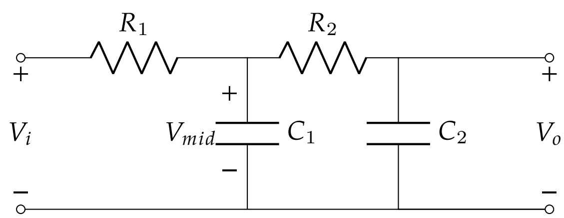
This homework is due on Thursday, October 8, 2020, at 10:59PM.

Self-grades are due on Thursday, October 15, 2020, at 10:59PM.

1 Transfer functions

Consider the circuit below.



The circuit has an input phasor voltage V_i at frequency ω rad/sec applied at the input terminals shown in the illustration above, causing an output phasor voltage V_o at output terminals.

We are going to construct the transfer function $H(\omega) = \frac{V_o}{V_i}$ in two steps. We will compute two intermediate transfer functions, $H_1(\omega) = \frac{V_{mid}}{V_i}$ and $H_2(\omega) = \frac{V_o}{V_{mid}}$. Then, we will find the overall transfer function as the product of these two intermediate transfer functions, i.e. $H(\omega) = \frac{V_o}{V_i} = H_1(\omega)H_2(\omega)$.

This approach is valid since the V_{mid} cancel.

a) For the first step, **find the intermediate transfer function** $H_2(\omega) = \frac{V_o}{V_{mid}}$**.** Have your expression be in terms of Z_{R2} and Z_{C2}, that is the impedances of R_2 and C_2.

Solution

V_{mid} and V_o are in a voltage divider configuration, with impedances Z_{R2} and Z_{C2}. This means

$$V_o = (V_{mid})\frac{Z_{C2}}{Z_{R2} + Z_{C2}}, \tag{1}$$

so we can say

$$H_2(\omega) = \frac{V_o}{V_{mid}} = \frac{Z_{C2}}{Z_{R2} + Z_{C2}}. \tag{2}$$

b) Now, **compute the other intermediate transfer function** $H_1(\omega) = \frac{V_{mid}}{V_i}$. Have your expression be in terms of Z_{R1}, Z_{R2}, Z_{C1}, and Z_{C2}. (i.e. Don't forget to consider the impact of loading by R_2 and C_2 in this transfer function.) *hint: Applying KCL at the V_{mid} node would be a good place to start. You should try to find an expression for H_1 that has factors that H_2 can cancel out.*

Solution

From KCL, we have

$$\frac{V_{mid}-V_i}{Z_{R1}}+\frac{V_{mid}}{Z_{C1}}+\frac{V_{mid}}{Z_{R2}+Z_{C2}}=0 \tag{3}$$

$$V_{mid}\left(\frac{1}{Z_{R1}}+\frac{1}{Z_{C1}}+\frac{1}{Z_{R2}+Z_{C2}}\right)=V_i\left(\frac{1}{Z_{R1}}\right) \tag{4}$$

$$H_1(\omega)=\frac{V_{mid}}{V_i}=\frac{1}{Z_{R1}\left(\frac{1}{Z_{R1}}+\frac{1}{Z_{C1}}+\frac{1}{Z_{R2}+Z_{C2}}\right)} \tag{5}$$

c) Then, **use these two intermediate transfer functions to calculate the overall transfer function** as $H(\omega)=\frac{V_o}{V_i}=H_1(\omega)H_2(\omega)$.

Solution

Combining the above formulas, we have

$$\begin{aligned}
H(\omega) &= \frac{Z_{C2}}{Z_{R2}+Z_{C2}}\frac{1}{Z_{R1}\left(\frac{1}{Z_{R1}}+\frac{1}{Z_{C1}}+\frac{1}{Z_{R2}+Z_{C2}}\right)} && (6)\\
&= \frac{Z_{C2}}{Z_{R2}+Z_{C2}}\frac{1}{1+\frac{Z_{R1}}{Z_{C1}}+\frac{Z_{R1}}{Z_{R2}+Z_{C2}}} && (7)\\
&= \frac{Z_{C2}}{Z_{R2}+Z_{C2}+\frac{Z_{R2}+Z_{C2}}{Z_{C1}}Z_{R_1}+Z_{R1}} && (8)\\
&= \frac{Z_{C1}Z_{C2}}{Z_{R1}Z_{C1}+Z_{C1}Z_{C2}+Z_{R1}Z_{C2}+Z_{R2}Z_{C1}+Z_{R1}Z_{R2}}. && (9)
\end{aligned}$$

d) Sometimes it is useful to collect all the frequency dependence into one place and to figure out how to think about what scale might be somewhat natural for the frequency. **Obtain an expression for $H(\omega)=V_o/V_i$ in the form**

$$H(\omega)=\frac{V_o}{V_i}=\frac{1}{1+2\xi\frac{j\omega}{\omega_c}+\frac{(j\omega)^2}{\omega_c^2}},$$

given that $R_1=2\,\Omega$, $R_2=4\,\Omega$, $C_1=\frac{9}{2}\,\mathrm{F}$, and $C_2=1\,\mathrm{F}$. What are the values of ξ and ω_c?

Solution

The impedance in the phasor domain is given by

$$Z_{R1}=R_1 \qquad Z_{R2}=R_2 \qquad Z_{C1}=\frac{1}{j\omega C_1} \qquad Z_{C2}=\frac{1}{j\omega C_2}$$

Using the result in part (a), we have

$$H(\omega) = \frac{\frac{1}{j\omega C_1}\frac{1}{j\omega C_2}}{\frac{R_1}{j\omega C_1} + \frac{1}{j\omega C_1}\frac{1}{j\omega C_2} + \frac{R_1}{j\omega C_2} + \frac{R_2}{j\omega C_1} + R_1R_2} \tag{10}$$

$$= \frac{1}{1 + j\omega\big(R_1C_1 + (R_1+R_2)C_2\big) + (j\omega)^2R_1R_2C_1C_2} \tag{11}$$

$$= \frac{1}{1 + j\omega(15) + (j\omega)^2(36)} \tag{12}$$

$$= \frac{1}{1 + 2(15/12)\frac{j\omega}{1/6} + (\frac{j\omega}{1/6})^2}, \tag{13}$$

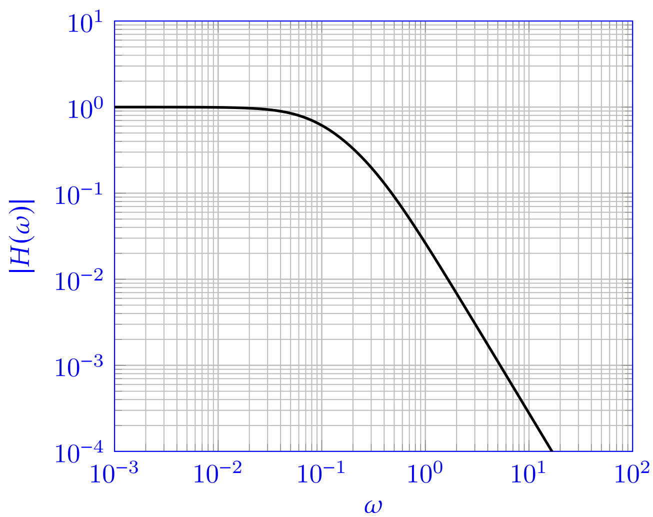
Which means $\xi = \frac{15}{12} = \frac{5}{4}$ and $\omega_c = \frac{1}{6}$ rad/s.

e) Use a computer to **draw Bode Plots of $|H(\omega)|$ and $\angle H(\omega)$.** What kind of filter (low-pass, high-pass, band-pass) is this? Comment on the slope of the log-log magnitude plot at low and high frequencies. Why would you use this instead of a simple RC filter?

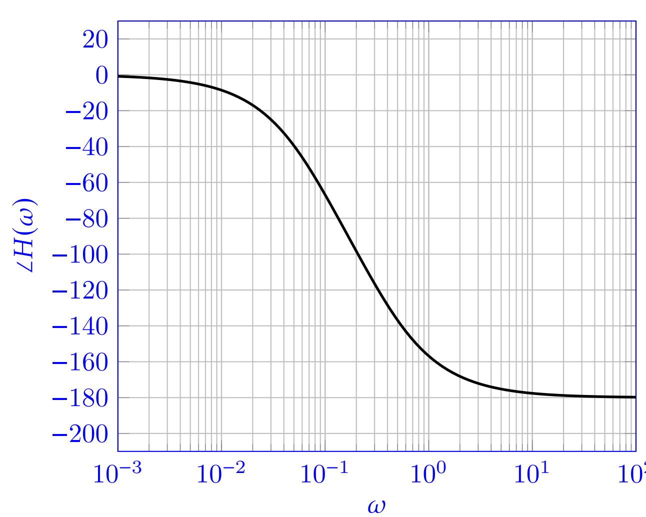
Solution

Evaluating the transfer function above

Log-log plot of transfer function magnitude



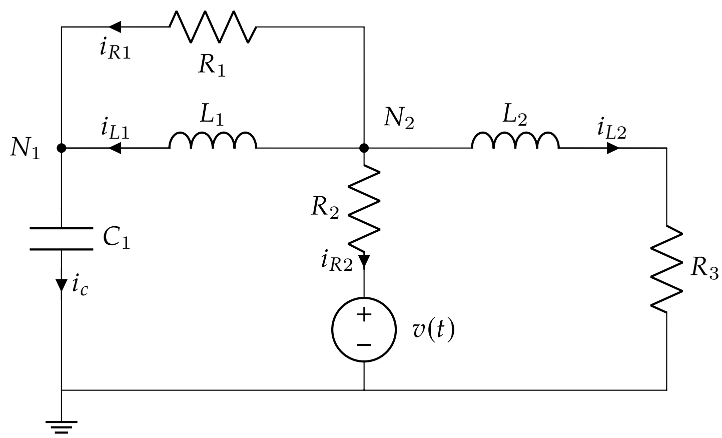
Semi-log plot of transfer function phase



This is a low pass filter. Note that at low frequencies, the slope of the magnitude is 0, and at high frequencies it is -2 decades/decade. Recall that a simple RC low-pass filter has a slope of -1 at high frequencies, so a higher order filter like this one provides a sharper drop-off.

2 Phasor-Domain Circuit Analysis

The analysis techniques you learned previously for resistive circuits are equally applicable for analyzing AC circuits (circuits driven by sinusoidal inputs) in the phasor domain. In this problem, we will walk you through the steps with a concrete example. Consider the circuit below.



The components in this circuit are given by:
Voltage source:

$$v(t) = 10\sqrt{2}\cos\left(100t - \frac{\pi}{4}\right)$$

Resistors:

$$R_1 = 5\,\Omega, \quad R_2 = 5\,\Omega, \quad R_3 = 1\,\Omega$$

Inductors:

$$L_1 = 50\,\text{mH}, \quad L_2 = 20\,\text{mH}$$

Capacitor:

$$C_1 = 2\,\text{mF}$$

a) Transform the given circuit to the phasor domain (components and sources).

Solution

$$Z_{L1} = j\omega L_1 = j100 \times 50 \times 10^{-3} = j5\ \Omega$$
$$Z_{L2} = j\omega L_2 = j100 \times 20 \times 10^{-3} = j2\ \Omega$$
$$Z_{C1} = \frac{1}{j\omega C_1} = \frac{1}{j100 \times 2 \times 10^{-3}} = -j5\ \Omega$$
$$V = |V|e^{j\angle V} = 10\sqrt{2}e^{-j\frac{\pi}{4}}$$

For the resistors $Z_R = R$.

b) Write out KCL for node N_1 and N_2 in the phasor domain in terms of the currents provided.

Solution

At node 1:

$$i_{L1} + i_{R1} = i_c$$

At node 2:

$$i_{R1} + i_{L1} + i_{R2} + i_{L2} = 0$$

c) Write the equations derived above in terms of the node voltages and the impedances. The node voltages V_1 and V_2 are the voltage drops from N_1 and N_2 to the ground.

Solution

We have

$$\frac{V_2 - V_1}{Z_{L1}} + \frac{V_2 - V_1}{R_1} = \frac{V_1}{Z_C}$$

$$\frac{V_2 - V_1}{R_1} + \frac{V_2 - V_1}{Z_{L1}} + \frac{V_2 - v}{R_2} + \frac{V_2}{R_3 + Z_{L2}} = 0$$

Plugging in values from part (a), we get

$$\frac{V_2 - V_1}{j5} + \frac{V_2 - V_1}{5} = \frac{V_1}{-j5}$$

$$\frac{V_2 - V_1}{5} + \frac{V_2 - V_1}{j5} + \frac{V_2 - 10\sqrt{2}e^{-j\frac{\pi}{4}}}{5} + \frac{V_2}{1 + j2} = 0$$

For future parts, we want the denominators of each current to be either purely real or purely imaginary. To put i_{L2} in this form, we can manipulate the expression by multiplying the denominator by its conjugate:

$$\frac{V_2}{1 + j2}\left(\frac{1 - j2}{1 - j2}\right) = \frac{V_2(1 - j2)}{1 - (-4)} = \frac{V_2(1 - j2)}{5}$$

Our final KCL equations at nodes N_1 and N_2 are

$$\frac{V_2 - V_1}{j5} + \frac{V_2 - V_1}{5} = \frac{V_1}{-j5}$$

$$\frac{V_2 - V_1}{5} + \frac{V_2 - V_1}{j5} + \frac{V_2 - 10\sqrt{2}e^{-j\frac{\pi}{4}}}{5} + \frac{V_2(1 - j2)}{5} = 0$$

d) Write the equations you derived in part (c) in a matrix form, i.e., $\mathbf{A}\begin{bmatrix} V_1 \\ V_2 \end{bmatrix} = \vec{b}$. Write out $\mathbf{A}$ and $\vec{b}$ numerically.

Solution

From the above two equations, we have

$$\mathbf{A} = \begin{bmatrix} -\frac{1}{5} & \frac{1}{5} - j\frac{1}{5} \\ -\frac{1}{5} + j\frac{1}{5} & \frac{3}{5} - j\frac{3}{5} \end{bmatrix} = \begin{bmatrix} -0.2 & 0.2 - j0.2 \\ -0.2 + j0.2 & 0.6 - j0.6 \end{bmatrix}$$

$$\vec{b} = \begin{bmatrix} 0 \\ \frac{10\sqrt{2}e^{-j\frac{\pi}{4}}}{5} \end{bmatrix} = \begin{bmatrix} 0 \\ 2 - j2 \end{bmatrix}$$

e) Solve the systems of linear equations you derived in part (d) with any method you prefer and then find $i_c(t)$.

Solution

The inverse of a 2×2 matrix is given by:

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix}^{-1} = \frac{1}{ad - bc} \begin{bmatrix} d & -b \\ -c & a \end{bmatrix}.$$

$$\mathbf{A}^{-1} = \begin{bmatrix} -6 + j3 & 2 - j1 \\ -2 + j1 & 1.5 - j0.5 \end{bmatrix}$$

With that, we find

$$\begin{bmatrix} V_1 \\ V_2 \end{bmatrix} = \mathbf{A}^{-1}\vec{b} = \begin{bmatrix} 2 - j6 \\ 4 - j2 \end{bmatrix} = \begin{bmatrix} \sqrt{40}e^{-j1.249} \\ \sqrt{20}e^{-j0.464} \end{bmatrix}$$

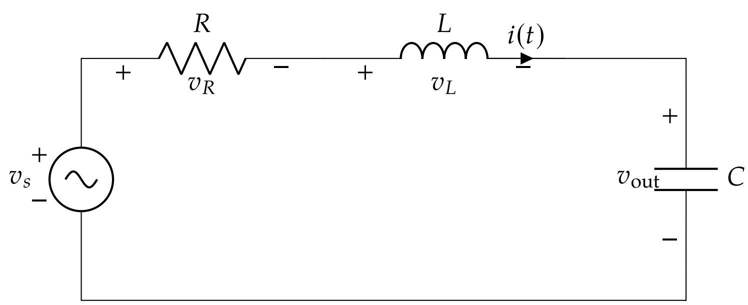
$$I_C = \frac{V_1}{-j5} = \frac{j}{5}V_1 = \frac{\sqrt{40}}{5}e^{j0.322} = 1.265e^{j0.322}$$

Transforming I_C back to time domain, we get

$$i_C(t) = 1.265\cos(100t + 0.322)$$

3 RLC Circuit

In this question, we will take a look at an electrical system described by a second order differential equations and analyze it using the phasor domain. Consider the circuit below, where $R = 8\,\mathrm{k\Omega}$, $L = 1\,\mathrm{mH}$, $C = 200\,\mathrm{nF}$, and $v_s = 2\cos\left(2000t + \frac{\pi}{4}\right)$.



a) What are the impedances of the resistor Z_R, inductor Z_L, and capacitor Z_C?

Solution

The impedance of a resistor is the same as its resistance.

$$Z_R = 8000\,\Omega$$

We can find the frequency of the circuit by looking at $v_s(t)$. We can write a sinousoidal signal as $A\cos(\omega t + \phi)$, where A is the amplitude, ω is the frequency, and ϕ is the phase. In this case, the frequency is $2000\,\frac{\mathrm{rad}}{\mathrm{s}}$.

$$Z_L = j\omega L = j2000 \cdot 10^{-3} = j2\Omega$$

$$Z_C = \frac{1}{j\omega C} = \frac{1}{j2000 \cdot 2 \cdot 10^{-7}} = -j25 \cdot 10^2\Omega$$

b) Solve for V_{out} in phasor form.

Solution

Converting v_s into phasor form, we have

$$V_s = |A|e^{j\phi} = 2e^{j\frac{\pi}{4}}$$

The circuit given is a voltage divider. Since impedances act like resistors, we can use the same equation as that for a resistive voltage divider.

$$V_{\text{out}} = V_s \frac{Z_C}{Z_R + Z_L + Z_C} = 2e^{j\frac{\pi}{4}} \frac{-j * 2.5 * 10^3}{8 * 10^3 + j * 2 - j * 25 * 10^2} = 2e^{j\frac{\pi}{4}} \frac{-j * 2500}{8000 - j * 2498}$$

We can solve for the magnitude and angle of the divider using

$$\left|2e^{j\frac{\pi}{4}} \frac{-j * 2500}{8000 - j * 2498}\right| = 2\frac{2500}{\sqrt{8000^2 + (-2498)^2}} = 0.597$$

$$\angle\left(2e^{j\frac{\pi}{4}} \frac{-j * 2500}{8000 - j * 2498}\right) = \angle(2e^{j\frac{\pi}{4}}) + \angle(-j * 2500) - \angle(8000 - j * 2498)$$

$$= \frac{\pi}{4} + \frac{-\pi}{2} - \text{atan2}(-2498, 8000) = -0.4827 \,\text{rad}$$

$$V_{\text{out}} = 0.597e^{-j0.4827}$$

c) What is V_{out} in the time domain?

Solution

We know for $v_{out}(t) = A\cos(\omega t + \phi)$, we have $v_{out}(t) = A(\cos(\phi) + j\sin(\phi)) = Ae^{j\phi}$. Thus, we have $A = 0.597, \phi = -0.4827$, which gives us $V_{out}(t) = 0.597\cos(\omega t - 0.4827)$

d) Solve for the current $i(t)$.

Solution

$$i = \frac{V_s}{Z_R + Z_L + Z_C} = \frac{|V_s|}{|Z_R + Z_L + Z_C|} e^{j(\angle V_s - \angle(Z_R + Z_L + Z_C))} = 2.38 \cdot 10^{-4} e^{j1.088}$$

Going back to the time domain:

$$i(t) = 2.38 \cdot 10^{-4} \cos(2000t + 1.088)$$

e) Solve for the transfer function $H(\omega) = \frac{V_{\text{out}}}{V_s}$

Leave your answer in terms of R, L, C, and ω.

Solution

Looking back at part (b),

$$V_{\text{out}} = V_{\text{s}} \frac{Z_C}{Z_R + Z_L + Z_C}$$

Rearranging, we get

$$H(\omega) = \frac{V_{\text{out}}}{V_{\text{s}}} = \frac{Z_C}{Z_R + Z_L + Z_C} = \frac{\frac{1}{j\omega C}}{R + j\omega L + \frac{1}{j\omega C}}$$

$$H(\omega) = \frac{1}{1 + j\omega RC + (j\omega)^2 LC}$$

f) Now assume $v_{\text{s}}(t) = 2\cos\left(2000t + \frac{\pi}{4}\right) + \cos\left(1000t + \frac{\pi}{2}\right)$ and find the time domain output voltage, $v_{out}(t)$.

Solution

By the linearity of phasors (principle of superposition), we can treat each component of the signal independently and sum the outputs. For $v_{s1}(t) = 2\cos\left(2000t + \frac{\pi}{4}\right)$ we've already found the output to be $v_{out,1} = 0.597\cos(\omega t - 0.4827)$ Consider $v_{s2}(t) = \cos\left(1000t + \frac{\pi}{2}\right)$, which we write in phasor form as $V_{s2} = 1e^{\frac{j\pi}{2}} = j$. Using the transfer function above at $\omega = 1000$ rad /s

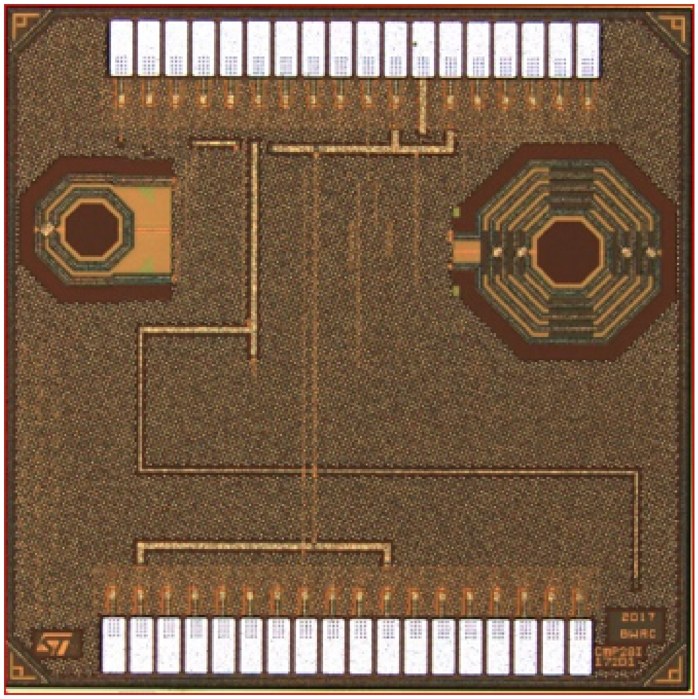
$$\begin{aligned}
V_{out,2} = V_{s2}H(\omega) &= \frac{V_{s2}}{1 + j\omega RC + (j\omega)^2 LC} \\
&= \frac{j}{1 + j(1000)(8\times 10^3)(2\times 10^{-7}) + (1000j)^2(10^{-3})(2\times 10^{-7})} \\
&= \frac{j}{1 + 1.6j - (2\times 10^{-4})} \\
&= \frac{j}{0.9998 + 1.6j} \\
&= 0.4495 + 0.2809j
\end{aligned}$$

Thus $|V_{out,2}| = 0.5300$ and the phase is $\tan^{-1}(0.2809/0.4494) = 0.5585$ radians, so $v_{out,2}(t) = 0.5300\cos(1000t + 0.5585)$. The combined output signal is

$$v_{out}(t) = 0.597\cos(\omega t - 0.4827) + 0.530\cos(1000t + 0.5585).$$

4 LC Tank

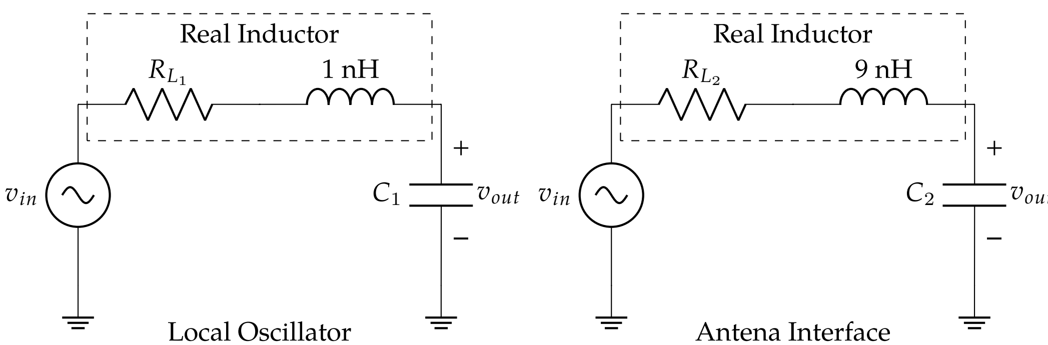
The chip below was designed in EE194 in Spring 2017. It is 1.1 mm on a side, and contains a 32 bit RISC-V microprocessor, 64kB RAM, and a 2.4 GHz Bluetooth Low Energy radio for communicating with cell phones.



There are two inductors on the chip. The inductor on the right is part of an impedance-matching circuit for the antenna interface. We will refer to this as the *antenna interface inductor*. Its inductance is 9 nH and the filter it is a part of has a Q of 8 at 2.4 GHz.

The inductor on the left has an inductance of 1nH, and participates in an LC filter with a Q of 15 at 4.8 GHz, and this LC filter is the "local oscillator" at the core of the radio's transmit- and receive-circuits. We will call this the *local oscillator inductor*.

There is a capacitor connected to each of the two inductors, which make resonators.



We are including the resistance that is present in real inductors in our model because our Q is not infinite. These circuit diagrams are meant to be representative of the two resonators described, which are a small part of the overall circuit seen in the chip above.

a) **For the antenna interface inductor, what is the capacitance (to one significant figure) needed to make it resonant at 2.4 GHz?**

Solution

We are given that L = 9 nH and $\omega_n = 2\pi 2.4 * 10^9$. Therefore, using $LC = \frac{1}{\omega_n^2}$ we get,

$$LC = \frac{1}{(2\pi 2.4 * 10^9)^2} \approx 4.4 * 10^{-21}$$
$$\implies C = \frac{4.4 * 10^{-21}}{9 * 10^{-9}} \approx 0.49 * 10^{-12} \approx 0.49pF$$

b) **For the local oscillator inductor on the left, what capacitance (to four significant figures) is needed to make it resonant at 4.8 GHz? How about at 5 GHz?**

Solution

We are given that L = 1 nH and $\omega_n = 2\pi 4.8 * 10^9$. Therefore, using $LC = \frac{1}{\omega_n^2}$ we get,

$$LC = \frac{1}{(2\pi 4.8 * 10^9)^2}$$
$$\implies C = \frac{1}{10^{-9} * (2\pi 4.8 * 10^9)^2} \approx 1.099pF$$

Similarly, if $\omega_n = 2\pi 5 * 10^9$,

$$C = \frac{1}{10^{-9} * (2\pi 5 * 10^9)^2} \approx 1.013pF$$

c) **For the local oscillator inductor on the left with resonance at 4.8 GHz, what is the inductor resistance, R_{L_1} such that the $Q = 8$?**
What happens to Q if we have a higher resistance?

Solution

We can write the transfer function as

$$H(jw) = \frac{1}{LC(j\omega)^2 + RC(j\omega) + 1} = \frac{1}{\frac{(j\omega)^2}{\omega_n^2} + \frac{j\omega(2\xi)}{\omega_n} + 1}$$

so $\omega_n = \frac{1}{\sqrt{LC}}$ and $\xi = \frac{RC\omega_n}{2} = \frac{R}{2\omega_n L}$.
So $Q = \frac{1}{2\xi} = \frac{\omega_n L}{R}$. Therefore,

$$R = \frac{\omega_n L}{Q} = \frac{2\pi * 4.8 * 10^9 \cdot 1 * 10^{-9}}{8} = 3.7\,\Omega$$

A higher resistance results in more damping and a lower quality factor Q.

5 Homework Process and Study Group

Citing sources and collaborators are an important part of life, including being a student!
We also want to understand what resources you find helpful and how much time homework is taking, so we can change things in the future if possible.

a) **What sources (if any) did you use as you worked through the homework?**

b) **If you worked with someone on this homework, who did you work with?**
List names and student ID's. (In case of homework party, you can also just describe the group.)

c) **Roughly how many total hours did you work on this homework?**

d) **Do you have any feedback on this homework assignment?**