This homework is due on Thursday, December 3, 2020, at 10:59PM. Self-grades are due on Thursday, December 10, 2020, at 10:59PM.

1 Sampling Theorem

Consider the following signal, \( x(t) \) defined as,

\[ x(t) = \cos(2\pi t) + \sin(4\pi t) \]

a) Find the maximum frequency, \( \omega_{\text{max}} \), of \( x(t) \) in radians per second.

**Solution**

There are two distinct frequencies in this signal at \( \omega = 2\pi \) and \( \omega = 4\pi \). Therefore, \( \omega_{\text{max}} = 4\pi \) in radians per second, which is equal to 2 Hz.

b) If I sample every \( T \) seconds, what is the sampling frequency in radians per second?

**Solution**

\( \omega_s = \frac{2\pi}{T} \).

c) What is the smallest sampling period \( T \) that may result in an imperfect reconstruction?

**Solution**

From the Nyquist Sampling Theorem, we must sample at \( \omega_s > 2\omega_{\text{max}} = 8\pi \) in order to perfectly reconstruct our signal. Therefore \( T = \frac{\omega_s}{\omega_s} \) must be strictly less than \( \frac{1}{4} \) to guarantee a perfect reconstruction. Hence the smallest \( T \) for which we cannot reconstruct the signal is \( T = \frac{1}{4} \).
2 Aliasing

Watch the following video: https://www.youtube.com/watch?v=jQDjJRYmeWg

Assume the video camera running at 30 frames per second. That is to say, the camera takes 30 photos within a second, with the time between photos being constant.

a) Given that the main rotor has 5 blades, list all the possible rates at which the main rotor is spinning in revolutions per second assuming no physical limitations.

*Hint: Your answer should depend on \( k \) where \( k \) can be any integer.*

**Solution**

Let \( \Delta = \frac{1}{30} \) s be the sample period. Since there are 5 blades, the rotor will look like itself after it finishes a fifth of a revolution.

This means that, in one \( \Delta \), the rotor could have completed \( \frac{k}{5} \) revolutions, where \( k \) is an integer.

This means that the rotor could be spinning at \( \frac{k}{5\Delta} \) revolutions per second, which is \((6 \times k) \text{ Hz}\).

b) Given that the back rotor has 3 blades and completes 2 revolutions in 1 second in the video, list all the possible rates at which the back rotor is spinning in revolutions per second assuming no physical limitations.

*Hint: Your answer should depend on \( k \) where \( k \) can be any integer.*

**Solution**

The video will always be sampling with sample period \( \Delta = \frac{1}{30} \) s. This means that the back rotor could have a period of \( \Delta, 2\Delta, 3\Delta \) and so on and the camera would not be able to distinguish between them. Moreover, since a third of a rotation looks exactly like a complete rotation, the camera would not be able to tell the difference between periods of the form \( k\frac{\Delta}{3} \) where \( k \) is an integer.

In the video, we see the back rotor moving roughly at the rate of 2 revolutions per second, which means that, in the video, it has a period of \( \frac{1}{2} \) seconds.

This means that, in \( \Delta \) seconds, it can finish \( \frac{k}{3} + 2\Delta \) revolutions, where the \( \frac{k}{3} \) revolutions gets hidden by the sample rate. This means that the rotor could be spinning at \( \frac{k}{3\Delta} + 2 \) revolutions per second, which is \((10k + 2) \text{ Hz}\).
3 LTI Inputs

We have an LTI system whose exact characteristics we do not know. However, we know that it has a finite impulse response that is not longer than 5 samples. We also observed two sequences, \( x_1 \) and \( x_2 \), pass through the system and observed the system’s responses \( y_1 \) and \( y_2 \).

\[
\begin{array}{cccccccc}
  x_1 & 0 & 2 & 2 & 0 & -1 & 0 & 0 & 0 & 0 \\
  y_1 & 0 & 4 & 10 & 8 & -2 & -3 & 1 & 1 & -1 \\
\end{array}
\]

\[
\begin{array}{cccccccc}
  x_2 & -2 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\
  y_2 & -4 & -6 & 0 & 5 & -1 & -1 & 1 & 0 & 0 \\
\end{array}
\]

a) Given the above sequences, what would be the output for the input \( x_3 \)?

\[
\begin{array}{cccccccc}
  x_3 & 0 & 0 & -2 & 0 & 1 & 0 & 0 & 0 & 0 \\
\end{array}
\]

**Solution**

This is a shifted version of \( x_2 \). Since the system is time-invariant, the output is:

\[
\begin{array}{cccccccc}
  y_3 & 0 & 0 & -4 & -6 & 0 & 5 & -1 & -1 & 1 & 0 \\
\end{array}
\]

b) What is the output of the system for the input \( x_1 - x_2 \)?

**Solution**

Since the system is linear, the output is \( y_1 - y_2 \):

\[
\begin{array}{cccccccc}
  y_1 - y_2 & 4 & 10 & 10 & 3 & -1 & -2 & 0 & 1 & -1 & 0 \\
\end{array}
\]

c) Given the above information, how could you find the impulse response of this system? What is the impulse response?
Solution

Since $\frac{1}{2}(x_1 + x_3)$ is an impulse at the second sample, $\frac{1}{2}(y_1 + y_3)$ is the system’s impulse response starting from the second sample.

| $y_1 + y_3$ | 0 | 4 | 6 | 2 | -2 | 2 | 0 | 0 | 0 | 0 |

so

<table>
<thead>
<tr>
<th>$n$</th>
<th>$h[n]$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$n &lt; 0$</td>
<td>0</td>
</tr>
<tr>
<td>0</td>
<td>2</td>
</tr>
<tr>
<td>1</td>
<td>3</td>
</tr>
<tr>
<td>2</td>
<td>1</td>
</tr>
<tr>
<td>3</td>
<td>-1</td>
</tr>
<tr>
<td>4</td>
<td>1</td>
</tr>
<tr>
<td>$n &gt; 4$</td>
<td>0</td>
</tr>
</tbody>
</table>

d) What is the output of this system for the following input:

| $x_4$ | 1 | 1 | 1 | 0 | 0 | 0 | 0 | 0 | 0 | 0 |

Solution

The input $x_4$ can be written as a sum of three impulses: $x_4 = \delta[n] + \delta[n - 1] + \delta[n - 2]$. Since the system is LTI and we have computed the impulse response in the previous part, the output is going to be


| $y_4$ | 2 | 5 | 6 | 3 | 1 | 0 | 1 | 0 | 0 | 0 |
4 LTI Low Pass Filters

Given a sequence of discrete samples with high frequency noise, we can de-noise our signal with a discrete low-pass filter. Two examples are given below:

\[ y[n] = 0.5y[n-1] + x[n] \quad (1) \]
\[ y[n] = 0.25x[n] + 0.25x[n-1] + 0.25x[n-2] + 0.25x[n-3] \quad (2) \]

a) Show that both systems (1) and (2) are LTI.

**Solution**

For (1),

(i) **Linearity:**

- **Additivity:**
  
  Let \( x_1[n] \) and \( x_2[n] \) be inputs with outputs \( y_1[n] \) and \( y_2[n] \).
  
  For input \( \hat{x}[n] = (x_1 + x_2)[n] \), we see that
  
  \[ \hat{y}[n] = x_1[n] + x_2[n] = y_1[n] - 0.5y_1[n-1] + y_2[n] - 0.5y_2[n-1] \]
  
  This shows that \( \hat{y}[n] = y_1[n] + y_2[n] \) is the output.

- **Scaling:**
  
  Let \( x[n] \) be an input with output \( y[n] \). For a scaled input \( \hat{x}[n] = \alpha x[n] \), we see that
  
  \[ \hat{y}[n] = \alpha x[n] = \alpha y[n] - 0.5\alpha y[n-1] \]

  This implies that \( \hat{y}[n] = \alpha y[n] \).

(ii) **Time-Invariance**

Let \( \hat{x}[n] = x[n - n_0] \) be a delayed input signal. We see that

\[ \hat{x}[n] = x[n - n_0] = y[n - n_0] - 0.5y[n - n_0 - 1] \]

As a result, the output \( \hat{y}[n] \) must be \( \hat{y}[n] = y[n - n_0] \).

We could have also proven that this system is LTI by solving for \( y[n] \) directly in terms of \( x[n] \) and then showing that it is Linear and Time-Invariant.

For (2),

(i) **Linearity:**

- **Additivity:**
  
  Let \( x_1[n] \) and \( x_2[n] \) be inputs with outputs \( y_1[n] \) and \( y_2[n] \).
  
  For input \( \hat{x}[n] = (x_1 + x_2)[n] \), we see that the output is
  
  \[ \hat{y}[n] = 0.25 \hat{x}[n] + 0.25 \hat{x}[n-1] + 0.25 \hat{x}[n-2] + 0.25 \hat{x}[n-3] \]
  
  \[ = 0.25(x_1 + x_2)[n] + 0.25(x_1 + x_2)[n-1] + 0.25(x_1 + x_2)[n-2] + 0.25(x_1 + x_2)[n-3] \]
  
  \[ = 0.25x_1[n] + 0.25x_1[n-1] + 0.25x_1[n-2] + 0.25x_1[n-3] \]
  
  \[ + 0.25x_2[n] + 0.25x_2[n-1] + 0.25x_2[n-2] + 0.25x_2[n-3] \]
  
  \[ = y_1[n] + y_2[n] \]

  This shows that \( \hat{y}[n] = y_1[n] + y_2[n] \) is the output.
• Scaling: Let \( x[n] \) be an input with output \( y[n] \). For a scaled input \( x'[n] = ax[n] \), we see that

\[
\hat{y}[n] = 0.25\hat{x}[n] + 0.25\hat{x}[n - 1] + 0.25\hat{x}[n - 2] + 0.25\hat{x}[n - 3]
= 0.25ax[n] + 0.25ax[n - 1] + 0.25ax[n - 2] + 0.25ax[n - 3]
= a(0.25x[n] + 0.25x[n - 1] + 0.25x[n - 2] + 0.25x[n - 3])
= ay[n]
\]

This implies that \( \hat{y}[n] = ay[n] \).

(ii) **Time-Invariance**

Let \( \hat{x}[n] = x[n - n_0] \) be a delayed input signal. Then the output \( \hat{y}[n] \) will be

\[
\hat{y}[n] = 0.25\hat{x}[n] + 0.25\hat{x}[n - 1] + 0.25\hat{x}[n - 2] + 0.25\hat{x}[n - 3]
= 0.25x[n - n_0] + 0.25x[n - n_0 - 1] + 0.25x[n - n_0 - 2] + 0.25x[n - n_0 - 3]
\]

On the other hand, \( y[n - n_0] \) is equal to

\[
y[n - n_0] = 0.25x[n - n_0] + 0.25x[n - n_0 - 1] + 0.25x[n - n_0 - 2] + 0.25x[n - n_0 - 3]
\]

Therefore, we see that \( \hat{y}[n] = y[n - n_0] \) and can conclude that the system is time-invariant.

b) Write the impulse responses \( h[n] \) for (1) and (2). You may assume that \( h[n] = 0 \) for \( n < 0 \).

**Solution**

For (1),

\[
h[n] = \begin{cases} 
    0.5^n, & n \geq 0 \\
    0, & n < 0 
\end{cases}
\]

This has an infinite impulse response (IIR).

For (2),

\[
h[n] = \begin{cases} 
    0.25, & n = 0, 1, 2, 3 \\
    0, & \text{otherwise} 
\end{cases}
\]

This has a finite impulse response (FIR).

c) Are either of these systems causal? Are either of these systems stable?

**Solution**

They are both causal since \( y[n] \) depends only on inputs from \( x[n] \) and before. One could also argue that they are causal since the system is LTI and \( h[n] = 0 \) for \( n < 0 \).

Both filters are stable because the impulse responses are absolutely summable.

d) Given the input sequence \( x[n] = 2\cos(\pi n) + n \) from \( n = 0 \) to \( n = 7 \), find the output \( y \) for each system from \( n = 0 \) to \( n = 7 \). Assume that \( y[n] = 0 \) for \( n < 0 \).
Solution

For both systems, we can compute the output $y[n] = 0.5y[n-1] + x[n]$ using the difference equation or through a convolution.

For (1), the output will be

$$y = [2, 0, 4, 3, 7.5, 6.75, 11.375, 10.6875]$$

Notice how the oscillations have been reduced since this system is a low-pass filter.

For (2), the output will be

$$y = [0.5, 0.25, 1.25, 1.5, 2.5, 3.5, 4.5, 5.5]$$
Again the oscillations $\cos(\pi n)$ have been significantly attenuated at the output. Notice how the output is approximately $y[n] \approx n$. 
5 Convolution Matching

Consider the following discrete time signals:

\[ a[n] \]

\[ b[n] \]

\[ c[n] \]

\[ d[n] \]

\[ e[n] \]

\[ f[n] \]

a) Which of the options below shows the correct plot for the convolution \( a[n] \ast a[n] \)?

\[ (A) \]

\[ (B) \]

\[ (C) \]

\[ (D) \]

**Solution**

The correct plot is B. A shows cross-correlation instead of convolution. C is close, but off by one timestep. D shows pointwise multiplication of the two signals.
b) Which of the options below shows the correct plot for the convolution $a[n] \ast b[n]$?

![Plots A, B, C, D]

**Solution**

The correct plot is C. A shows pointwise multiplication of the two signals. B would be the output if $a[n]$ was a unit step as opposed to a boxcar. D shows cross-correlation instead of convolution.

c) Which of the options below shows the correct plot for the convolution $c[n] \ast d[n]$?

![Plots A, B, C, D]

**Solution**

The correct plot is A. This demonstrates that convolution with a shifted impulse shifts the input signal. That is,

$$x[n] \ast \delta[n - n_0] = \sum_{m=-\infty}^{\infty} \delta[m - n_0]x[n - m] = x[n - n_0]$$

From this we can argue that the impulse response of a delay block would be a shifted unit impulse.
B shows the output if \( d[n] \) were convolved with the identity element of convolution \( \delta[n] \), i.e. a unit impulse with a delay of \( n_0 = 0 \). C shows cross-correlation instead of convolution. D shows convolution with \( d[4-n] \).

d) Which of the options below shows the correct plot for the convolution \( d[n] * e[n] \)?

![Plot A](image1)

![Plot B](image2)

![Plot C](image3)

![Plot D](image4)

**Solution**
The correct plot is still A. This is because convolution is commutative!

e) Which of the options below shows the correct plot for the convolution \( e[n] * f[n] \)?

![Plot A](image5)

![Plot B](image6)

![Plot C](image7)

![Plot D](image8)

**Solution**
The correct plot is A. We can either show this algebraically or use the convolution formula. Since \( e[n] = \delta[n + 1] + \delta[n - 5] \) and convolution is distributive,

\[
(e * f)[n] = (f * e) = (f * (\delta[n + 1] + \delta[n - 5]))
\]

\[
\]

So it follows that the \( (e * f)[n] \) is the sum of two copies of \( f \). One shifted by 1 to the left and the other shifted 5 to the right.
6 Homework Process and Study Group

Citing sources and collaborators are an important part of life, including being a student! We also want to understand what resources you find helpful and how much time homework is taking, so we can change things in the future if possible.

a) What sources (if any) did you use as you worked through the homework?

b) If you worked with someone on this homework, who did you work with?
   List names and student ID’s. (In case of homework party, you can also just describe the group.)

c) Roughly how many total hours did you work on this homework?

d) Do you have any feedback on this homework assignment?