This homework is due on Thursday, October 15, 2020, at 10:59PM. Self-grades are due on Thursday, October 22, 2020, at 10:59PM.

Solutions have been provided for Problems 1-3 but Questions 4-6 are NOT optional and must be turned in on Gradescope.

1 One circuit, Many analyses

In this problem, we will recap several different methods that we have learnt and apply them all to the same circuit but under different situations.

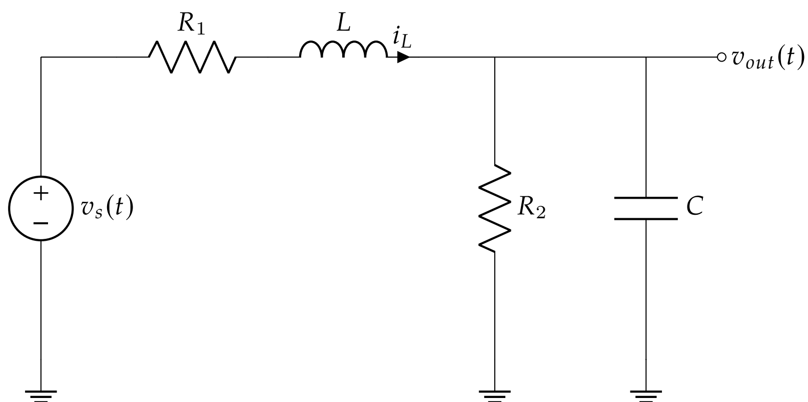


Figure 1: A model for a transmission line.

We are trying to transmit different signals across a very long wire. At longer lengths, the various electromagnetic losses incurred by a wire can be modeled using a model shown in Figure 1. We will take a look at the different kinds of signals that we want to transmit and which analyses techniques we should apply for assessing the output at receiving terminal v_{out}.

a) First we want to send a constant voltage value. We can do this by applying a constant voltage v_s as our input. If we apply $v_s = 12V$, find the capacitor voltage v_{out} and the inductor current i_L at equilibrium (or what we often refer to as DC steady-state). Use $R_1 = 100\Omega$, $R_2 = 100\Omega$, $C = 12\mu F$ and $L = 1mH$.

Solution

Applying a voltage $v_s = 12V$, at steady state, we know that the inductor acts as a short-circuit. Simultaneously, the capacitor acts an open circuit. The resulting equivalent circuit in steady-state will be

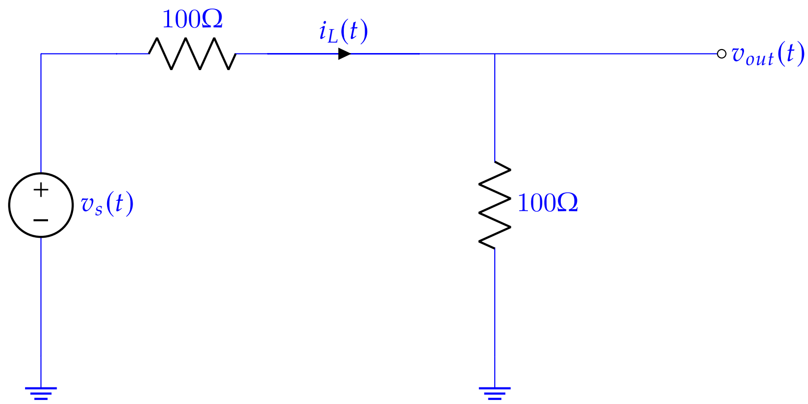


Figure 2: Equivalent circuit in steady state.

Using the voltage divider formulation, we have

$$\begin{aligned} v_{out} &= \frac{R_1}{R_1+R_2} v_s \\ &= \frac{100}{100+100} v_s && = 6V \end{aligned}$$

b) If $v_s(t)$ is a time-varying signal, write a system of differential equations using the inductor current i_L and capacitor voltage v_{out} as state variables. The equation system should be in the form

$$\frac{d}{dt}\begin{bmatrix} i_L \\ v_{out} \end{bmatrix} = A \begin{bmatrix} i_L \\ v_{out} \end{bmatrix} + B v_s(t) \tag{1}$$

Solution

Figure 3 shows the circuit we are analyzing. The devices have been named and additional currents and voltages have been labelled for clarity.

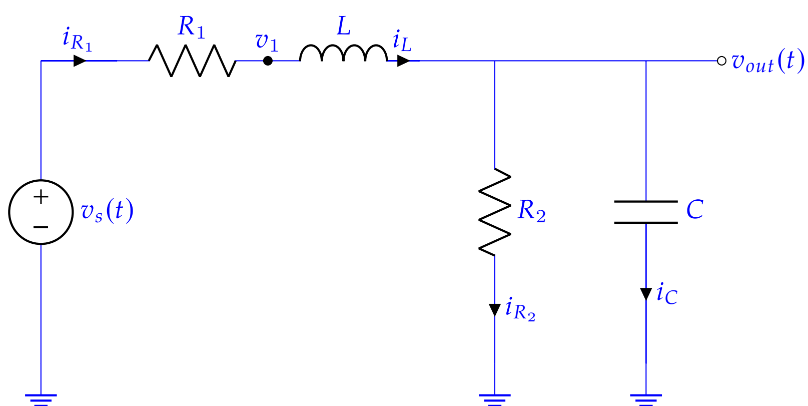


Figure 3: Transmission line model marked with additional currents and voltages.

KCL on the output node gives us

$$-i_L + i_C + i_{R_2} = 0. \tag{2}$$

Similarly, KCL at node 1, labelled with voltage v_1 gives us

$$-i_{R_1} + i_L = 0 \tag{3}$$

The inductor, capacitor and resistor $I-V$ relationships can be written using our circuit variables.

$$\begin{aligned} i_C &= C\frac{d}{dt}v_{out}(t) \\ i_{R_2} &= \frac{v_{out}(t)}{R_2} \\ L\frac{d}{dt}i_L(t) &= v_1 - v_{out}(t) \\ i_{R_1} &= \frac{v_s - v_1}{R_1} \end{aligned}$$

Combining Equations 2 and 3 with the $I-V$ relationships above, we can eliminate the additional variables i_{R_1}, v_1 and i_{R_2}. For Equation 2, this gives us

$$\begin{aligned} -i_L + C\frac{d}{dt}v_{out} + \frac{v_{out}}{R_2} = 0 \\ \frac{d}{dt}v_{out} &= -\frac{1}{R_2C}v_{out} + \frac{1}{C}i_L \end{aligned}$$

We can rewrite the $I-V$ relationship for the inductor above to obtain

$$\begin{aligned} L\frac{d}{dt}i_L &= v_s - Ri_L - v_{out} \\ \frac{d}{dt}i_L &= \frac{-R}{L}i_L - \frac{1}{L}v_{out} + \frac{1}{L}v_s \end{aligned}$$

We can now combine these to obtain our matrix equation system

$$\frac{d}{dt}\begin{bmatrix} i_L \\ v_{out} \end{bmatrix} = \begin{bmatrix} -\frac{R_1}{L} & -\frac{1}{L} \\ \frac{1}{C} & -\frac{1}{R_2C} \end{bmatrix}\begin{bmatrix} i_L \\ v_{out} \end{bmatrix} + \begin{bmatrix} \frac{1}{L} \\ 0 \end{bmatrix} v_s \tag{4}$$

c) Find the eigenvalues for the matrix A found above and comment on them.

Solution

The eigenvalues for matrix A are given by

$$\det\left(\begin{bmatrix} \lambda + \frac{R_1}{L} & \frac{1}{L} \\ -\frac{1}{C} & \lambda + \frac{1}{R_2C} \end{bmatrix}\right) = 0 \tag{5}$$

Expanding out the determinant for the matrix above, we get

$$\left(\lambda + \frac{R_1}{L}\right)\left(\lambda + \frac{1}{R_2C}\right) + \frac{1}{LC} = 0$$

$$\lambda^2 + \lambda\left(\frac{R_1}{L} + \frac{1}{R_2C}\right) + \frac{1}{LC}\left(1 + \frac{R_1}{R_2}\right) = 0$$

$$\lambda = -\frac{1}{2}\left(\frac{R_1}{L} + \frac{1}{R_2C}\right) \pm \frac{1}{2}\sqrt{\left(\frac{R_1}{L} + \frac{1}{R_2C}\right)^2 - \frac{4}{LC}\left(1 + \frac{R_1}{R_2}\right)}$$

$$= -\frac{1}{2}\left(\frac{R_1}{L} + \frac{1}{R_2C}\right) \pm \frac{1}{2}\sqrt{\left(\frac{R_1}{L} - \frac{1}{R_2C}\right)^2 - \frac{4}{LC}}$$

We can make a few observations about the eigenvalues

- The eigenvalues are real if $\left(\frac{R_1}{L} - \frac{1}{R_2C}\right) > \sqrt{\frac{4}{LC}}$.
- If the eigenvalues are not purely real, they occur as complex conjugate pairs.
- The real part of the eigenvalues is negative. This implies that in the absence of an external input $v_s(t)$, the state variables will settle to a steady-state value of 0.

d) We want to send a pulse instead of a steady value. For this case, $v_s(t)$ is shown in Figure 4.

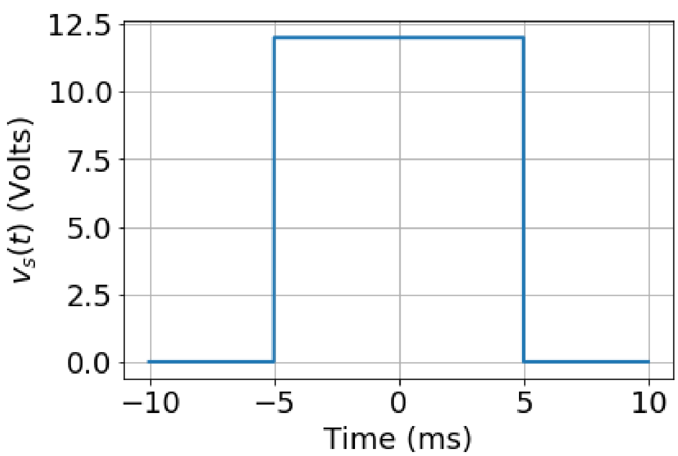
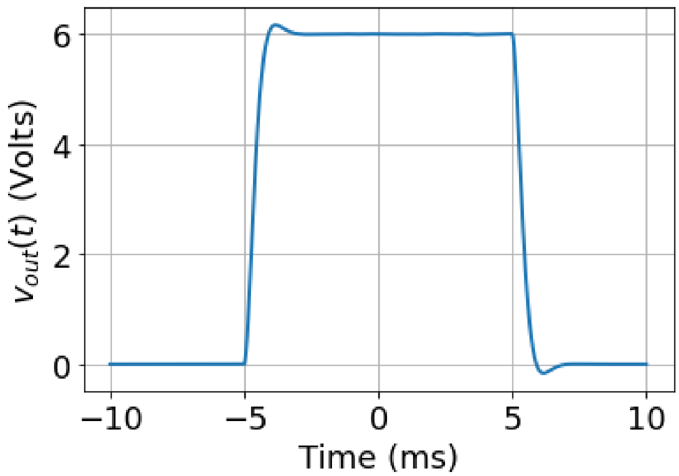


Figure 4: Pulse input to be transmitted across the wire.

It is not always possible or convenient to solve differential equations by hand using eigenvalues and the guess-and-check methods we have developed so far in the class. In the supplied Jupyter notebook *RLC_Circuit_Analsysis.ipynb*, fill out the entries for matrices A and B. The notebook has an implementation of a numerical solution to differential equations.

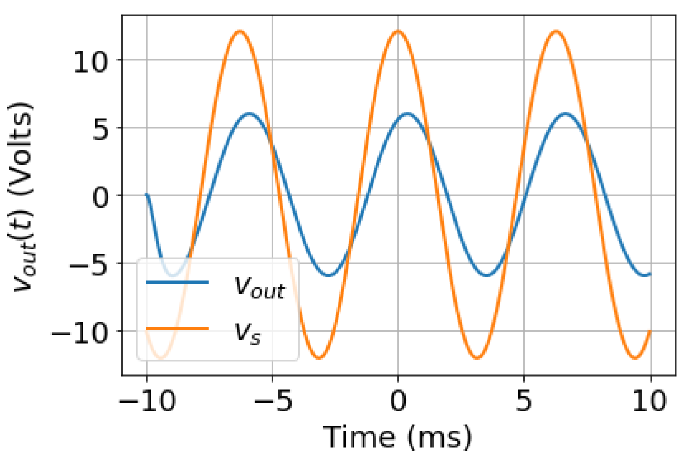
Sketch the output $v_{out}(t)$ for the pulse input $v_s(t)$ shown in Figure 4 using the supplied python notebook. The circuit parameters for this problem have been specified in the notebook: $R_1 = 10\Omega$, $R_2 = 10\Omega$, $C = 24\mu F$ and $L = 5mH$.

Solution



e) Finally, we want to test how our transmission line will carry a sinusoidal input. First, we will use the numerical technique that we saw in the previous part to evaluate the output $v_{out}(t)$. Using the code provided in the supplied iPython notebook, plot the output $v_{out}(t)$ for a sinusoidal input $v_s(t)$. Look at the last 2 cycles of the input and plot the corresponding output. Note the difference in amplitude and phase between the input $v_s(t)$ and the output $v_{out}(t)$. We will come back to these when we repeat this calculation for the sinusoidal steady state using phasor analysis.

Solution



f) We now want to transmit a sine wave, $v_s(t) = 12\sin(\omega t)$ along the transmission. Using phasor analysis, find the transfer function

$$H(\omega) = \frac{V_{out}}{V_s}, \tag{6}$$

where V_s is a phasor representing the input voltage $v_s(t)$ and V_{out} is a phasor representing the output voltage $v_{out}(t)$. Find the output phasor V_{out}. If we use the same circuit parameters from part (d), comment on how the transfer function relates the time-waveforms $v_s(t)$ and $v_{out}(t)$ in (e).

HINT: Try using different initial conditions to see how the numerical solution changes.

Solution

In sinusoidal steady-state, we can analyze various currents and voltages in the circuit using phasors.

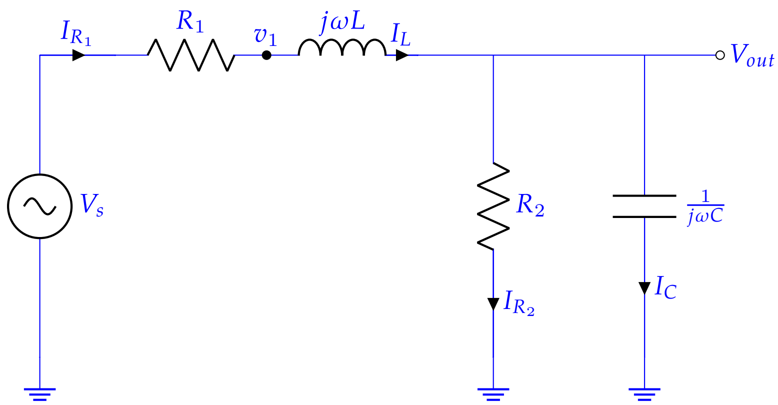


Figure 5: Transmission line model with phasors.

The impedances Z_{R_2} and Z_C can be combined in parallel

$$Z_{R_2||C} = \frac{\frac{R_2}{j\omega C R_2}}{R_2 + \frac{1}{j\omega C}} = \frac{R_2}{1 + j\omega R_2 C}. \tag{7}$$

Impedances Z_{R_1} and Z_L can be combined in series to give

$$Z_{R_1,L} = R_1 + j\omega L. \tag{8}$$

The transfer function, $H(\omega) = \frac{V_{out}}{V_s}$ can be found using the voltage divider formulation

$$V_{out} = V_{in}\left(\frac{Z_{R_2||C}}{Z_{R_2||C} + Z_{R_1,L}}\right). \tag{9}$$

Plugging in the equivalent impedances $Z_{R_2||C}$ from Equation 7 and $Z_{R_1,L}$ from Equation 8, we get

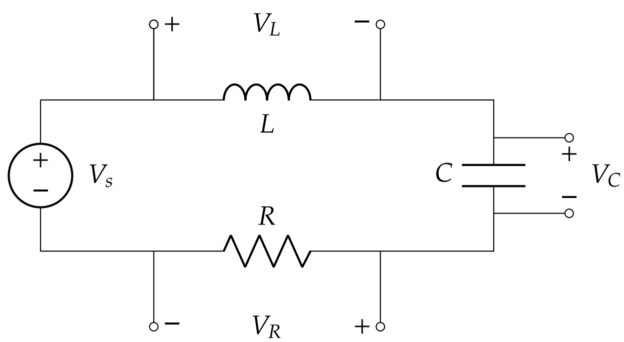
$$
\begin{aligned}
H(\omega) &= \frac{V_{out}}{V_s} \\
&= \frac{Z_{R_2||C}}{Z_{R_2||C} + Z_{R_1,L}} \\
&= \frac{\frac{R_2}{1+j\omega R_2 C}}{\frac{R_2}{1+j\omega R_2 C} + R_1 + j\omega L} \\
&= \frac{R_2}{R_2 + \left(1 + j\omega R_2 C\right)\left(R_1 + j\omega L\right)} \\
&= \frac{R_2}{R_1 + R_2 + j\omega\left(L + R_1 R_2 C\right) + (j\omega)^2\left(R_2 LC\right)} \\
&= \frac{\frac{R_2}{R_1+R_2}}{1 + j\omega\left(\frac{L}{R_1+R_2} + \frac{R_1 R_2}{R_1+R_2}C\right) + (j\omega)^2\left(\frac{R_2}{R_1+R_2}LC\right)}
\end{aligned}
$$

In periodic steady-state, the ratio of the peak amplitudes for $v_{out}(t)$ and $v_s(t)$ is given by the magnitude $|\,H(\omega)\,|$ of the transfer function. On the other hand, the phase difference between the waveforms for $v_{out}(t)$ and $v_s(t)$ is given by the phase $\angle H(\omega)$ of the transfer function.

2 RLC circuit as passive filters

As originally conceived by Bode in the 1930s, Bode plot is only an asymptotic approximation of the frequency response, using straight line segments. It relies on using a logarithmic scale for the input frequency ω to express the magnitude of the transfer functions on a logarithmic scale $\log_{10}|\,H(\omega)\,|$.

In this question, we will go through some examples to appreciate the beauty and simplicity of Bode plots. In the iPython notebook *BodePlots.ipynb*, you will see how well the approximation of Bode plots is in different regions. In particular, we will work with the RLC circuit shown below:



In the following questions, we will be exploring how to use the above RLC circuit to construct highpass, lowpass, and bandpass filters. As the name suggests, a highpass filter will suppress the low frequency components while keeping the high frequency components of the input unblocked. Since the circuit contains only passive elements, namely resistors, capacitors, and inductors, these filters are called *passive filters*. On the other hand, if the circuit contains op amps, transistors, or other active devices, it will become *active filters*.

a) **Lowpass filter.** Treat V_s as the input and V_C as the output. Obtain the transfer function $H_{LP} = \frac{V_C}{V_s}$, and its magnitude and phase. Draw the Bode plot for the magnitude. Explain why this is a lowpass filter.

Solution

To obtain the transfer function, we use the definition:

$$H_{LP}(\omega) = \frac{V_C}{V_s} = \frac{(1/j\omega C)I}{V_s} = \frac{1}{(1-\omega^2 LC) + j\omega RC}$$

where we used $V_s = I(R + j\omega L + \frac{1}{j\omega C})$, so $\frac{I}{V_s} = \frac{1}{R+j\omega L+\frac{1}{j\omega C}}$. For the magnitude:

$$M_{LP}(\omega) = |H_{LP}(\omega)| = \frac{1}{\sqrt{(1-\omega^2 LC)^2 + \omega^2 R^2 C^2}}$$

$$\phi_{LP} = \begin{cases} -\pi - \tan^{-1}\left(\frac{\omega RC}{1-\omega^2 LC}\right) & \omega > \frac{1}{\sqrt{LC}} \\ -\tan^{-1}\left(\frac{\omega RC}{1-\omega^2 LC}\right) & \omega < \frac{1}{\sqrt{LC}} \\ -\frac{\pi}{2} & \omega = \frac{1}{\sqrt{LC}} \end{cases}$$

where we calculate the phase by first identifying which quadrant the complex transfer function lies in and then use arctan to get the angle.

For the bode plots, let $\omega_0 = \frac{1}{\sqrt{LC}}$ and $Q = \frac{\omega_0 L}{R}$, then we have:

$$H_{LP}(\omega) = \frac{1}{(1-(\omega/\omega_0)^2) + j\frac{\omega}{Q\omega_0}}$$

The magnitude response for this transfer function is shown in Figure 6.

It is a low pass filter because the transfer has large magnitude when the frequency is low (on the left side of the graph), and the magnitude is reducing exponentially as the frequency is above the resonance frequency ω_0.

If we change the resistance R, you can see that it doesn't change the resonance frequency ω_0, but it does change $Q = \frac{\omega_0 L}{R}$, which is known as the quality factor. As we increase it, we have more dramatic peaks around the resonance frequency, and it becomes more pronounced relative to the level of the asymptotes. In the plot, we show several cases for the Q, you can see that for large Q, the asymptotic approximation by bode plots are less accurate especially around the resonance frequency.

b) **Highpass filter.** Let V_L be the output. Obtain the transfer function $H_{HP} = \frac{V_L}{V_s}$, and its magnitude and phase. Draw the Bode plot for the magnitude. Explain why this is a highpass filter.

Solution

To obtain the transfer function, we use the definition:

$$H_{HP}(\omega) = \frac{V_L}{V_s} = \frac{j\omega LI}{V_s} = \frac{-\omega^2 LC}{(1-\omega^2 LC) + j\omega RC}$$

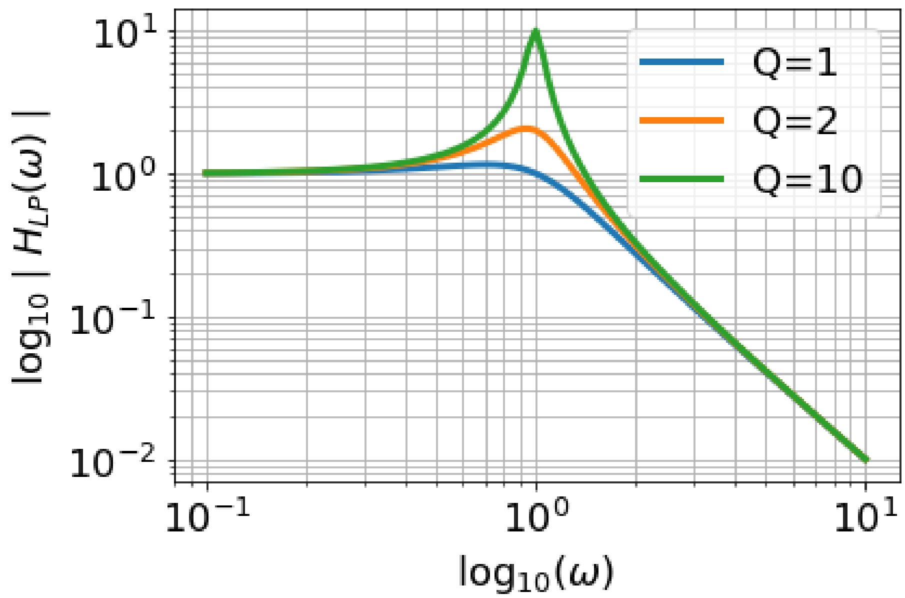


Figure 6: Magnitude response of a low-pass filter

where we used $V_s = I(R + j\omega L + \frac{1}{j\omega C})$, so $\frac{I}{V_s} = \frac{1}{R+j\omega L+\frac{1}{j\omega C}}$. For the magnitude:

$$M_{HP}(\omega) = |H_{HP}(\omega)| = \frac{\omega^2 LC}{\sqrt{(1-\omega^2 LC)^2 + \omega^2 R^2 C^2}}$$

$$\phi_{HP} = \begin{cases} -\tan^{-1}\left(\frac{\omega RC}{1-\omega^2 LC}\right) & \omega > \frac{1}{\sqrt{LC}} \\ \pi - \tan^{-1}\left(\frac{\omega RC}{1-\omega^2 LC}\right) & \omega < \frac{1}{\sqrt{LC}} \\ \frac{\pi}{2} & \omega = \frac{1}{\sqrt{LC}} \end{cases}$$

where we calculate the phase by first identifying which quadrant the complex transfer function lies in and then use arctan to get the angle.

$$\phi_{HP} = \pi - \tan^{-1}\left(\frac{\omega RC}{1-\omega^2 LC}\right)$$

For the bode plots, let $\omega_0 = \frac{1}{\sqrt{LC}}$ and $Q = \frac{\omega_0 L}{R}$, then we have:

$$H_{LP}(\omega) = \frac{-(\omega/\omega_0)^2}{(1-(\omega/\omega_0)^2) + j\frac{\omega}{Q\omega_0}}$$

Figure 7 show the magnitude plot for this transfer function.

It is a high pass filter because the transfer has large magnitude when the frequency is high (on the right side of the graph), and the magnitude is reducing exponentially as the frequency is less than the resonance frequency ω_0.

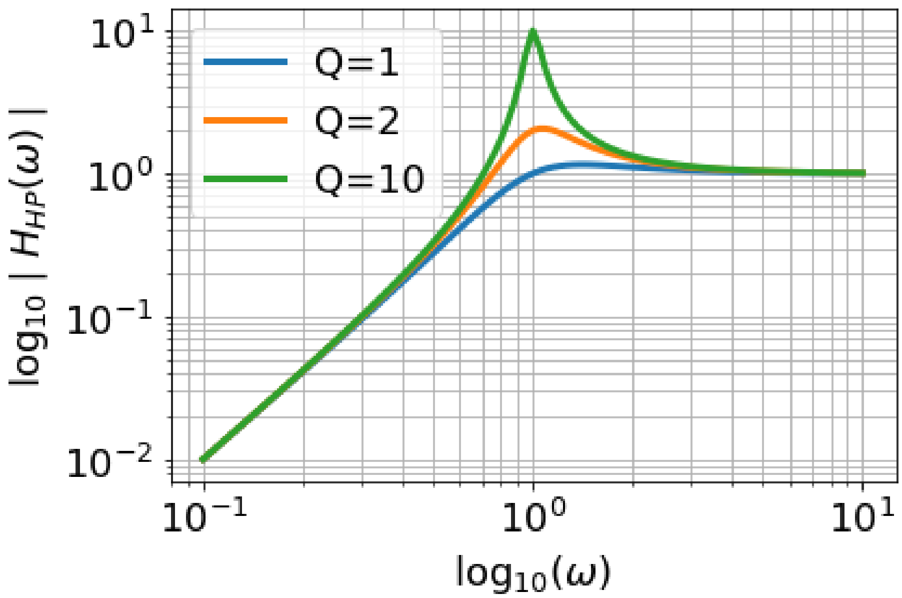


Figure 7: Magnitude response of a high pass filter.

Similar to the previous part, if we increase Q, we have more dramatic peaks around the resonance frequency, and it becomes more pronounced relative to the level of the asymptotes.

c) **Bandpass filter.** How can you obtain a bandpass filter based on your findings above? Write out the transfer function and its magnitude and phase.

Solution

Yes we can. The output will be V_R to construct a bandpass filter:

$$H_{BP}(\omega) = \frac{V_R}{V_s} = \frac{RI}{V_s} = \frac{j\omega RC}{(1-\omega^2 LC) + j\omega RC}$$

where we used $V_s = I(R + j\omega L + \frac{1}{j\omega C})$, so $\frac{I}{V_s} = \frac{1}{R+j\omega L+\frac{1}{j\omega C}}$.

$$M_{BP}(\omega) = |H_{BP}(\omega)| = \frac{\omega RC}{\sqrt{(1-\omega^2 LC)^2 + \omega^2 R^2 C^2}}$$

$$\phi_{LP} = \begin{cases} -\frac{\pi}{2} - \tan^{-1}\left(\frac{\omega RC}{1-\omega^2 LC}\right) & \omega > \frac{1}{\sqrt{LC}} \\ \frac{\pi}{2} - \tan^{-1}\left(\frac{\omega RC}{1-\omega^2 LC}\right) & \omega < \frac{1}{\sqrt{LC}} \\ 0 & \omega = \frac{1}{\sqrt{LC}} \end{cases}$$

where we calculate the phase by first identifying which quadrant the complex transfer function lies in and then use arctan to get the angle.

d) The **resonant frequency**, ω_0, is the input frequency (other than 0 and ∞) that leads to the elimination of the imaginary part of the circuit impedance, i.e., the impedance is purely real. Find the resonant frequency for the RLC circuit above.

Solution

The impedance of the circuit as measured from both sides of the voltage source is given by:

$$Z_R + Z_L + Z_C = R + j\omega L + \frac{1}{j\omega C} = R + j(\omega L - \frac{1}{\omega C}) \tag{10}$$

which is the same for all the lowpass, highpass, and bandpass filters. To eliminate the imaginary part, we can set:

$$\omega_0 L = \frac{1}{\omega_0 C} \Rightarrow \omega_0 = \sqrt{\frac{1}{LC}} \tag{11}$$

which, by definition, is the resonant frequency.

e) For mobile communications, the center frequency is approximately 800 MHz. In the IPython notebook, experiment with different L and C to center the bandpass filter.

Solution

To center the bandpass filter, we need to find L and C such that $\omega_0 = \sqrt{\frac{1}{LC}} = 8 \times 10^8$. Please refer to the IPython notebook for suitable values.

You might have noticed that the advantage of Bode plot is that it makes it easier to work with transfer functions that have multiple factors. We can write $H(\omega)$ as a product of such factors:

$$H(\omega) = A_1(\omega)A_2(\omega)...A_n(\omega) \tag{12}$$

In this class, we will focus on functions A_1 to A_n that assume one of the possible forms.

$$\begin{aligned} \text{Constant factor: } H &= K \\ \text{Zero @ origin: } H &= (j\omega)^N \\ \text{Pole @ origin: } H &= 1/(j\omega)^N \\ \text{Zero @ } \omega_c\text{: } H &= (1 + j\omega/\omega_c)^N \\ \text{Pole @ } \omega_c\text{: } H &= 1/(1 + j\omega/\omega_c)^N \end{aligned}$$

The construction thus becomes simple addition or subtraction of these forms. For instance, $H(\omega) = 10\frac{1+j\omega/\omega_Z}{1+j\omega/\omega_p}$, where $A_1 = 10$, $A_2 = 1 + j\omega/\omega_Z$, $A_3 = \frac{1}{1+j\omega/\omega_p}$.

(f) For transfer function $H(\omega) = M(\omega)e^{j\phi(\omega)}$, how to represent the magnitude $M(\omega)$ and phase $\phi(\omega)$ with the magnitudes $|A_i(\omega)|$ and phase $\phi_{A_i}(\omega)$?

Solution

Since we have $H(\omega) = A_1(\omega)A_2(\omega)...A_n(\omega)$, and $A_i(\omega) = |A_i(\omega)|e^{j\phi_{A_i}(\omega)}$ in the polar representation, we have:

$$H(\omega) = M(\omega)e^{j\phi(\omega)} = |A_1(\omega)||A_2(\omega)| \cdots |A_n(\omega)|e^{j(\phi_{A_1}(\omega)+\phi_{A_2}(\omega)+\cdots+\phi_{A_n}(\omega))} \tag{13}$$

Therefore, we have:

$$\begin{aligned} M(\omega) &= |A_1(\omega)||A_2(\omega)| \cdots |A_n(\omega)| \\ \phi(\omega) &= \phi_{A_1}(\omega) + \phi_{A_2}(\omega) + \cdots + \phi_{A_n}(\omega) \end{aligned}$$

by comparison.

(g) Consider the transfer function

$$H(\omega) = \frac{(j10\omega + 30)^2}{(300 - 3\omega^2 + j90\omega)} \tag{14}$$

Refer to the IPython notebook for constructing the Bode plots for this transfer function.

Solution

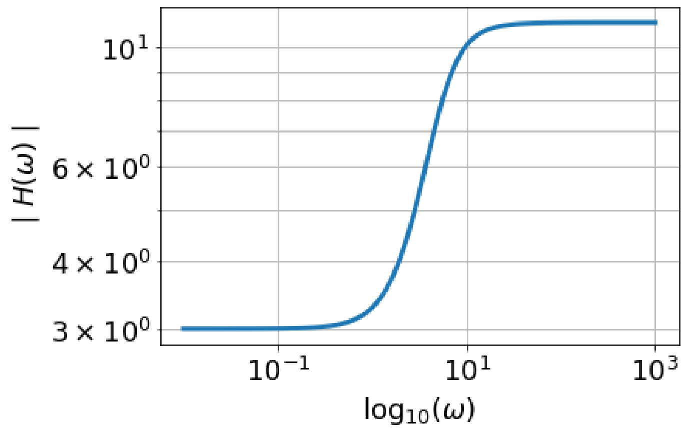


Figure 8: Magnitude response for the transfer function $H(\omega)$.

(h) In addition to the Bode plots, we also plotted the magnitude of the transfer functions without approximations. Please comment on the differences.

Solution

We can see that Bode plots are linear approximations of the magnitude of the transfer functions. The approximation is generally good at regions without corners. Around the corners, the exact plot usually have some "shootings" or smooth curves.

This homework problem addresses the essential concepts of Bode plots. Please check the table in Figure 9 to improve your understanding.

3 Otto the Pilot

Otto has devised a control algorithm, so that his plane climbs to the desired altitude by itself. However, he is having oscillatory transients as shown in the figure. Prof. Sanders told him that if his system has complex eigenvalues

$$\lambda_{1,2} = v \pm j\omega,$$

then his altitude would indeed oscillate with frequency ω about the steady state value, $1\,\text{km}$, and that the time trace of his altitude would be tangent to the curves $1 + e^{vt}$ and $1 - e^{vt}$ near its maxima and minima respectively.

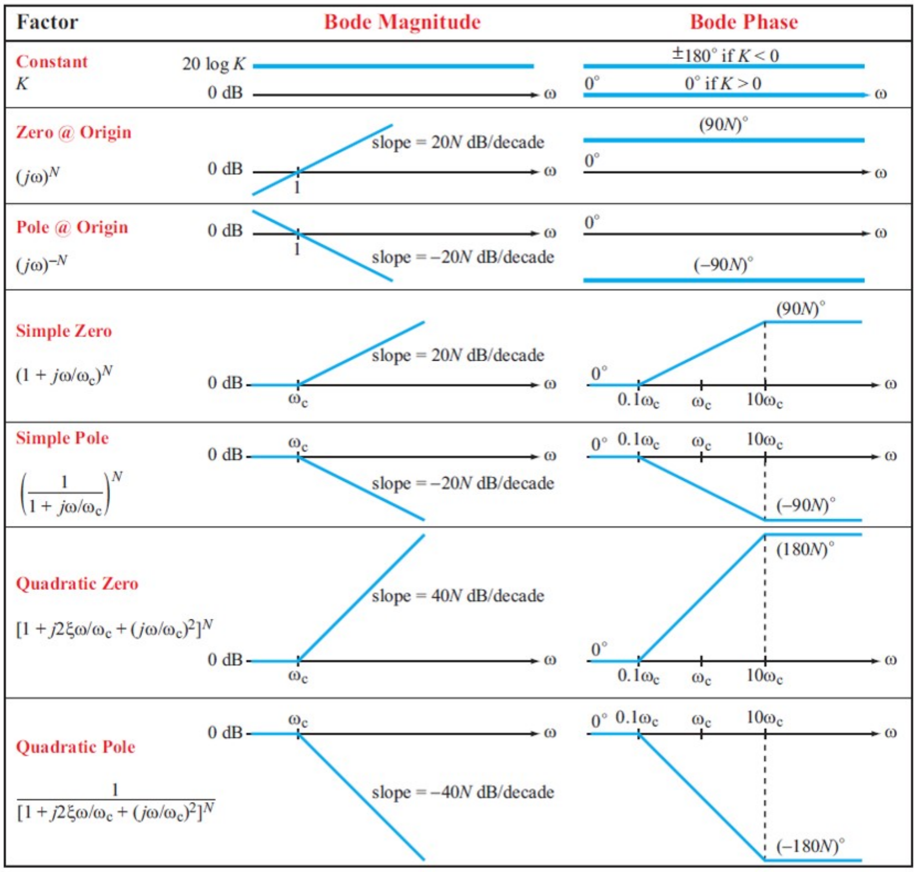
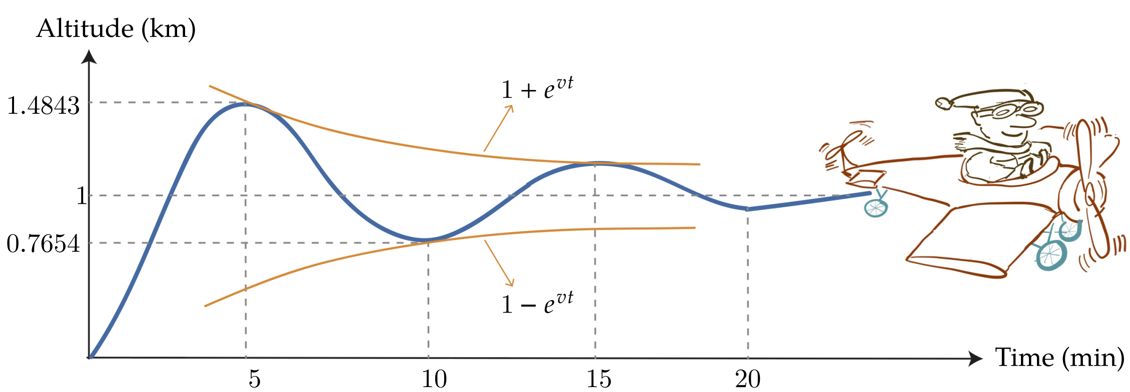


Figure 9: Reference for sketching Bode plots.



a) Find the real part v and the imaginary part ω from the altitude plot.

Solution

Solving $1 + e^{5v} = 1.4843$ gives us $v = -0.1450\,\frac{1}{\text{min}}$. Then, comparing the maxima that are separated by an interval of 10 minutes gives $\omega = \frac{2\pi}{10} = 0.628\,32\,\frac{\text{rad}}{\text{min}}$.

If you solved in units of $\frac{1}{\text{min}}$ and $\frac{\text{rad}}{\text{s}}$, then $v = -0.0024\,\frac{1}{\text{s}}$ and $\omega = 0.0105\,\frac{\text{rad}}{\text{s}}$.

b) Let the dynamical model for the altitude be

$$\frac{d}{dt}\begin{bmatrix} y(t) \\ \dot{y}(t) \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ a_1 & a_2 \end{bmatrix}\begin{bmatrix} y(t) \\ \dot{y}(t) \end{bmatrix},$$

where $y(t)$ is the deviation of the altitude from the steady state value, $\dot{y}(t)$ is the time derivative of $y(t)$, and a_1 and a_2 are constants. Using your answer to part (a), find what a_1 and a_2 are.

Solution

The eigenvalues of $A = \begin{bmatrix} 0 & 1 \\ a_1 & a_2 \end{bmatrix}$ are given by $0 = \lambda^2 - a_2\lambda - a_1$, or equivalently,

$$\lambda = \frac{a_2 \mp \sqrt{a_2^2 + 4a_1}}{2} = v \mp j\omega.$$

Solving for a_1 and a_2 (using the $\frac{1}{\text{min}}$ and $\frac{\text{rad}}{\text{min}}$ values of v and ω), we get

$$a_2 = 2v = -0.2900 \text{ and } a_1 = -\omega^2 - \frac{a_2^2}{4} = -0.4158.$$

If you solved using the $\frac{1}{\text{s}}$ and $\frac{\text{rad}}{\text{s}}$ values of v and ω, then

$$a_2 = 2v = -0.0048 \text{ and } a_1 = -\omega^2 - \frac{a_2^2}{4} = -1.16 \cdot 10^{-4}.$$

c) Otto can change a_2 by turning a knob. Tell him what value he should pick so that he has a "critically damped" ascent with two real negative eigenvalues at the same location.

Solution

To get two real identical eigenvalues, Otto should choose a_2 to make $a_2^2 + 4a_1 = 0$. This means that $a_2 = \pm 2\sqrt{-a_1}$. Under this condition, there will be a single eigenvalue, $\frac{a_2}{2}$. Since this is required to be negative, we only look at the negative root.

Solving with the a_1 derived from the $\frac{1}{\text{min}}$ and $\frac{\text{rad}}{\text{min}}$ values of v and ω, he should tune his knob to

$$a_2 = -2\sqrt{-a_1} = -2\sqrt{0.4158} = -1.2897.$$

If you solved using a_1 derived from the $\frac{1}{\text{s}}$ and $\frac{\text{rad}}{\text{s}}$ values of v and ω, then you get

$$a_2 = -2\sqrt{-a_1} = -2\sqrt{1.16 \cdot 10^{-4}} = -0.0215.$$

4 Predator-Prey Model

Ellen is a keen ecologist studying life on a planet far, far away. She found two species on the planet, one predator and the other prey. You found her journal back on Earth and are trying to decipher life on this other planet. In Ellen's journal, she states the following observations about the predator and prey populations

- The prey population grows at a rate proportional to the current prey population.
- For a given prey population, the rate of increase of prey population decreases linearly with the predator population.

As the predator population increases, it is harder for the prey population to grow.

- The predator population also grows at a rate proportional to the current predator population.
- For a fixed predator population, predator population increases linearly with prey population. As the amount of prey increases, predators have more food and their population increases faster.

Let's call these species **x (prey) and y (predator)** to keep things simple. A friend suggests using the following model for predator and prey populations:

$$\frac{d}{dt}x(t) = (a - by)x \tag{15}$$

$$\frac{d}{dt}y(t) = (cx - d)y \tag{16}$$

a) Comment on the validity of the model described in Equations 15 and 16.

Solution

We are given a set of proportionality and linearity constraints on the growth rates of the two populations. Since the growth rates are specified, we can model the populations of these species as a continuous system. Modeling often comes with a large set of assumptions.

Here, we assume that the growth rate of the predator and prey species can be said to be the derivative of our variables x and y with respect to time. For a fixed level of *other* population, the model shows a proportionality between the rate of change of the population of a species and its current population. This stems from our knowledge that population growth and decay are often exponential in nature. When resources are abundant, populations rise exponentially. While on the other hand, when resources are scarce, populations decay exponentially.

The underlying biological reality also constrains the model parameters. Consider the scenario when there are no predators ($y = 0$). In this case, the prey population should increase since it is not checked by the predators. This would imply that a in the model is a positive number.

Similarly, in the absence of prey $x = 0$, the predators do not have a food source and going back to the model, this would imply that $d > 0$.

b) You find a data entry in Ellen's journal measuring the growth rates of predator and prey populations. At a predator population of 10 and prey population of 10, both populations increase at $10/s$. With a single prey remaining, the predators starve and their population decreases at $8/s$ at a population of 10 predators. At population levels of 1 predator and 10 prey, the prey population rapidly increases at $100/s$.

Using these observations, **find the model parameters a, b, c and d**.

Solution

Plugging in the measurements from the journal, for the prey population, we have

$$\begin{aligned} 10 &= (a - 10b)10 \\ 100 &= (a - 1b)10 \end{aligned}$$

Solving the simultaneous system of equations, we get

$$\begin{aligned} a &= 11 \\ b &= 1 \end{aligned}$$

For the predator population, we have

$$\begin{aligned} 10 &= (10c - d) * 10 \\ -8 &= (1c - d) * 10 \end{aligned}$$

Solving this system of equations, we get

$$\begin{aligned} c &= 0.2 \\ d &= 1 \end{aligned}$$

Putting our results together, the predator-prey populations evolve as

$$\begin{aligned} \frac{d}{dt}x(t) &= (11 - 1y)x \\ \frac{d}{dt}y(t) &= (0.2x - 1)y \end{aligned}$$

c) If we had 1000 predators on the planet and no prey. What will happen to the predator population?

Solution

Intuitively, from the lack of prey, the predator population will start decrease. When we plug in the numbers, we see that for $x = 0$ and $y = 1000$,

$$\begin{aligned} \frac{d}{dt}y(t) &= (0.2 \cdot 0 - 1)1000 \\ &= -1000 \end{aligned}$$

Since $\frac{dy}{dt}$ is a large negative number, the predator population will start decreasing rapidly in the absence of prey.

d) At what levels will the predator and prey populations not change any more. As we have seen in class, this set of population levels is called an equilibrium.

Solution

For the predator and prey population to be stable, or in other words, not be changing over time, we need

$$\frac{dx}{dt} = 0\,,$$
$$\frac{dy}{dt} = 0.$$

From the expressions we have previously derived for the predator and prey population, we get

$$(11 - 1y)x = 0$$
$$(0.2x - 1)y = 0$$

We can see that $x = 0$ and $y = 0$ is a solution to this system of equations and intuitively, if there we no predator and prey to begin with, their populations (or the lack of) would stay stable at those levels. However, there is another more interesting situation. If there were 5 prey and 11 predators, their populations could still be balanced.

e) We want to analyze the model at a set of population levels. We don't know how to solve the model described in Equations 15 and 16. We can, however, linearize the model around a set of population levels and analyze the population changes in that vicinity.

If the predator population is 10 and the prey population is 1, linearize the system around these populations and express the new, linearized system in the form

$$\frac{d}{dt}\begin{bmatrix}\tilde{x}\\ \tilde{y}\end{bmatrix} = A\begin{bmatrix}\tilde{x}\\ \tilde{y}\end{bmatrix} + B \tag{17}$$

Solution

We can linearize the system using partial derivatives. In general, for a system with 2 state-variables x and y and a general state-space model

$$\frac{dx}{dt} = f(x, y)$$
$$\frac{dy}{dt} = g(x, y),$$

the linearized model around x^*, y^* can by found using a Taylor expansion of f and g around the linearization point. This gives us

$$f(x^* + \Delta x, y^* + \Delta y) = f(x^*, y^*) + \left.\frac{\partial f}{\partial x}\right|_{x*}\Delta x + \left.\frac{\partial f}{\partial y}\right|_{y*}\Delta y$$
$$g(x^* + \Delta x, y^* + \Delta y) = g(x^*, y^*) + \left.\frac{\partial g}{\partial x}\right|_{x*}\Delta x + \left.\frac{\partial g}{\partial y}\right|_{y*}\Delta y$$

Renaming our deviations from the linearization point x^* (Δx) and y^* (Δy) to be the new state variable $\tilde{x}$ and $\tilde{y}$, we get our linearized model

$$\frac{d}{dt}\begin{bmatrix}\tilde{x}\\ \tilde{y}\end{bmatrix} = \begin{bmatrix}\frac{\partial f}{\partial x} & \frac{\partial f}{\partial y}\\ \frac{\partial g}{\partial x} & \frac{\partial g}{\partial y}\end{bmatrix}\begin{bmatrix}\tilde{x}\\ \tilde{y}\end{bmatrix} + \begin{bmatrix}f(x^*, y^*)\\ g(x^*, y^*)\end{bmatrix} \tag{18}$$

With the model given to use in Equations 15 and 16,

$$f(x,y) = (11 - 1y)x$$
$$g(x,y) = (0.2x - 1)y$$

We have the partial derivatives

$$\frac{\partial f}{\partial x} = 11 - 1y$$
$$\frac{\partial f}{\partial y} = -x$$
$$\frac{\partial g}{\partial x} = 0.2y$$
$$\frac{\partial g}{\partial y} = 0.2x - 1$$

Evaluating these at $x^* = 1$ and $y^* = 10$, we get

$$A = \begin{bmatrix} 1 & -1 \\ 2 & -0.8 \end{bmatrix} \tag{19}$$

In this linearized model, B is given by $\begin{bmatrix} f(x^*, y^*) \\ g(x^*, y^*) \end{bmatrix}$.

$$B = \begin{bmatrix} 1 \\ -8 \end{bmatrix} \tag{20}$$

5 Checkpoint Feedback Form

a) Please fill out the survey.

If the survey link doesn't work, please refer to Piazza to complete the survey.

6 Homework Process and Study Group

Citing sources and collaborators are an important part of life, including being a student!
We also want to understand what resources you find helpful and how much time homework is taking, so we can change things in the future if possible.

a) **What sources (if any) did you use as you worked through the homework?**

b) **If you worked with someone on this homework, who did you work with?**
List names and student ID's. (In case of homework party, you can also just describe the group.)

c) **Roughly how many total hours did you work on this homework?**

d) **Do you have any feedback on this homework assignment?**