In this problem, we will recap several different methods that we have learnt and apply them all to the same circuit but under different situations.

![Circuit Diagram]

Figure 1: A model for a transmission line.

We are trying to transmit different signals across a very long wire. At longer lengths, the various electromagnetic losses incurred by a wire can be modeled using a model shown in Figure 1. We will take a look at the different kinds of signals that we want to transmit and which analyses techniques we should apply for assessing the output at receiving terminal $v_{out}$.

a) First we want to send a constant voltage value. We can do this by applying a constant voltage $v_s$ as our input. If we apply $v_s = 12V$, find the capacitor voltage $v_{out}$ and the inductor current $i_L$ at equilibrium (or what we often refer to as DC steady-state). Use $R_1 = 100\,\Omega$, $R_2 = 100\,\Omega$, $C = 12\mu F$ and $L = 1\,mH$.

**Solution**

Applying a voltage $v_s = 12V$, at steady state, we know that the inductor acts as a short-circuit. Simultaneously, the capacitor acts an open circuit. The resulting equivalent circuit in steady-state will be
Using the voltage divider formulation, we have

\[
v_{\text{out}} = \frac{R_1}{R_1 + R_2} v_s = \frac{100}{100 + 100} v_s = 6V
\]

b) If \(v_s(t)\) is a time-varying signal, write a system of differential equations using the inductor current \(i_L\) and capacitor voltage \(v_{\text{out}}\) as state variables. The equation system should be in the form

\[
\frac{d}{dt} \begin{bmatrix} i_L \\ v_{\text{out}} \end{bmatrix} = A \begin{bmatrix} i_L \\ v_{\text{out}} \end{bmatrix} + Bv_s(t)
\]

\(1\)

**Solution**

Figure 3 shows the circuit we are analyzing. The devices have been named and additional currents and voltages have been labelled for clarity.
KCL on the output node gives us

\[ -i_L + i_C + i_{R_2} = 0. \]  \hspace{1cm} (2)

Similarly, KCL at node 1, labelled with voltage \( v_1 \), gives us

\[ -i_{R_1} + i_L = 0 \]  \hspace{1cm} (3)

The inductor, capacitor and resistor \( I-V \) relationships can be written using our circuit variables.

\[ i_C = C \frac{d}{dt} v_{out}(t) \]
\[ i_{R_2} = \frac{v_{out}(t)}{R_2} \]
\[ L \frac{d}{dt} i_L(t) = v_1 - v_{out}(t) \]
\[ i_{R_1} = \frac{v_s - v_1}{R_1} \]

Combining Equations 2 and 3 with the \( I-V \) relationships above, we can eliminate the additional variables \( i_{R_1}, v_1 \) and \( i_{R_2} \). For Equation 2, this gives us

\[ -i_L + C \frac{d}{dt} v_{out} + \frac{v_{out}}{R_2} = 0 \]
\[ \frac{d}{dt} v_{out} = -\frac{1}{R_2C} v_{out} + \frac{1}{C} i_L \]

We can rewrite the \( I-V \) relationship for the inductor above to obtain

\[ L \frac{d}{dt} i_L = v_s - R i_L - v_{out} \]
\[ \frac{d}{dt} i_L = -\frac{R}{L} i_L - \frac{1}{L} v_{out} + \frac{1}{L} v_s \]

We can now combine these to obtain our matrix equation system

\[ \frac{d}{dt} \begin{bmatrix} i_L \\ v_{out} \end{bmatrix} = \begin{bmatrix} -\frac{R}{L} & -\frac{1}{LC} \\ \frac{1}{LC} & -\frac{1}{R_2L} \end{bmatrix} \begin{bmatrix} i_L \\ v_{out} \end{bmatrix} + \begin{bmatrix} \frac{1}{L} \\ 0 \end{bmatrix} v_s \]  \hspace{1cm} (4)

c) Find the eigenvalues for the matrix \( A \) found above and comment on them.

**Solution**

The eigenvalues for matrix \( A \) are given by

\[ \det \left( \lambda + \frac{R}{L} \begin{bmatrix} 1 & \frac{1}{L} \\ \frac{1}{L} & \lambda \end{bmatrix} \right) = 0 \]  \hspace{1cm} (5)
Expanding out the determinant for the matrix above, we get

\[
\begin{align*}
\left( \lambda + \frac{R_1}{L} \right) \left( \lambda + \frac{1}{R_2C} \right) + \frac{1}{LC} &= 0 \\
\lambda^2 + \lambda \left( \frac{R_1}{L} + \frac{1}{R_2C} \right) + \frac{1}{LC} \left( 1 + \frac{R_1}{R_2} \right) &= 0
\end{align*}
\]

\[
\lambda = -\frac{1}{2} \left( \frac{R_1}{L} + \frac{1}{R_2C} \right) \pm \frac{1}{2} \sqrt{\left( \frac{R_1}{L} + \frac{1}{R_2C} \right)^2 - 4 \left( 1 + \frac{R_1}{R_2} \right) \frac{1}{LC}}
\]

\[
= -\frac{1}{2} \left( \frac{R_1}{L} + \frac{1}{R_2C} \right) \pm \frac{1}{2} \sqrt{\left( \frac{R_1}{L} - \frac{1}{R_2C} \right)^2 - 4 \frac{1}{LC}}
\]

We can make a few observations about the eigenvalues

- The eigenvalues are real if \( \left( \frac{R_1}{L} - \frac{1}{R_2C} \right) > \frac{1}{\sqrt{LC}} \).
- If the eigenvalues are not purely real, they occur as complex conjugate pairs.
- The real part of the eigenvalues is negative. This implies that in the absence of an external input \( v_s(t) \), the state variables will settle to a steady-state value of 0.

d) We want to send a pulse instead of a steady value. For this case, \( v_s(t) \) is shown in Figure 4.

![Figure 4: Pulse input to be transmitted across the wire.](image)

It is not always possible or convenient to solve differential equations by hand using eigenvalues and the guess-and-check methods we have developed so far in the class. In the supplied Jupyter notebook `RLC_Circuit_Analysis.ipynb`, fill out the entries for matrices \( A \) and \( B \). The notebook has an implementation of a numerical solution to differential equations.

Sketch the output \( v_{out}(t) \) for the pulse input \( v_s(t) \) shown in Figure 4 using the supplied python notebook. The circuit parameters for this problem have been specified in the notebook: \( R_1 = 10 \Omega, R_2 = 10 \Omega, C = 24 \mu F \) and \( L = 5mH \).
e) Finally, we want to test how our transmission line will carry a sinusoidal input. First, we will use the numerical technique that we saw in the previous part to evaluate the output $v_{out}(t)$. Using the code provided in the supplied iPython notebook, plot the output $v_{out}(t)$ for a sinusoidal input $v_s(t)$. Look at the last 2 cycles of the input and plot the corresponding output. Note the difference in amplitude and phase between the input $v_s(t)$ and the output $v_{out}(t)$. We will come back to these when we repeat this calculation for the sinusoidal steady state using phasor analysis.

Solution

\[ H(\omega) = \frac{V_{out}}{V_s}, \]  

f) We now want to transmit a sine wave, $v_s(t) = 12 \sin(\omega t)$ along the transmission. Using phasor analysis, find the transfer function
where $V_s$ is a phasor representing the input voltage $v_s(t)$ and $V_{out}$ is a phasor representing the output voltage $v_{out}(t)$. Find the output phasor $V_{out}$. If we use the same circuit parameters from part (d), comment on how the transfer function relates the time-waveforms $v_s(t)$ and $v_{out}(t)$ in (e).

**HINT:** Try using different initial conditions to see how the numerical solution changes.

**Solution**

In sinusoidal steady-state, we can analyze various currents and voltages in the circuit using phasors.

The impedances $Z_{R_2}$ and $Z_C$ can be combined in parallel

$$Z_{R_2||C} = \frac{R_2}{\frac{1}{j\omega C} + \frac{1}{Z_{R_2}}},$$

(7)

Impedances $Z_{R_1}$ and $Z_L$ can be combined in series to give

$$Z_{R_1,L} = R_1 + j\omega L.$$  

(8)

The transfer function, $H(\omega) = \frac{V_{out}}{V_s}$ can be found using the voltage divider formulation

$$V_{out} = V_{in} \left( \frac{Z_{R_2||C}}{Z_{R_2||C} + Z_{R_1,L}} \right).$$

(9)

Plugging in the equivalent impedances $Z_{R_2||C}$ from Equation 7 and $Z_{R_1,L}$ from Equation 8, we
get

\[ H(\omega) = \frac{V_{out}}{V_s} = \frac{Z_{R_2||C}}{Z_{R_2||C} + Z_{R_1,L}} \]
\[ = \frac{\frac{R_2}{1+j\omega R_2 C}}{\frac{R_2}{1+j\omega R_2 C} + R_1 + j\omega L} \]
\[ = \frac{R_2}{R_2 + (1 + j\omega R_2 C) (R_1 + j\omega L)} \]
\[ = \frac{R_2}{R_1 + R_2 + j\omega (L + R_1 R_2 C) + (j\omega)^2 (R_2 LC)} \]
\[ = \frac{\frac{R_3}{R_1 + R_2}}{1 + j\omega \left( \frac{L}{R_1 R_2} + \frac{R_1 R_2 C}{R_1 + R_2} \right) + (j\omega)^2 \left( \frac{R_3}{R_1 + R_2} \right) \}} \]

In periodic steady-state, the ratio of the peak amplitudes for \( v_{out}(t) \) and \( v_s(t) \) is given by the magnitude \(| H(\omega) |\) of the transfer function. On the other hand, the phase difference between the waveforms for \( v_{out}(t) \) and \( v_s(t) \) is given by the phase \( \angle H(\omega) \) of the transfer function.

2. RLC circuit as passive filters

As originally conceived by Bode in the 1930s, Bode plot is only an asymptotic approximation of the frequency response, using straight line segments. It relies on using a logarithmic scale for the input frequency \( \omega \) to express the magnitude of the transfer functions on a logarithmic scale \( \log_{10} | H(\omega) | \).

In this question, we will go through some examples to appreciate the beauty and simplicity of Bode plots. In the iPython notebook \textit{BodePlots.ipynb}, you will see how well the approximation of Bode plots is in different regions. In particular, we will work with the RLC circuit shown below:

\[
\begin{array}{c}
\text{\scriptsize -} \\
V_s \\
\text{\scriptsize +} \\
\hline
| \\
\hline
| \\
\hline
\text{\scriptsize -} \\
V_R \\
\text{\scriptsize +} \\
\hline
| \\
\hline
| \\
\hline
\text{\scriptsize +} \\
V_L \\
\text{\scriptsize -} \\
\hline
| \\
\hline
| \\
\hline
\text{\scriptsize +} \\
V_C \\
\text{\scriptsize -} \\
\hline
| \\
\hline
| \\
\hline
\end{array}
\]

In the following questions, we will be exploring how to use the above RLC circuit to construct highpass, lowpass, and bandpass filters. As the name suggests, a highpass filter will suppress the low frequency components while keeping the high frequency components of the input unblocked. Since the circuit contains only passive elements, namely resistors, capacitors, and inductors, these filters are called \textit{passive filters}. On the other hand, if the circuit contains op amps, transistors, or other active devices, it will become \textit{active filters}. 


a) **Lowpass filter.** Treat $V_s$ as the input and $V_C$ as the output. Obtain the transfer function $H_{LP} = \frac{V_C}{V_s}$, and its magnitude and phase. Draw the Bode plot for the magnitude. Explain why this is a lowpass filter.

**Solution**

To obtain the transfer function, we use the definition:

$$H_{LP}(\omega) = \frac{V_C}{V_s} = \frac{(1/j\omega C)I}{V_s} = \frac{1}{(1 - \omega^2 LC) + j\omega RC}$$

where we used $V_s = I(R + j\omega L + \frac{1}{j\omega C})$, so $\frac{1}{V_s} = \frac{1}{R + j\omega L + \frac{1}{j\omega C}}$. For the magnitude:

$$M_{LP}(\omega) = |H_{LP}(\omega)| = \frac{1}{\sqrt{(1 - \omega^2 LC)^2 + \omega^2 R^2 C^2}}$$

$$\phi_{LP} = \begin{cases} -\pi - \tan^{-1}\left(\frac{\omega RC}{1 - \omega^2 LC}\right) & \omega > \frac{1}{\sqrt{LC}} \\ -\tan^{-1}\left(\frac{\omega RC}{1 - \omega^2 LC}\right) & \omega < \frac{1}{\sqrt{LC}} \\ -\frac{\pi}{2} & \omega = \frac{1}{\sqrt{LC}} \end{cases}$$

where we calculate the phase by first identifying which quadrant the complex transfer function lies in and then use arctan to get the angle.

For the bode plots, let $\omega_0 = \frac{1}{\sqrt{LC}}$ and $Q = \frac{\omega_0 L}{R}$, then we have:

$$H_{LP}(\omega) = \frac{1}{(1 - (\omega/\omega_0)^2) + j\frac{\omega}{Q\omega_0}}$$

The magnitude response for this transfer function is shown in Figure 6.

It is a low pass filter because the transfer has large magnitude when the frequency is low (on the left side of the graph), and the magnitude is reducing exponentially as the frequency is above the resonance frequency $\omega_0$.

If we change the resistance $R$, you can see that it doesn’t change the resonance frequency $\omega_0$, but it does change $Q = \frac{\omega_0 L}{R}$, which is known as the quality factor. As we increase it, we have more dramatic peaks around the resonance frequency, and it becomes more pronounced relative to the level of the asymptotes. In the plot, we show several cases for the $Q$, you can see that for large $Q$, the asymptotic approximation by bode plots are less accurate especially around the resonance frequency.

b) **Highpass filter.** Let $V_i$ be the output. Obtain the transfer function $H_{HP} = \frac{V_i}{V_s}$, and its magnitude and phase. Draw the Bode plot for the magnitude. Explain why this is a highpass filter.

**Solution**

To obtain the transfer function, we use the definition:

$$H_{HP}(\omega) = \frac{V_i}{V_s} = \frac{j\omega LI}{V_s} = \frac{-\omega^2 LC}{(1 - \omega^2 LC) + j\omega RC}$$
where we used \( V_s = I(R + j\omega L + \frac{1}{j\omega C}) \), so \( \frac{1}{v_s} = \frac{1}{R + j\omega L + \frac{1}{j\omega C}} \). For the magnitude:

\[
M_{HP}(\omega) = |H_{HP}(\omega)| = \frac{\omega^2 LC}{\sqrt{(1 - \omega^2 LC)^2 + \omega^2 R^2 C^2}}
\]

\[
\phi_{HP} = \begin{cases} 
-\tan^{-1}\left( \frac{\omega RC}{1 - \omega^2 LC} \right) & \omega > \frac{1}{\sqrt{LC}} \\
\pi - \tan^{-1}\left( \frac{\omega RC}{1 - \omega^2 LC} \right) & \omega < \frac{1}{\sqrt{LC}} \\
\frac{\pi}{2} & \omega = \frac{1}{\sqrt{LC}} 
\end{cases}
\]

where we calculate the phase by first identifying which quadrant the complex transfer function lies in and then use arctan to get the angle.

\[
\phi_{HP} = \pi - \tan^{-1}\left( \frac{\omega RC}{1 - \omega^2 LC} \right)
\]

For the bode plots, let \( \omega_0 = \frac{1}{\sqrt{LC}} \) and \( Q = \frac{\omega_0 L}{R} \), then we have:

\[
H_{LP}(\omega) = \frac{-(\omega/\omega_0)^2}{(1 - (\omega/\omega_0)^2) + j \frac{\omega}{\omega_0}}
\]

Figure 7 show the magnitude plot for this transfer function.

It is a high pass filter because the transfer has large magnitude when the frequency is high (on the right side of the graph), and the magnitude is reducing exponentially as the frequency is less than the resonance frequency \( \omega_0 \).
Similar to the previous part, if we increase $Q$, we have more dramatic peaks around the resonance frequency, and it becomes more pronounced relative to the level of the asymptotes.

c) Bandpass filter. How can you obtain a bandpass filter based on your findings above? Write out the transfer function and its magnitude and phase.

**Solution**

Yes we can. The output will be $V_R$ to construct a bandpass filter:

$$H_{BP}(\omega) = \frac{V_R}{V_s} = \frac{RI}{Vs} = \frac{j\omega RC}{(1 - \omega^2LC) + j\omega RC}$$

where we used $V_s = I(R + j\omega L + \frac{1}{j\omega C})$, so $\frac{1}{Vs} = \frac{1}{R + j\omega L + \frac{1}{j\omega C}}$.

$$M_{BP}(\omega) = |H_{BP}(\omega)| = \frac{\omega RC}{\sqrt{(1 - \omega^2LC)^2 + \omega^2R^2C^2}}$$

$$\phi_{LP} = \begin{cases} \pi - \tan^{-1}\left(\frac{\omega RC}{1 - \omega^2LC}\right) & \omega > \frac{1}{\sqrt{LC}} \\ \pi - \tan^{-1}\left(\frac{\omega RC}{1 - \omega^2LC}\right) & \omega < \frac{1}{\sqrt{LC}} \\ 0 & \omega = \frac{1}{\sqrt{LC}} \end{cases}$$

where we calculate the phase by first identifying which quadrant the complex transfer function lies in and then use arctan to get the angle.

d) The resonant frequency, $\omega_0$, is the input frequency (other than 0 and $\infty$) that leads to the elimination of the imaginary part of the circuit impedance, i.e., the impedance is purely real. Find the resonant frequency for the RLC circuit above.
Solution

The impedance of the circuit as measured from both sides of the voltage source is given by:

\[ Z_R + Z_L + Z_C = R + j\omega L + \frac{1}{j\omega C} = R + j(\omega L - \frac{1}{\omega C}) \quad (10) \]

which is the same for all the lowpass, highpass, and bandpass filters. To eliminate the imaginary part, we can set:

\[ \omega_0 L = \frac{1}{\omega_0 C} \Rightarrow \omega_0 = \sqrt{\frac{1}{LC}} \quad (11) \]

which, by definition, is the resonant frequency.

e) For mobile communications, the center frequency is approximately 800 MHz. In the IPython notebook, experiment with different \( L \) and \( C \) to center the bandpass filter.

Solution

To center the bandpass filter, we need to find \( L \) and \( C \) such that \( \omega_0 = \sqrt{\frac{1}{LC}} = 8 \times 10^8 \). Please refer to the IPython notebook for suitable values.

You might have noticed that the advantage of Bode plot is that it makes it easier to work with transfer functions that have multiple factors. We can write \( H(\omega) \) as a product of such factors:

\[ (\omega) = A_1(\omega)A_2(\omega)\cdots A_n(\omega) \quad (12) \]

In this class, we will focus on functions \( A_1 \) to \( A_n \) that assume one of the possible forms.

- Constant factor: \( H = K \)
- Zero @ origin: \( H = (j\omega)^N \)
- Pole @ origin: \( H = 1/(j\omega)^N \)
- Zero @ \( \omega_c \): \( H = (1 + j\omega/\omega_c)^N \)
- Pole @ \( \omega_c \): \( H = 1/(1 + j\omega/\omega_c)^N \)

The construction thus becomes simple addition or subtraction of these forms. For instance, 

\[ H(\omega) = 10^{1+\omega/\omega_2} \frac{1+\omega/\omega_1}{1+\omega/\omega_3} \frac{1+\omega/\omega_4}{1+\omega/\omega_5} \] where \( A_1 = 10, A_2 = 1 + j\omega/\omega_2, A_3 = \frac{1}{1+j\omega/\omega_3} \).

(f) For transfer function \( H(\omega) = M(\omega)e^{j\phi(\omega)} \), how to represent the magnitude \( M(\omega) \) and phase \( \phi(\omega) \) with the magnitudes \( |A_1(\omega)| \) and phase \( \phi_{A_1}(\omega) \)?

Solution

Since we have \( H(\omega) = A_1(\omega)A_2(\omega)\cdots A_n(\omega) \), and \( A_i(\omega) = |A_i(\omega)|e^{j\phi_{A_i}(\omega)} \) in the polar representation, we have:

\[ H(\omega) = M(\omega)e^{j\phi(\omega)} = |A_1(\omega)||A_2(\omega)|\cdots|A_n(\omega)|e^{j(\phi_{A_1}(\omega)+\phi_{A_2}(\omega)+\cdots+\phi_{A_n}(\omega))} \quad (13) \]

Therefore, we have:

\[ M(\omega) = |A_1(\omega)||A_2(\omega)|\cdots|A_n(\omega)| \]
\[ \phi(\omega) = \phi_{A_1}(\omega) + \phi_{A_2}(\omega) + \cdots + \phi_{A_n}(\omega) \]

by comparison.
(g) Consider the transfer function

\[
H(\omega) = \frac{(j10\omega + 30)^2}{(300 - 3\omega^2 + j90\omega)}
\]  

(14)

Refer to the IPython notebook for constructing the Bode plots for this transfer function.

Solution

![Bode plot](image)

Figure 8: Magnitude response for the transfer function \(H(\omega)\).

(h) In addition to the Bode plots, we also plotted the magnitude of the transfer functions without approximations. Please comment on the differences.

Solution

We can see that Bode plots are linear approximations of the magnitude of the transfer functions. The approximation is generally good at regions without corners. Around the corners, the exact plot usually have some “shootings” or smooth curves.

This homework problem addresses the essential concepts of Bode plots. Please check the table in Figure 9 to improve your understanding.

3 Otto the Pilot

Otto has devised a control algorithm, so that his plane climbs to the desired altitude by itself. However, he is having oscillatory transients as shown in the figure. Prof. Sanders told him that if his system has complex eigenvalues

\[ \lambda_{1,2} = v \pm j\omega, \]

then his altitude would indeed oscillate with frequency \(\omega\) about the steady state value, 1 km, and that the time trace of his altitude would be tangent to the curves \(1 + e^{\omega t}\) and \(1 - e^{\omega t}\) near its maxima and minima respectively.
a) Find the real part $v$ and the imaginary part $\omega$ from the altitude plot.

**Solution**

Solving $1 + e^{\omega t} = 1.4843$ gives us $v = -0.1450 \, \frac{1}{\text{min}}$. Then, comparing the maxima that are separated by an interval of 10 minutes gives $\omega = \frac{2\pi}{10} = 0.62832 \, \text{rad/\text{min}}.$
If you solved in units of $\frac{1}{\text{min}}$ and $\frac{\text{rad}}{\text{rad}}$, then $v = -0.0024 \frac{1}{s}$ and $\omega = 0.0105 \frac{\text{rad}}{s}$.

b) Let the dynamical model for the altitude be

$$\frac{d}{dt} \begin{bmatrix} y(t) \\ \dot{y}(t) \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ a_1 & a_2 \end{bmatrix} \begin{bmatrix} y(t) \\ \dot{y}(t) \end{bmatrix},$$

where $y(t)$ is the deviation of the altitude from the steady state value, $\dot{y}(t)$ is the time derivative of $y(t)$, and $a_1$ and $a_2$ are constants. Using your answer to part (a), find what $a_1$ and $a_2$ are.

**Solution**

The eigenvalues of $A = \begin{bmatrix} 0 & 1 \\ a_1 & a_2 \end{bmatrix}$ are given by $0 = \lambda^2 - a_2 \lambda - a_1$, or equivalently,

$$\lambda = \frac{a_2 \pm \sqrt{a_2^2 + 4a_1}}{2} = v \mp j\omega.$$

Solving for $a_1$ and $a_2$ (using the $\frac{1}{\text{min}}$ and $\frac{\text{rad}}{\text{min}}$ values of $v$ and $\omega$), we get

$$a_2 = 2v = -0.2900 \text{ and } a_1 = -\omega^2 - \frac{a_2^2}{4} = -0.4158.$$

If you solved using the $\frac{1}{s}$ and $\frac{\text{rad}}{s}$ values of $v$ and $\omega$, then

$$a_2 = 2v = -0.0048 \text{ and } a_1 = -\omega^2 - \frac{a_2^2}{4} = -1.16 \cdot 10^{-4}.$$

c) Otto can change $a_2$ by turning a knob. Tell him what value he should pick so that he has a “critically damped” ascent with two real negative eigenvalues at the same location.

**Solution**

To get two real identical eigenvalues, Otto should choose $a_2$ to make $a_2^2 + 4a_1 = 0$. This means that $a_2 = \pm 2\sqrt{-a_1}$. Under this condition, there will be a single eigenvalue, $\frac{a_2}{2}$. Since this is required to be negative, we only look at the negative root.

Solving with the $a_1$ derived from the $\frac{1}{\text{min}}$ and $\frac{\text{rad}}{\text{min}}$ values of $v$ and $\omega$, he should tune his knob to

$$a_2 = -2\sqrt{-a_1} = -2\sqrt{0.4158} = -1.2897.$$

If you solved using $a_1$ derived from the $\frac{1}{s}$ and $\frac{\text{rad}}{s}$ values of $v$ and $\omega$, then you get

$$a_2 = -2\sqrt{-a_1} = -2\sqrt{1.16 \cdot 10^{-4}} = -0.0215.$$