This homework is due on Thursday, October 15, 2020, at 10:59PM. Self-grades are due on Thursday, October 22, 2020, at 10:59PM.

Solutions have been provided for Problems 1-3 but Questions 4-6 are NOT optional and must be turned in on Gradescope.

1 One circuit, Many analyses

In this problem, we will recap several different methods that we have learnt and apply them all to the same circuit but under different situations.

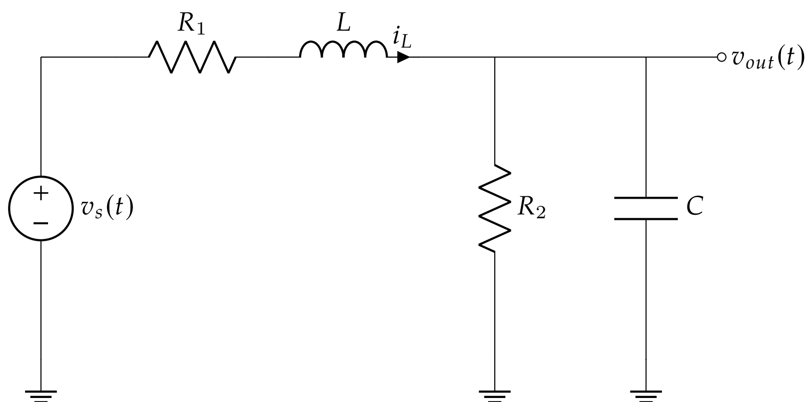


Figure 1: A model for a transmission line.

We are trying to transmit different signals across a very long wire. At longer lengths, the various electromagnetic losses incurred by a wire can be modeled using a model shown in Figure 1. We will take a look at the different kinds of signals that we want to transmit and which analyses techniques we should apply for assessing the output at receiving terminal v_{out}.

a) First we want to send a constant voltage value. We can do this by applying a constant voltage v_s as our input. If we apply $v_s = 12V$, find the capacitor voltage v_{out} and the inductor current i_L at equilibrium (or what we often refer to as DC steady-state). Use $R_1 = 100\Omega$, $R_2 = 100\Omega$, $C = 12\mu F$ and $L = 1mH$.

b) If $v_s(t)$ is a time-varying signal, write a system of differential equations using the inductor current i_L and capacitor voltage v_{out} as state variables. The equation system should be in the form

$$\frac{d}{dt}\begin{bmatrix} i_L \\ v_{out} \end{bmatrix} = A \begin{bmatrix} i_L \\ v_{out} \end{bmatrix} + Bv_s(t) \tag{1}$$

c) Find the eigenvalues for the matrix A found above and comment on them.

d) We want to send a pulse instead of a steady value. For this case, $v_s(t)$ is shown in Figure 2.

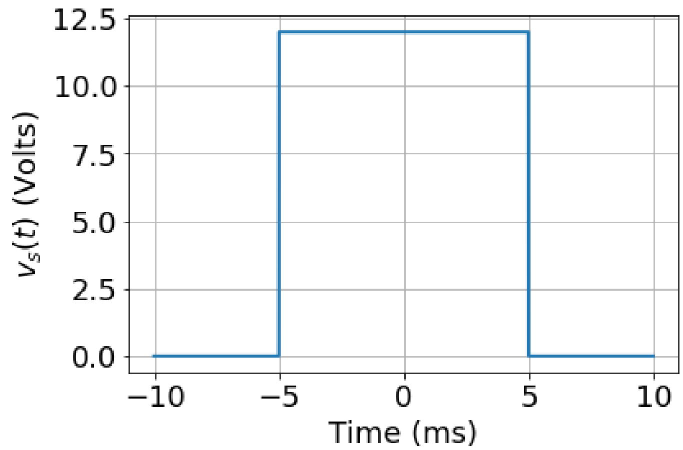


Figure 2: Pulse input to be transmitted across the wire.

It is not always possible or convenient to solve differential equations by hand using eigenvalues and the guess-and-check methods we have developed so far in the class. In the supplied Jupyter notebook *RLC_Circuit_Analsysis.ipynb*, fill out the entries for matrices A and B. The notebook has an implementation of a numerical solution to differential equations.

Sketch the output $v_{out}(t)$ for the pulse input $v_s(t)$ shown in Figure 2 using the supplied python notebook. The circuit parameters for this problem have been specified in the notebook: $R_1 = 10\Omega$, $R_2 = 10\Omega$, $C = 24\mu F$ and $L = 5mH$.

e) Finally, we want to test how our transmission line will carry a sinusoidal input. First, we will use the numerical technique that we saw in the previous part to evaluate the output $v_{out}(t)$. Using the code provided in the supplied iPython notebook, plot the output $v_{out}(t)$ for a sinusoidal input $v_s(t)$. Look at the last 2 cycles of the input and plot the corresponding output. Note the difference in amplitude and phase between the input $v_s(t)$ and the output $v_{out}(t)$. We will come back to these when we repeat this calculation for the sinusoidal steady state using phasor analysis.

f) We now want to transmit a sine wave, $v_s(t) = 12\sin(\omega t)$ along the transmission. Using phasor analysis, find the transfer function

$$H(\omega) = \frac{V_{out}}{V_s}, \tag{2}$$

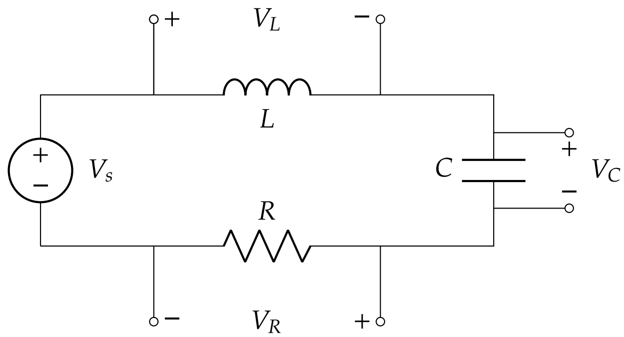
where V_s is a phasor representing the input voltage $v_s(t)$ and V_{out} is a phasor representing the output voltage $v_{out}(t)$. Find the output phasor V_{out}. If we use the same circuit parameters from part (d), comment on how the transfer function relates the time-waveforms $v_s(t)$ and $v_{out}(t)$ in (e).

HINT: Try using different initial conditions to see how the numerical solution changes.

2 RLC circuit as passive filters

As originally conceived by Bode in the 1930s, Bode plot is only an asymptotic approximation of the frequency response, using straight line segments. It relies on using a logarithmic scale for the input frequency ω to express the magnitude of the transfer functions on a logarithmic scale $\log_{10} \mid H(\omega) \mid$.

In this question, we will go through some examples to appreciate the beauty and simplicity of Bode plots. In the iPython notebook *BodePlots.ipynb*, you will see how well the approximation of Bode plots is in different regions. In particular, we will work with the RLC circuit shown below:



In the following questions, we will be exploring how to use the above RLC circuit to construct highpass, lowpass, and bandpass filters. As the name suggests, a highpass filter will suppress the low frequency components while keeping the high frequency components of the input unblocked. Since the circuit contains only passive elements, namely resistors, capacitors, and inductors, these filters are called *passive filters*. On the other hand, if the circuit contains op amps, transistors, or other active devices, it will become *active filters*.

a) **Lowpass filter.** Treat V_s as the input and V_C as the output. Obtain the transfer function $H_{LP} = \frac{V_C}{V_s}$, and its magnitude and phase. Draw the Bode plot for the magnitude. Explain why this is a lowpass filter.

b) **Highpass filter.** Let V_L be the output. Obtain the transfer function $H_{HP} = \frac{V_L}{V_s}$, and its magnitude and phase. Draw the Bode plot for the magnitude. Explain why this is a highpass filter.

c) **Bandpass filter.** How can you obtain a bandpass filter based on your findings above? Write out the transfer function and its magnitude and phase.

d) The **resonant frequency**, ω_0, is the input frequency (other than 0 and ∞) that leads to the elimination of the imaginary part of the circuit impedance, i.e., the impedance is purely real. Find the resonant frequency for the RLC circuit above.

e) For mobile communications, the center frequency is approximately 800 MHz. In the IPython notebook, experiment with different L and C to center the bandpass filter.

You might have noticed that the advantage of Bode plot is that it makes it easier to work with transfer functions that have multiple factors. We can write $H(\omega)$ as a product of such factors:

$$H(\omega) = A_1(\omega)A_2(\omega)...A_n(\omega) \tag{3}$$

In this class, we will focus on functions A_1 to A_n that assume one of the possible forms.

$$\text{Constant factor: } H = K$$
$$\text{Zero @ origin: } H = (j\omega)^N$$
$$\text{Pole @ origin: } H = 1/(j\omega)^N$$
$$\text{Zero @ } \omega_c\text{: } H = (1 + j\omega/\omega_c)^N$$
$$\text{Pole @ } \omega_c\text{: } H = 1/(1 + j\omega/\omega_c)^N$$

The construction thus becomes simple addition or subtraction of these forms. For instance, $H(\omega) = 10\frac{1+j\omega/\omega_Z}{1+j\omega/\omega_p}$, where $A_1 = 10$, $A_2 = 1 + j\omega/\omega_Z$, $A_3 = \frac{1}{1+j\omega/\omega_p}$.

(f) For transfer function $H(\omega) = M(\omega)e^{j\phi(\omega)}$, how to represent the magnitude $M(\omega)$ and phase $\phi(\omega)$ with the magnitudes $|A_i(\omega)|$ and phase $\phi_{A_i}(\omega)$?

(g) Consider the transfer function

$$H(\omega) = \frac{(j10\omega + 30)^2}{(300 - 3\omega^2 + j90\omega)} \tag{4}$$

Refer to the IPython notebook for constructing the Bode plots for this transfer function.

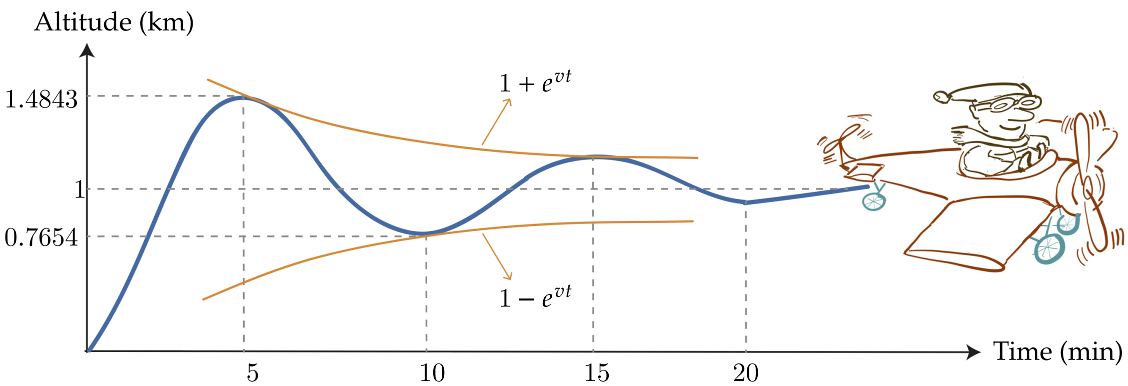
(h) In addition to the Bode plots, we also plotted the magnitude of the transfer functions without approximations. Please comment on the differences.

3 Otto the Pilot

Otto has devised a control algorithm, so that his plane climbs to the desired altitude by itself. However, he is having oscillatory transients as shown in the figure. Prof. Sanders told him that if his system has complex eigenvalues

$$\lambda_{1,2} = v \pm j\omega,$$

then his altitude would indeed oscillate with frequency ω about the steady state value, 1 km, and that the time trace of his altitude would be tangent to the curves $1 + e^{vt}$ and $1 - e^{vt}$ near its maxima and minima respectively.



a) Find the real part v and the imaginary part ω from the altitude plot.

b) Let the dynamical model for the altitude be

$$\frac{d}{dt}\begin{bmatrix} y(t) \\ \dot{y}(t) \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ a_1 & a_2 \end{bmatrix}\begin{bmatrix} y(t) \\ \dot{y}(t) \end{bmatrix},$$

where $y(t)$ is the deviation of the altitude from the steady state value, $\dot{y}(t)$ is the time derivative of $y(t)$, and a_1 and a_2 are constants. Using your answer to part (a), find what a_1 and a_2 are.

c) Otto can change a_2 by turning a knob. Tell him what value he should pick so that he has a "critically damped" ascent with two real negative eigenvalues at the same location.

4 Predator-Prey Model

Ellen is a keen ecologist studying life on a planet far, far away. She found two species on the planet, one predator and the other prey. You found her journal back on Earth and are trying to decipher life on this other planet. In Ellen's journal, she states the following observations about the predator and prey populations

- The prey population grows at a rate proportional to the current prey population.
- For a given prey population, the rate of increase of prey population decreases linearly with the predator population.

As the predator population increases, it is harder for the prey population to grow.

- The predator population also grows at a rate proportional to the current predator population.
- For a fixed predator population, predator population increases linearly with prey population. As the amount of prey increases, predators have more food and their population increases faster.

Let's call these species **x (prey) and y (predator)** to keep things simple. A friend suggests using the following model for predator and prey populations:

$$\frac{d}{dt}x(t) = (a - by)x \tag{5}$$

$$\frac{d}{dt}y(t) = (cx - d)y \tag{6}$$

a) Comment on the validity of the model described in Equations 5 and 6.

b) You find a data entry in Ellen's journal measuring the growth rates of predator and prey populations. At a predator population of 10 and prey population of 10, both populations increase at $10/s$. With a single prey remaining, the predators starve and their population decreases at $8/s$ at a population of 10 predators. At population levels of 1 predator and 10 prey, the prey population rapidly increases at $100/s$.

   Using these observations, **find the model parameters a, b, c and d**.

c) If we had 1000 predators on the planet and no prey. What will happen to the predator population?

d) At what levels will the predator and prey populations not change any more. As we have seen in class, this set of population levels is called an equilibrium.

e) We want to analyze the model at a set of population levels. We don't know how to solve the model described in Equations 5 and 6. We can, however, linearize the model around a set of population levels and analyze the population changes in that vicinity.

   If the predator population is 10 and the prey population is 1, linearize the system around these populations and express the new, linearized system in the form

$$\frac{d}{dt}\begin{bmatrix}\tilde{x}\\ \tilde{y}\end{bmatrix} = A\begin{bmatrix}\tilde{x}\\ \tilde{y}\end{bmatrix} \tag{7}$$

5 Checkpoint Feedback Form

a) Please fill out the survey.

If the survey link doesn't work, please refer to Piazza to complete the survey.

6 Homework Process and Study Group

Citing sources and collaborators are an important part of life, including being a student!
We also want to understand what resources you find helpful and how much time homework is taking, so we can change things in the future if possible.

a) **What sources (if any) did you use as you worked through the homework?**

b) **If you worked with someone on this homework, who did you work with?**
List names and student ID's. (In case of homework party, you can also just describe the group.)

c) **Roughly how many total hours did you work on this homework?**

d) **Do you have any feedback on this homework assignment?**