{\theta_L} + k_R\frac{\Delta\theta_R}{\theta_R}}$$

The answer is correct because plugging in zero into all the model mismatch terms into c causes $c = 0$, so $\delta_{ss} = 0$ if there is no model mismatch. Compared to the open-loop result of $\delta_{ss} = \pm\infty$, the closed loop $\delta_{ss} = c\frac{1}{1-\lambda}$ is a much-desired improvement.

What does this mean for the car? It means that the car will turn initially for a bit but eventually converge to a fixed heading and keep going straight from there.

6 Homework Process and Study Group

Citing sources and collaborators are an important part of life, including being a student!
We also want to understand what resources you find helpful and how much time homework is taking, so we can change things in the future if possible.

a) **What sources (if any) did you use as you worked through the homework?**

b) **If you worked with someone on this homework, who did you work with?**
List names and student ID's. (In case of homework party, you can also just describe the group.)

c) **Roughly how many total hours did you work on this homework?**

d) **Do you have any feedback on this homework assignment?**