This homework is due on Thursday, October 1, 2020, at 10:59PM.

Self-grades are due on Thursday, October 8, 2020, at 10:59PM.

1 Complex Numbers

A common way to visualize complex numbers is to use the complex plane. Recall that a complex number z is often represented in Cartesian form.

$$z = x + jy \text{ with } \mathrm{Re}\{z\} = x \text{ and } \mathrm{Im}\{z\} = y$$

See Figure 1 for a visualization of z in the complex plane.

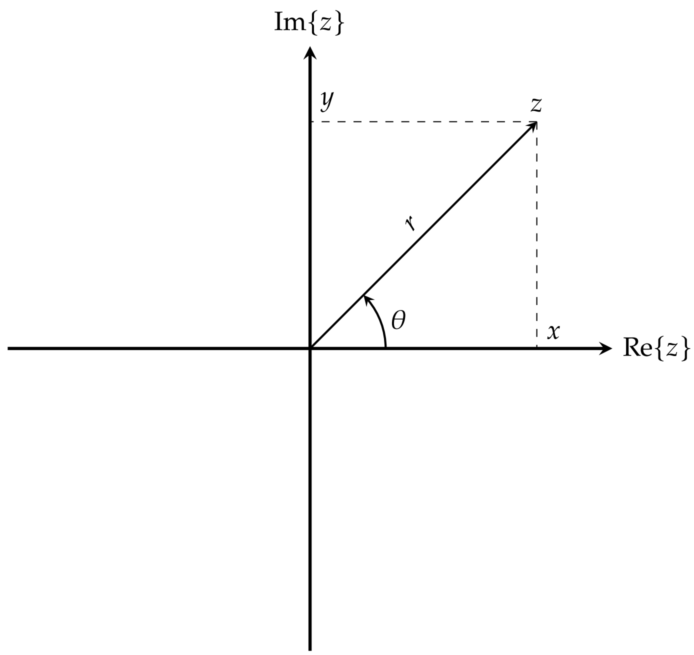


Figure 1: Complex Plane

In this question, we will derive the polar form of a complex number and use this form to make some interesting conclusions.

a) **Calculate the length of z in terms of x and y as shown in Figure 1.** This is the magnitude of a complex number and is denoted by $|z|$ or r. *Hint.* Use the Pythagorean theorem.

Solution

$$r = \sqrt{x^2 + y^2} = |z|$$

b) **Represent x, the real part of z, and y, the imaginary part of z, in terms of r and θ.**

Solution

$$x = r\cos(\theta) \text{ and } y = r\sin(\theta)$$

c) **Substitute for x and y in z.** Use Euler's identity[1] $e^{j\theta} = \cos\theta + j\sin\theta$ to **conclude that,**

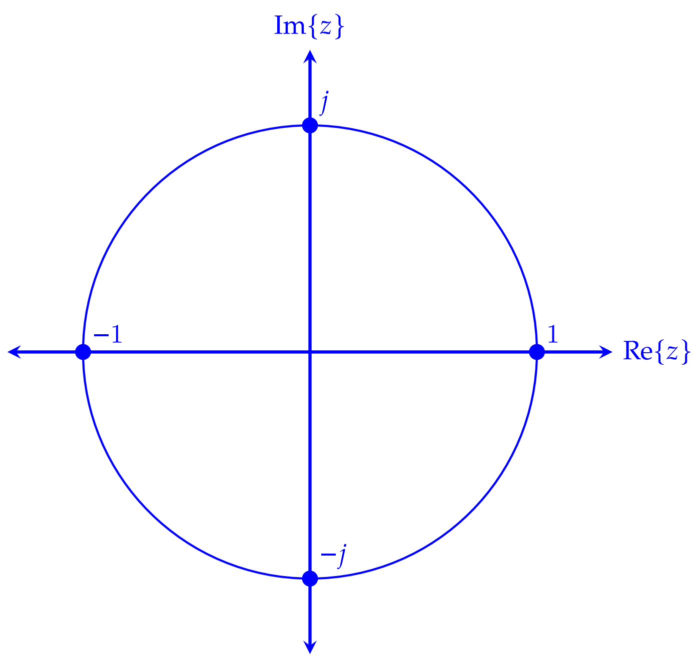
$$z = re^{j\theta}$$

Solution

$$\begin{aligned} z &= r\cos(\theta) + jr\sin(\theta) \\ &= r(\cos(\theta) + j\sin(\theta)) \\ &= re^{j\theta} \end{aligned}$$

d) In the complex plane, **draw out all the complex numbers such that $|z| = 1$. What are the z values where the figure intersects the real axis and the imaginary axis?**

Solution



e) If $z = re^{j\theta}$, **prove that** $\bar{z} = re^{-j\theta}$. Recall that the complex conjugate of a complex number $z = x + jy$ is $\bar{z} = x - jy$.

[1] also known as de Moivre's Theorem.

Solution

$$\begin{aligned}\bar{z} &= \overline{(r(\cos(\theta) + j\sin(\theta)))}\\ &= r(\cos(\theta) - j\sin(\theta))\\ &= r(\cos(-\theta) + j\sin(-\theta))\\ &= re^{-j\theta}\end{aligned}$$

f) **Show that,**

$$r^2 = z\bar{z}$$

Solution

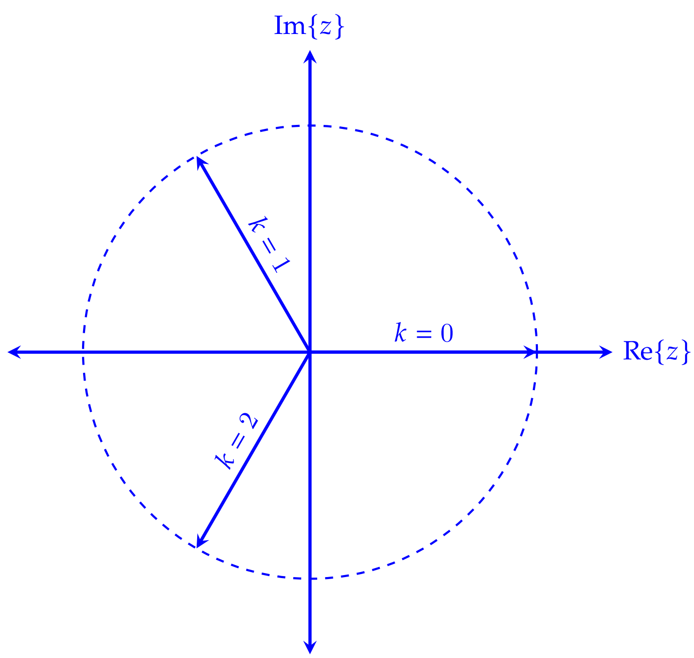
$$z\bar{z} = re^{j\theta}re^{-j\theta} = r^2e^{j\theta-j\theta} = r^2e^0 = r^2$$

g) Intuitively **argue that**

$$\sum_{k=0}^{2} e^{j\frac{2\pi}{3}k} = 0$$

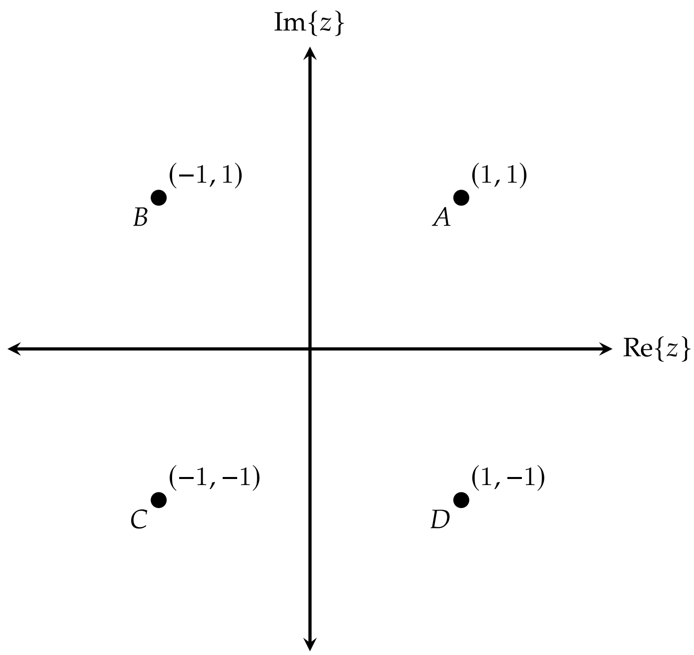
Do so by drawing out the different values of the sum in the complex plane and making an argument based on the vector sum.

Solution



The three vectors are of the same magnitude and are equally spaced from each other in a way that sums to zero. Intuitively, this is because we have three directions pointing perfectly away from each other. This can also be verified with Euler's formula (if you did that, give yourself full credit), but an intuitive argument based on the direction of the vectors is sufficient.

h) In modern wireless communication, signals are sent as complex exponentials $e^{j\omega t}$, with receivers detecting both the cosine and sine components of the signal. One common scheme for encoding digital data, such as what is used in your phone and in WiFi, is known as quadrature phase-shift keying (QPSK). In this technique, there are four points of interest on the complex plane (see figure below):



Write each of the points A, B, C, and D in polar form.

Solution

In Cartesian form, the complex numbers are given by:

$$A = 1 + j \qquad C = -1 - j$$
$$B = -1 + j \qquad D = 1 - j$$

For a given complex number $z = x + jy$, we employ the following two formulas to convert to polar form:

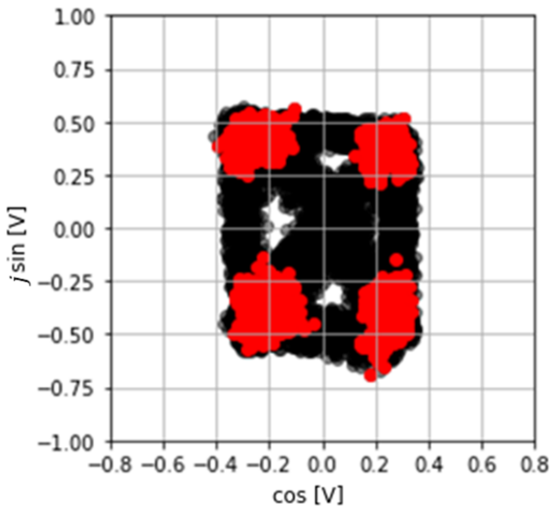
$$r = \sqrt{x^2 + y^2}$$
$$\theta = \text{atan2}(y, x)$$

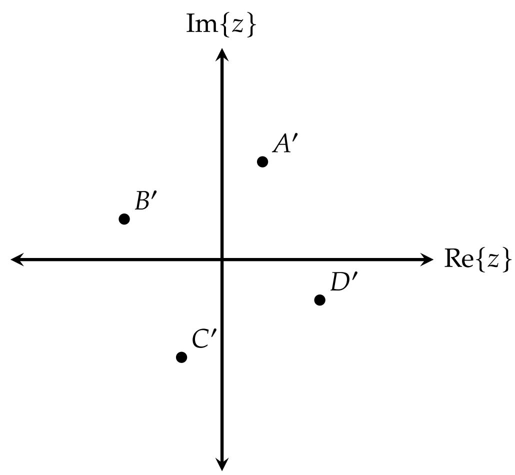
Plugging in, the final answer is:

$$A = \sqrt{2}e^{j\frac{\pi}{4}} \qquad C = \sqrt{2}e^{j\frac{-3\pi}{4}}$$
$$B = \sqrt{2}e^{j\frac{3\pi}{4}} \qquad D = \sqrt{2}e^{j\frac{-\pi}{4}}$$

i) Due to amplitude error and phase noise, in practice these points often arrive at the receiver scaled and rotated (see the figure on the left). Suppose the received points are as follows:

$$A' = 1e^{j\frac{3\pi}{8}} \qquad C' = 1e^{j\frac{-5\pi}{8}}$$
$$B' = 1e^{j\frac{7\pi}{8}} \qquad D' = 1e^{j\frac{-\pi}{8}}$$





Find a corrective value $r_x e^{j\theta_x}$ that when multiplied with A', B', C', D' recovers the original points A, B, C, D. What is the original noise $r_n e^{j\theta_n}$ that created the shift?

Solution

All the points are scaled and rotated by the same amount, so we pick A' arbitrarily. The problem boils down to finding $r_x e^{j\theta_x}$ such that

$$r_x e^{j\theta_x} A' = A$$

Plugging in the polar form of A and A',

$$r_x e^{j\theta_x} 1e^{j\frac{3\pi}{8}} = r_x e^{j(\theta_x + \frac{3\pi}{8})} = \sqrt{2}e^{j\frac{\pi}{4}}$$

From which we can read off the solution,

$$r_x = \sqrt{2}$$
$$\theta_x = -\frac{\pi}{8}$$

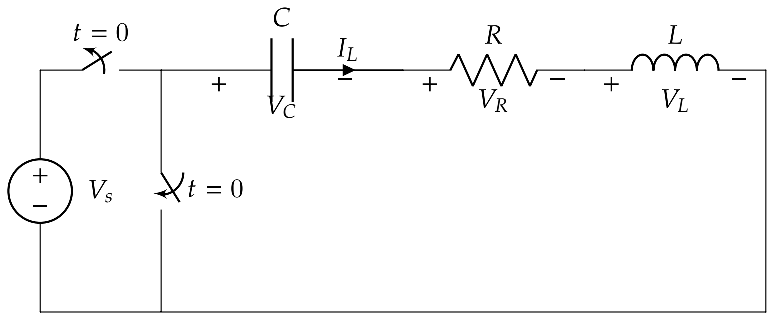
This corrective value cancels out the rotation and scaling caused by the original noise, and thus the error is given by the multiplicative inverse:

$$r_n e^{j\theta_n} = \frac{1}{\sqrt{2}} e^{j\frac{\pi}{8}}$$

Note that the phase correction $e^{j\theta_x}$ is the complex conjugate of the phase noise $e^{j\theta_n}$ that created the error.

2 RLC Responses

Consider the following circuit like you saw in lecture:



Assume the circuit above has reached steady state for $t < 0$. At time $t = 0$, the switch changes state and disconnects the voltage source, replacing it with a short.

In this problem, the current through the inductor and the voltage across the capacitor are the natural physical state variables since these are what correlate to how energy is actually stored in the system. (A magnetic field through the inductor and an electric field within the capacitor.)

a) **Write the system of differential equations in terms of state variables $x_1(t) = I_L(t)$ and $x_2(t) = V_C(t)$ that describes this circuit for $t \geq 0$. Leave the system symbolic in terms of V_s, L, R, and C.**

Solution

For this part, we need to find two differential equations, each including a derivative of one of the state variables.

First, let's consider the capacitor equation $I_C(t) = C\frac{d}{dt}V_C(t)$. In this circuit, $I_C(t) = I_L(t)$, so we can write

$$I_C(t) = C\frac{d}{dt}V_C(t) = I_L(t) \tag{1}$$

$$\frac{d}{dt}V_C(t) = \frac{1}{C}I_L(t). \tag{2}$$

If we use the state variable names, we can write this as

$$\frac{d}{dt}x_2(t) = \frac{1}{C}x_1(t), \tag{3}$$

so now we have one differential equation.

For the other differential equation, we can apply KVL around the single loop in this circuit. (Alternatively, we could just solve it directly and substitute in for the desired voltage on the capacitor, which is a state variable.) Going clockwise, we have

$$V_C(t) + V_R(t) + V_L(t) = 0. \tag{4}$$

Using Ohm's Law and the inductor equation $V_L = L\frac{d}{dt}I_L(t)$, we can write this as

$$V_C(t) + RI_L(t) + L\frac{d}{dt}I_L(t) = 0, \tag{5}$$

which we can rewrite as

$$\frac{d}{dt}I_L(t) = -\frac{R}{L}I_L(t) - \frac{1}{L}V_C(t). \tag{6}$$

If we use the state variable names, this becomes

$$\frac{d}{dt}x_1(t) = -\frac{R}{L}x_1(t) - \frac{1}{L}x_2(t), \tag{7}$$

and we have a second differential equation.

To summarize the final system is

$$\frac{d}{dt}x_1(t) = -\frac{R}{L}x_1(t) - \frac{1}{L}x_2(t) \tag{8}$$

$$\frac{d}{dt}x_2(t) = \frac{1}{C}x_1(t). \tag{9}$$

b) Write the system of equations in vector/matrix form with the vector state variable $\vec{x}(t) = \begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix}$. This should be in the form $\frac{d}{dt}\vec{x}(t) = A\vec{x}(t)$ with a 2×2 matrix A.

Solution

By inspection from the previous part, we have

$$\begin{bmatrix} \frac{d}{dt}x_1(t) \\ \frac{d}{dt}x_2(t) \end{bmatrix} = \begin{bmatrix} -\frac{R}{L} & -\frac{1}{L} \\ \frac{1}{C} & 0 \end{bmatrix} \begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix}, \tag{10}$$

which is in the form $\frac{d}{dt}\vec{x}(t) = A\vec{x}(t)$, with

$$A = \begin{bmatrix} -\frac{R}{L} & -\frac{1}{L} \\ \frac{1}{C} & 0 \end{bmatrix}. \tag{11}$$

c) Find the eigenvalues of the A matrix symbolically.
(Hint: the quadratic formula will be involved.)

Solution

To find the eigenvalues, we'll solve $\det(A - \lambda I) = 0$. In other words, we want to find λ such that

$$\det(A - \lambda I) = \det\left(\begin{bmatrix} -\frac{R}{L} - \lambda & -\frac{1}{L} \\ \frac{1}{C} & -\lambda \end{bmatrix}\right) \tag{12}$$

$$= -\lambda\left(-\frac{R}{L} - \lambda\right) + \frac{1}{LC} \tag{13}$$

$$= \lambda^2 + \frac{R}{L}\lambda + \frac{1}{LC} = 0. \tag{14}$$

The Quadratic Formula gives

$$\lambda = -\frac{1}{2}\frac{R}{L} \pm \frac{1}{2}\sqrt{\left(\frac{R}{L}\right)^2 - \frac{4}{LC}}. \tag{15}$$

d) Under what condition on the circuit parameters R, L, C are the eigenvalue(s) of A

i) distinct and real eigenvalues.
ii) distinct purely imaginary eigenvalues.
iii) distinct complex eigenvalues with nonzero real and imaginary parts.
iv) a single real eigenvalue.

Solution

a) For both eigenvalues to be real and distinct, we need the quantity inside the square root to be positive. In other words, we need

$$\frac{R^2}{L^2} - \frac{4}{LC} > 0, \tag{16}$$

or, equivalently,

$$R > 2\sqrt{\frac{L}{C}}. \tag{17}$$

b) The only way for both eigenvalues to be purely imaginary is to have $R = 0$. In this case, the eigenvalues would be

$$\lambda = \pm j\sqrt{\frac{1}{LC}}. \tag{18}$$

c) For the eigenvalues to be complex with nonzero real and imaginary parts, we need to have $R > 0$ and

$$\frac{R^2}{L^2} - \frac{4}{LC} < 0, \tag{19}$$

or, equivalently,

$$R < 2\sqrt{\frac{L}{C}}. \tag{20}$$

d) For the solution of the Quadratic Formula to give single real solution, we need to have

$$\frac{R^2}{L^2} - \frac{4}{LC} = 0, \tag{21}$$

or, equivalently,

$$R = 2\sqrt{\frac{L}{C}}. \tag{22}$$

e) **Overdamped Case.** In the provided Jupyter notebook, move the sliders to approximately $R = 1k\Omega$ and $C = 10nF$. Sketch $V_c(t)$ and comment on its appearance. Additionally, sketch the location of the eigenvalues on the complex plane.

Solution

$V_c(t)$ should look like a decaying exponential. The eigenvalues lie on the real axis at coordinates $(-1 \times 10^5, 0)$ and $(-4 \times 10^7, 0)$.

f) **Undamped Case.** In the provided Jupyter notebook, move the sliders to approximately $R = 0\Omega$ and $C = 10nF$. Sketch $V_c(t)$ and comment on its appearance. Additionally, sketch the location of the eigenvalues on the complex plane. Are the waveforms for $x_1(t)$ and $x_2(t)$ "transient" — do they die out with time?

Note: Because there is no resistance, this is called the "undamped" case.

Solution

No, these waveforms are sinusoids and do not die out over time. They are not transient. The eigenvalues are located on the imaginary axis at coordinates $(0, -2 \times 10^6)$ and $(0, 2 \times 10^6)$.

g) **Underdamped Case.** In the provided Jupyter notebook, move the sliders to approximately $R = 1\Omega$ and $C = 10nF$. Sketch $V_c(t)$ and comment on its appearance. Additionally, sketch the location of the eigenvalues on the complex plane. Are the waveforms for $x_1(t)$ and $x_2(t)$ "transient" — do they die out with time?

Note: Because the resistance is so small, this is called the "underdamped" case. It is good to reflect upon these waveforms to see why engineers consider such behavior to be reflective of systems that don't have enough damping.

Solution

Yes, the waveforms are transient. They appear to be sinusoids that are decaying exponentially. The eigenvalues should be located at coordinates $(-0.02\times10^6, 2\times10^6)$ and $(-0.02\times10^6, -2\times10^6)$.

h) **Critically Damped Case**. In the provided Jupyter notebook, move the sliders to the resistance value of $R = 100\Omega$ and $C = 10nF$. Sketch $V_c(t)$ and comment on its appearance. Additionally, sketch the location of the eigenvalues on the complex plane. What happens to the voltage and eigenvalues as you slightly increase or decrease R?

Solution

At the $R = 100$, $V_c(t)$ appears to decay exponentially. A slight increase in R causes the voltage to decay more slowly. A slight decrease in R causes a voltage undershoot and eventually oscillations. The eigenvalues have converged into the same point at $(-2 \times 10^6, 0)$. Increasing R makes them split into two points, and both points remain on the real axis. One point goes towards the origin, while the other goes towards negative infinity. Decreasing R splits the eigenvalues back into their complex conjugates.

3 Phasors for circuit

In the lectures, we introduced phasors. That is, for a given sinusoidal $x(t) = A\cos(\omega t + \phi)$ with $\omega \neq 0$ and A being the magnitude, we can represent it as a phasor $X = Ae^{j\phi}$ such that $x(t) = \text{Re}(Ae^{j\phi}e^{j\omega t}) = \text{Re}(Xe^{j\omega t})$.

a) Show that $x(t)$ can be written as $x(t) = \frac{1}{2}(Xe^{j\omega t} + \bar{X}e^{-j\omega t})$, where $\bar{X}$ is the complex conjugate of X.

Solution

$$\begin{aligned} x(t) &= A\cos(\omega t + \phi) \\ &= A\left(\frac{e^{j(\omega t+\phi)} + e^{-j(\omega t+\phi)}}{2}\right) \\ &= \frac{1}{2}\left(Ae^{j(\omega t+\phi)} + Ae^{-j(\omega t+\phi)}\right) \\ &= \frac{1}{2}\left(Ae^{j\phi}e^{j\omega t} + Ae^{-j\phi}e^{-j\omega t}\right) \\ &= \frac{1}{2}\left(Xe^{j\omega t} + \bar{X}e^{-j\omega t}\right) \end{aligned}$$

b) It is straight forward that if the phasor $X = 0$ then $x(t) = \text{Re}(Xe^{j\omega t}) = 0$ for every t. The converse is also true. Show if $x(t) = 0$ for all t, then $X = 0$.

Hint: Pick appropriate t to show that the real part and imaginary part of the phasor is 0. That is, what t makes $e^{j\omega t} = 1$ and what t makes $e^{j\omega t} = j$?

Solution

For $t = 0$, we have

$$x(0) = \frac{1}{2}(X + \bar{X}) = \text{Re}\{X\} = 0.$$

For $t = \frac{\pi}{2\omega}$, we have

$$x(\frac{\pi}{2\omega}) = \frac{1}{2}(Xe^{j\frac{\pi}{2}} - \bar{X}e^{-j\frac{\pi}{2}}) = \frac{1}{2}(jX - j\bar{X}) = -\text{Im}\{X\} = 0.$$

Since both the real part and imaginary part of X are 0, we conclude that $X = 0$.

c) With the results from the previous part, show that KCL on phasors are equivalent to KCL on the time waveform. That is, for current $i_1(t), i_2(t), \ldots, i_n(t)$ and their phasors $I_1, I_2, \ldots, I_n$, $I_1 + I_2 + \cdots + I_n = 0$ if and only if $i_1(t) + i_2(t) + \cdots + i_n(t) = 0$ for any t.

Hint: The linearity holds that the time waveform $x_1(t) + x_2(t)$ has corresponding phasor $X_1 + X_2$.

Solution

Let $i(t) = i_1(t) + i_2(t) + \cdots + i_n(t)$, then its phasor is given by $I = I_1 + I_2 + \cdots + I_n$. From the previous part, we know that $I = 0$ if and only if $i(t) = 0$ for any t. That is, $I_1 + I_2 + \cdots + I_n = 0$ if and only if $i_1(t) + i_2(t) + \cdots + i_n(t) = 0$ for any t. The KCL is equivalent on time waveform and phasor.

d) Show the same result holds for KVL. That is, for voltages $v_1(t), v_2(t), \ldots, v_n(t)$ and their phasors $V_1, V_2, \ldots, V_n$, $V_1 + V_2 + \cdots + V_n = 0$ if and only if $v_1(t) + v_2(t) + \cdots + v_n(t) = 0$ for any t.

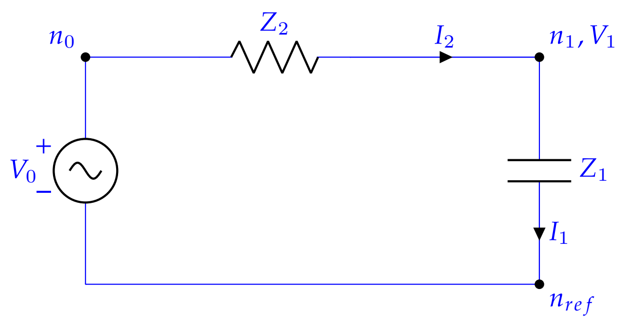
Solution

Let $v(t) = v_1(t) + v_2(t) + \cdots + v_n(t)$, then its phasor is given by $V = V_1 + V_2 + \cdots + V_n$. We know that $V = 0$ if and only if $v(t) = 0$ for any t. That is, $V_1 + V_2 + \cdots + V_n = 0$ if and only if $v_1(t) + v_2(t) + \cdots + v_n(t) = 0$ for any t. The KVL is equivalent on time waveform and phasor.

Since both KCL and KVL hold on phasor form, we may employ the same circuit analysis tools on phasors.

e) Consider the following circuit where the power supply outputs $u(t) = \mathrm{Re}\left\{V_0 e^{j\omega t}\right\}$ for some $\omega \neq 0$. Employ nodal analysis using phasors and only the voltage-current relation, $V = IZ$. In the circuit, $Z_1 = \frac{1}{j\omega C}$ is the impedance of the capacitor and $Z_2 = R$ is the impedance of the resistor. Then express V_1 in terms of V_0, Z_1, and Z_2. Note that V_0, V_1 are both phasors.

Solution



With reference to n_{ref}, the potential for node n_0 is V_0. And the potential for node n_1 is V_1. From the voltage-current relation, we have

$$I_1 = \frac{V_1}{Z_1}$$
$$I_2 = \frac{V_0 - V_1}{Z_2}$$

Now apply KCL on node n_1, $I_1 = I_2$.

$$\frac{V_1}{Z_1} = \frac{V_0 - V_1}{Z_2}$$
$$V_1\left(\frac{1}{Z_1} + \frac{1}{Z_2}\right) = \frac{V_0}{Z_2}$$
$$V_1 \frac{Z_1 + Z_2}{Z_1 Z_2} = \frac{V_0}{Z_2}$$
$$V_1 = V_0 \frac{Z_1}{Z_1 + Z_2}$$
$$= V_0 \frac{1}{1 + j\omega RC}$$

This is a voltage divider.

f) For $V_0 = A$ where A is a real number, the power supply outputs time waveform $u(t) = A\cos(\omega t)$. Write the time waveform $v_1(t)$ for V_1 based on the phasor you obtained for the circuit in the previous part.

Solution

From the previous part, we have

$$\begin{aligned}
V_1 &= V_0 \frac{1}{1 + j\omega RC} \\
&= A \frac{1}{1 + j\omega RC} \\
&= A \frac{1 - j\omega RC}{(1 + j\omega RC)(1 - j\omega RC)} \\
&= A \frac{1 - j\omega RC}{1 + (\omega RC)^2} \\
&= \frac{A}{\sqrt{1 + (\omega RC)^2}} \left(\frac{1}{\sqrt{1 + (\omega RC)^2}} - \frac{j\omega RC}{\sqrt{1 + (\omega RC)^2}} \right) \\
&= \frac{A}{\sqrt{1 + (\omega RC)^2}} e^{j\phi}
\end{aligned}$$

In the final expression above, $\phi = -\tan^{-1}(\omega RC)$. Note that $-\tan(\theta) = \tan(-\theta)$ and that $\tan^{-1}$ maps any real number to $\left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$.

Now the time waveform writes as follows.

$$\begin{aligned}
v_1(t) &= \mathrm{Re}\left\{ V_1 e^{j\omega t} \right\} \\
&= \mathrm{Re}\left\{ \frac{A}{\sqrt{1 + (\omega RC)^2}} e^{j\phi} e^{j\omega t} \right\} \\
&= \frac{A}{\sqrt{1 + (\omega RC)^2}} \cos\left(\omega t + \phi\right)
\end{aligned}$$

4 Phasors

a) Consider a resistor ($R = 1.5\Omega$), a capacitor ($C = 1$F), and an inductor ($L = 1$H) connected in series. **Give expressions for the impedances of Z_R, Z_C, Z_L for each of these elements as a function of the angular frequency ω.**

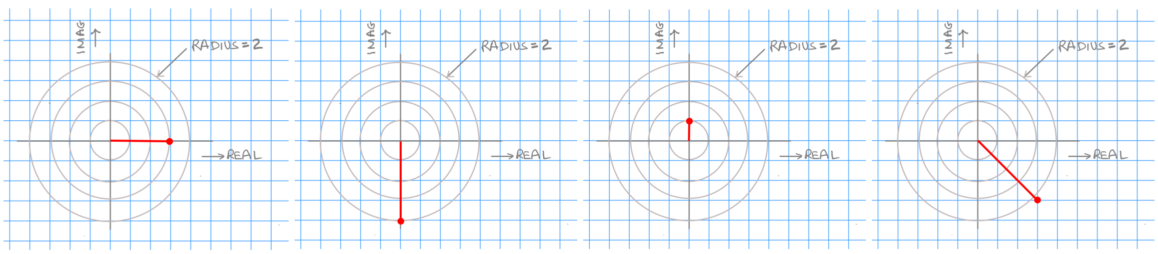
Solution

The impedances are as follows: $Z_R = R = 1.5$, $Z_C = \frac{1}{j\omega C} = \frac{1}{j\omega} = -\frac{j}{\omega}$ and $L = j\omega L = j\omega$.

b) **Draw the individual impedances as "vectors" on the same complex plane for the case $\omega = \frac{1}{2}$ rad/sec. Also draw the combined impedance Z_{total} of their series combination. Then give the magnitude and phase of Z_{total}.**

Solution

Substituting for $\omega = \frac{1}{2}$ in the above answers, we get, $Z_R = 1.5$, $Z_C = -2j$ and $Z_L = 0.5j$. Since the elements are in series, $Z_{total} = Z_L + Z_C + Z_R = 1.5 - 1.5j$. This has magnitude $1.5\sqrt{2}$ and phase $-\frac{\pi}{4}$. Following are the plots:



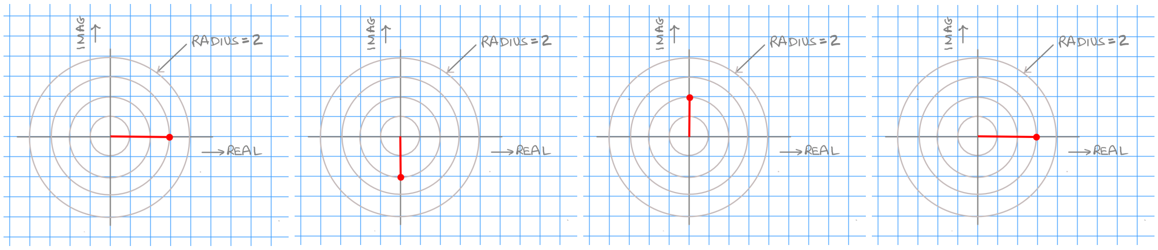
(a) $Z_R(@\omega = 0.5)$ (b) $Z_C(@\omega = 0.5)$ (c) $Z_L(@\omega = 0.5)$ (d) $Z_{total}(@\omega = 0.5)$

Figure 2: Impedances at $\omega = 0.5$.

c) **Draw the individual impedances as "vectors" on the same complex plane for the case $\omega = 1$ rad/sec. Also draw the combined impedance Z_{total} of their series combination. Then give the magnitude and phase of Z_{total}.**

Solution

Following the same method as last time, with $\omega = 1$, $Z_R = 1.5$, $Z_C = -j$, $Z_L = j$ and $Z_{total} = 1.5$ Thishas magnitude 1.5 and phase 0.



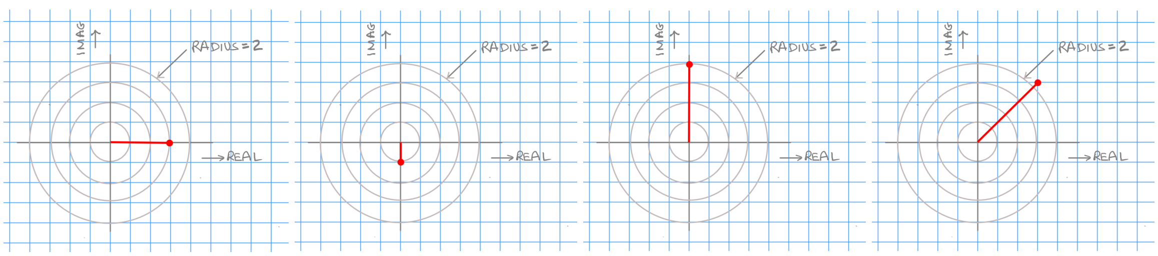
(a) $Z_R(@\omega = 1)$ (b) $Z_C(@\omega = 1)$ (c) $Z_L(@\omega = 1)$ (d) $Z_{total}(@\omega = 1)$

Figure 3: Impedances at $\omega = 1$.

d) **Draw the individual impedances as "vectors" on the same complex plane for the case $\omega = 2$ rad/sec. Also draw the combined impedance Z_{total} of their series combination. Then give the magnitude and phase of Z_{total}.**

Solution

Again, following the same method as last time, with $\omega = 2$, $Z_R = 1.5$, $Z_C = -0.5j$, $Z_L = 2j$ and $Z_{total} = 1.5 + 1.5j$. This has magnitude $1.5\sqrt{2}$ and phase $+\frac{\pi}{4}$.



(a) $Z_R(@\omega = 2)$ (b) $Z_C(@\omega = 2)$ (c) $Z_L(@\omega = 2)$ (d) $Z_{total}(@\omega = 2)$

Figure 4: Impedances at $\omega = 2$.

e) **For the previous series combination of RLC elements, what is the "natural frequency" ω_n where the series impedance is purely real?**

Solution

From our above answers, clearly the natural frequency, $\omega_n = 1$ rad/s . This is where the imaginary parts of the impedance cancel each other.

5 Homework Process and Study Group

Citing sources and collaborators are an important part of life, including being a student!
We also want to understand what resources you find helpful and how much time homework is taking, so we can change things in the future if possible.

a) **What sources (if any) did you use as you worked through the homework?**

b) **If you worked with someone on this homework, who did you work with?**
List names and student ID's. (In case of homework party, you can also just describe the group.)

c) **Roughly how many total hours did you work on this homework?**

d) **Do you have any feedback on this homework assignment?**