# Homework 12

**This homework is due on Saturday, November 18, 2023 at 11:59PM. Self-grades and HW Resubmissions are due the following Saturday, December 2, 2023 at 11:59PM.**

**1. SVD Proofs**

(a) **Show, for a general matrix $A \in \mathbb{R}^{m\times n}$ with $\mathrm{rank}(A) = r$ and $A = U\Sigma V^\top$, that $\mathrm{Null}(A) = \mathrm{Col}(V_{n-r})$.** *(HINT: How do we show two sets are equal? Try and use that approach here. Consider the outer product summation form for the SVD. Also, consider using the rank-nullity theorem that $\dim \mathrm{Col}(A) + \dim \mathrm{Null}(A) = n$.)*

**Solution:** $\mathrm{Null}(A) \subseteq \mathrm{Col}(V_{n-r})$

We can start by writing the SVD of $A$ in outer product form:

$$A = \sum_{i=1}^{r} \sigma_i \vec{u}_i \vec{v}_i^\top \tag{1}$$

Let us say that we want a (nonzero) vector $\vec{x}$ such that $A\vec{x} = \vec{0}$. This means that

$$A\vec{x} = \sum_{i=1}^{r} \sigma_i \vec{u}_i \vec{v}_i^\top \vec{x} \tag{2}$$

Here, we require $\vec{v}_i^\top \vec{x} = 0$ for $i = 1$ to $r$. Hence, $\vec{x}$ must be orthogonal to $\vec{v}_i$ for $i = 1$ to $r$, since each $\sigma_i \neq 0$ and $\vec{u}_i$ are linearly independent. This means that $\vec{x} \in \mathrm{Col}(V_{n-r})$. Since $\vec{x}$ is an arbitrary null space vector of $A$ by construction, it must be the case that $\mathrm{Null}(A) \subseteq \mathrm{Col}(V_{n-r})$.

Having shown this, we can use rank-nullity theorem to see that $\dim \mathrm{Null}(A) = n - r$ since $\mathrm{rank}(A) = r$. Hence, $\dim \mathrm{Null}(A) = \dim \mathrm{Col}(V_{n-r})$ and $\mathrm{Null}(A) = \mathrm{Col}(V_{n-r})$. If you want to go further, you can show the other direction as well:

$\mathrm{Null}(A) \supseteq \mathrm{Col}(V_{n-r})$ (Optional)

If we had an arbitrary $\tilde{\vec{v}} \in \mathrm{Col}(V_{n-r})$, it must be orthogonal to each of $\vec{v}_i$ for $i = 1$ to $r$. Hence,

$$A\tilde{\vec{v}} = \sum_{i=1}^{r} \sigma_i \vec{u}_i \underbrace{\vec{v}_i^\top \tilde{\vec{v}}}_{=0} = \vec{0} \tag{3}$$

Thus, $\tilde{\vec{v}} \in \mathrm{Null}(A)$ and $\mathrm{Null}(A) \supseteq \mathrm{Col}(V_{n-r})$.

This completes the proof that $\mathrm{Null}(A) = \mathrm{Col}(V_{n-r})$.

(b) **Show, for a general matrix $A \in \mathbb{R}^{m\times n}$ with $\mathrm{rank}(A) = r$ and $A = U\Sigma V^\top$, that $\mathrm{Col}(A) = \mathrm{Col}(U_r)$.**

**Solution:** $\mathrm{Col}(A) \subseteq \mathrm{Col}(U_r)$

We can again start by writing the SVD of $A$ in outer product form:

$$A = \sum_{i=1}^{r} \sigma_i \vec{u}_i \vec{v}_i^\top \tag{4}$$

We know that $\text{Col}(A) := \{\vec{b} : A\vec{x} = \vec{b}\}$. Hence, if we multiply $A$ by $\vec{x}$:

$$A\vec{x} = \sum_{i=1}^{r} \sigma_i \vec{u}_i \vec{v}_i^\top \vec{x} \tag{5}$$

$$= \sum_{i=1}^{r} \sigma_i \vec{u}_i \left(\vec{v}_i^\top \vec{x}\right) \tag{6}$$

$$= \sum_{i=1}^{r} \left(\vec{v}_i^\top \vec{x}\right) \sigma_i \vec{u}_i \tag{7}$$

which is a linear combination of $\vec{u}_1, \ldots, \vec{u}_r$. Hence, $A\vec{x} \in \text{Col}(U_r)$ and so $\text{Col}(A) \subseteq \text{Col}(U_r)$.

Having shown this, we can see that $\dim \text{Col}(A) = \dim \text{Col}(U_r) = r$, so it must be the case that $\text{Col}(A) = \text{Col}(U_r)$. If you want to go further, you can show the other direction as well:

$\text{Col}(A) \supseteq \text{Col}(U_r)$ (Optional)
Suppose $\vec{b} \in \text{Col}(U_r)$, so $U_r\vec{x} = \vec{b}$ for some vector $\vec{x}$. We can show that $\vec{b} \in \text{Col}(A)$. First, we can consider the compact SVD form of $A$:

$$A = U_r \Sigma_r V_r^\top \tag{8}$$

where $\Sigma_r \in \mathbb{R}^{r\times r}$ is diagonal. Now, we can define the vector $\tilde{\vec{x}} = V_r \Sigma_r^{-1} \vec{x}$. Notice that

$$A\tilde{\vec{x}} = U_r \Sigma_r V_r^\top \tilde{\vec{x}} \tag{9}$$

$$= U_r \Sigma_r V_r^\top V_r \Sigma_r^{-1} \vec{x} \tag{10}$$

$$= U_r \vec{x} = \vec{b} \tag{11}$$

so $\vec{b} \in \text{Col}(A)$, and thus, $\text{Col}(A) \supseteq \text{Col}(U_r)$.

This completes the proof that $\text{Col}(A) = \text{Col}(U_r)$.

(c) **Show, for a general matrix $A \in \mathbb{R}^{m\times n}$ with $\text{rank}(A) = r$ and $A = U\Sigma V^\top$, that $\text{Null}(A^\top) = \text{Col}(U_{m-r})$ and $\text{Col}(A^\top) = \text{Col}(V_r)$. Then show:**

  **i.** $\dim \text{Col}(A) + \dim \text{Null}(A^\top) = m$,

  **ii. and $\text{Col}(A)$ and $\text{Null}(A^\top)$ are orthogonal.**

**Solution:** We know $\text{Null}(A^\top) = \text{Col}\left(\tilde{V}_{m-r}\right) = \text{Col}(U_{m-r})$ and $\text{Col}(A^\top) = \text{Col}\left(\tilde{U}_r\right) = \text{Col}(V_r)$.

We know that $m = \dim \text{Col}(U_r) + \dim \text{Col}(U_{m-r}) = \dim \text{Col}(A) + \dim \text{Null}(A^\top)$. Since $\text{Col}(A) = \text{Col}(U_r)$ and $\text{Null}(A^\top) = \text{Col}(U_{m-r})$, $\text{Col}(A)$ and $\text{Null}(A^\top)$ must be orthogonal since all the columns of $U_r$ and all the columns of $U_{m-r}$ are orthogonal.

 

**2. SVD System ID**

Previously, we saw instances for how to solve system ID problems when $D \in \mathbb{R}^{m\times n}$ is full rank (for $m > n$). Now, let us consider more generally the following problem of estimating $\vec{p}$ in

$$D\vec{p} = \vec{s} \tag{12}$$

where $\vec{p} \in \mathbb{R}^n$, $\vec{s} \in \mathbb{R}^m$, and $D \in \mathbb{R}^{m\times n}$. We assume that $\text{rank}(D) = r < \min(m, n)$, and we do not make any further assumptions on the relationship between $m$ and $n$. Let's assume that $D$ has an SVD given by

$$D = U\Sigma V^\top \tag{13}$$

where $U \in \mathbb{R}^{m\times m}$ and $V \in \mathbb{R}^{n\times n}$ are orthonormal matrices. $\Sigma \in \mathbb{R}^{m\times n}$ has the following form:

$$\Sigma = \begin{bmatrix} \Sigma_r & 0_{r\times(n-r)} \\ 0_{(m-r)\times r} & 0_{(m-r)\times(n-r)} \end{bmatrix} \tag{14}$$

where $\Sigma_r = \begin{bmatrix} \sigma_1 & & & \\ & \sigma_2 & & \\ & & \ddots & \\ & & & \sigma_r \end{bmatrix}$ is a $r \times r$ diagonal matrix with nonzero elements along its diagonal.

Using this problem setup, we can rewrite our original system ID problem as

$$U\Sigma V^\top \vec{p} = \vec{s} \tag{15}$$

Our goal is to find $\vec{p}$ with smallest norm that best estimates $\vec{s}$.

For notational convenience, denote $U := \begin{bmatrix} U_r & U_{m-r} \end{bmatrix}$ where $U_r \in \mathbb{R}^{m\times r}$ is a matrix with the first $r$ columns of $U$ and $U_{m-r} \in \mathbb{R}^{m\times(m-r)}$ is a matrix with the last $m - r$ columns of $U$. Also, denote $V := \begin{bmatrix} V_r & V_{n-r} \end{bmatrix}$ where $V_r \in \mathbb{R}^{n\times r}$ is a matrix that has the first $r$ columns of $V$, and $V_{n-r} \in \mathbb{R}^{n\times(n-r)}$ is a matrix that has the last $n - r$ columns of $V$. From SVD properties, we know that the columns of $U_r$ form an orthonormal basis for $\text{Col}(D)$ and that the columns of $V_{n-r}$ form an orthonormal basis for $\text{Null}(D)$.

(a) **Using the fact that $U$ is orthonormal, show that $\Sigma V^\top \vec{p} = U^\top \vec{s}$.**

**Solution:** We have that

$$D\vec{p} = \vec{s} \tag{16}$$
$$U\Sigma V^\top \vec{p} = \vec{s} \tag{17}$$
$$U^\top U\Sigma V^\top \vec{p} = U^\top \vec{s} \tag{18}$$
$$\Sigma V^\top \vec{p} = U^\top \vec{s} \tag{19}$$

(b) **Show that we can write $\vec{p} = \begin{bmatrix} V_r & V_{n-r} \end{bmatrix} \begin{bmatrix} \vec{\alpha} \\ \vec{\beta} \end{bmatrix}$ for some vectors $\vec{\alpha}$ and $\vec{\beta}$ (i.e., find $\vec{\alpha} \in \mathbb{R}^r$ and $\vec{\beta} \in \mathbb{R}^{n-r}$). Show that changing $\vec{\beta}$ will not affect the result of $D\vec{p}$ and that we should set $\vec{\beta} = \vec{0}$ if we want to minimize $\|p\|$.** Since $V_{n-r}\vec{\beta}$ is the component of $\vec{p}$ that is in the nullspace of $D$, we can set $\vec{\beta}$ to be whatever we want. To choose the $\vec{p}$ with smallest norm, we will set $\vec{\beta} = \vec{0}$.

 

**Solution:** We have that

$$\vec{p} = V \begin{bmatrix} \vec{\alpha} \\ \vec{\beta} \end{bmatrix} \tag{20}$$

$$V^\top \vec{p} = \begin{bmatrix} \vec{\alpha} \\ \vec{\beta} \end{bmatrix} \tag{21}$$

$$\begin{bmatrix} V_r^\top \\ V_{n-r}^\top \end{bmatrix} \vec{p} = \begin{bmatrix} \vec{\alpha} \\ \vec{\beta} \end{bmatrix} \tag{22}$$

$$\begin{bmatrix} V_r^\top \vec{p} \\ V_{n-r}^\top \vec{p} \end{bmatrix} = \begin{bmatrix} \vec{\alpha} \\ \vec{\beta} \end{bmatrix} \tag{23}$$

so $\vec{\alpha} = V_r^\top \vec{p}$ and $\vec{\beta} = V_{n-r}^\top \vec{p}$.

(c) From the previous part, we can rewrite $\vec{p} = V \begin{bmatrix} \vec{\alpha} \\ \vec{0} \end{bmatrix}$. This simplifies our system ID problem as follows:

$$\Sigma V^\top V \begin{bmatrix} \vec{\alpha} \\ \vec{0} \end{bmatrix} = U^\top \vec{s} \tag{24}$$

$$\Sigma \begin{bmatrix} \vec{\alpha} \\ \vec{0} \end{bmatrix} = U^\top \vec{s} \tag{25}$$

**Simplify the left hand side of eq. (25) using eq. (14). Rewrite $U^\top \vec{s}$ as $\begin{bmatrix} U_r^\top \vec{s} \\ U_{m-r}^\top \vec{s} \end{bmatrix}$ and find an expression for $\vec{\alpha}$.** *(HINT: Block matrix multiplication will work like normal matrix-vector multiplication here since $\Sigma_r \in \mathbb{R}^{r \times r}$ and $\vec{\alpha} \in \mathbb{R}^r$.)*

**Solution:** Following the hint, we have

$$\Sigma \begin{bmatrix} \vec{\alpha} \\ \vec{0} \end{bmatrix} = \begin{bmatrix} \Sigma_r & 0_{r \times (n-r)} \\ 0_{(m-r) \times r} & 0_{(m-r) \times (n-r)} \end{bmatrix} \begin{bmatrix} \vec{\alpha} \\ \vec{0} \end{bmatrix} \tag{26}$$

$$= \begin{bmatrix} \Sigma_r \vec{\alpha} \\ \vec{0} \end{bmatrix} \tag{27}$$

Now, we set this expression equal to $U^\top \vec{s}$ to obtain

$$\begin{bmatrix} \Sigma_r \vec{\alpha} \\ \vec{0} \end{bmatrix} = \begin{bmatrix} U_r^\top \vec{s} \\ U_{m-r}^\top \vec{s} \end{bmatrix} \tag{28}$$

By looking at the first entry on the left and right hand sides, we have

$$\Sigma_r \vec{\alpha} = U_r^\top \vec{s} \tag{29}$$

$$\vec{\alpha} = \Sigma_r^{-1} U_r^\top \vec{s} \tag{30}$$

where $\Sigma_r$ is invertible since it is a diagonal matrix with nonzero elements along the diagonal. Explicitly, one may write

$$\Sigma_r^{-1} = \begin{bmatrix} \frac{1}{\sigma_1} & & & \\ & \frac{1}{\sigma_2} & & \\ & & \ddots & \\ & & & \frac{1}{\sigma_r} \end{bmatrix} \tag{31}$$

 

(d) **Use the previous part to come up with a solution for $\vec{p}$.**

**Solution:** From part (b), we have

$$\vec{p} = V \begin{bmatrix} \vec{\alpha} \\ \vec{0} \end{bmatrix} \tag{32}$$

and from the previous part we have

$$\vec{\alpha} = \Sigma_r^{-1} U_r^\top \vec{s} \tag{33}$$

Putting these two together, we have

$$\vec{p} = V \begin{bmatrix} \Sigma_r^{-1} U_r^\top \vec{s} \\ \vec{0} \end{bmatrix} \tag{34}$$

(e) From the concept of projections, we know that the optimal solution for $\vec{p}$ satisfies the property that the projection error, namely $\vec{s} - D\vec{p}$, is orthogonal to the projection itself, namely $D\vec{p}$. Write $\vec{s} := \begin{bmatrix} U_r & U_{m-r} \end{bmatrix} \begin{bmatrix} \vec{w} \\ \vec{z} \end{bmatrix}$ for some vectors $\vec{w} \in \mathbb{R}^r$ and $\vec{z} \in \mathbb{R}^{m-r}$. **Find $\vec{w}$ and $\vec{z}$. Using this, show that our solution for $\vec{p}$ is optimal.**

**Solution:** We have that $\vec{s} = U \begin{bmatrix} \vec{w} \\ \vec{z} \end{bmatrix}$. Simplifying this,

$$\vec{s} = U \begin{bmatrix} \vec{w} \\ \vec{z} \end{bmatrix} \tag{35}$$

$$U^\top \vec{s} = \begin{bmatrix} \vec{w} \\ \vec{z} \end{bmatrix} \tag{36}$$

$$\begin{bmatrix} U_r^\top \\ U_{m-r}^\top \end{bmatrix} \vec{s} = \begin{bmatrix} \vec{w} \\ \vec{z} \end{bmatrix} \tag{37}$$

$$\begin{bmatrix} U_r^\top \vec{s} \\ U_{m-r}^\top \vec{s} \end{bmatrix} = \begin{bmatrix} \vec{w} \\ \vec{z} \end{bmatrix} \tag{38}$$

so $\vec{w} = U_r^\top \vec{s}$ and $\vec{z} = U_{m-r}^\top \vec{s}$. Now, from the previous part, we have

$$\vec{p} = V \begin{bmatrix} \Sigma_r^{-1} U_r^\top \vec{s} \\ \vec{0} \end{bmatrix} \tag{39}$$

Plugging this back into the equation $D\vec{p}$ (where we substitute $D = U\Sigma V^\top$),

$$D\vec{p} = U\Sigma V^\top V \begin{bmatrix} \Sigma_r^{-1} U_r^\top \vec{s} \\ \vec{0} \end{bmatrix} \tag{40}$$

$$= U\Sigma \begin{bmatrix} \Sigma_r^{-1} U_r^\top \vec{s} \\ \vec{0} \end{bmatrix} \tag{41}$$

$$= U \begin{bmatrix} \Sigma_r & 0_{r\times(n-r)} \\ 0_{(m-r)\times r} & 0_{(m-r)\times(n-r)} \end{bmatrix} \begin{bmatrix} \Sigma_r^{-1} U_r^\top \vec{s} \\ \vec{0} \end{bmatrix} \tag{42}$$

$$= U \begin{bmatrix} U_r^\top \vec{s} \\ \vec{0} \end{bmatrix} \tag{43}$$

Thus, we have that

$$\vec{s} - D\vec{p} = U \begin{bmatrix} U_r^\top \vec{s} \\ U_{m-r}^\top \vec{s} \end{bmatrix} - U \begin{bmatrix} U_r^\top \vec{s} \\ \vec{0} \end{bmatrix} = U \begin{bmatrix} \vec{0} \\ U_{m-r}^\top \vec{s} \end{bmatrix} \tag{44}$$

Using this, we have that

$$\begin{aligned}
\langle \vec{s} - D\vec{p},\ D\vec{p} \rangle &= (\vec{s} - D\vec{p})^\top (D\vec{p}) && (45) \\
&= \left( U \begin{bmatrix} \vec{0} \\ U_{m-r}^\top \vec{s} \end{bmatrix} \right)^\top \left( U \begin{bmatrix} U_r^\top \vec{s} \\ \vec{0} \end{bmatrix} \right) && (46) \\
&= \begin{bmatrix} \vec{0}^\top & \vec{s}^\top U_{m-r} \end{bmatrix} U^\top U \begin{bmatrix} U_r^\top \vec{s} \\ \vec{0} \end{bmatrix} && (47) \\
&= \begin{bmatrix} \vec{0}^\top & \vec{s}^\top U_{m-r} \end{bmatrix} \begin{bmatrix} U_r^\top \vec{s} \\ \vec{0} \end{bmatrix} && (48) \\
&= \vec{0}^\top U_r^\top \vec{s} + \vec{s}^\top U_{m-r} \vec{0} && (49) \\
&= 0 && (50)
\end{aligned}$$

so $\vec{p}$ is optimal.

**3. SVD**

(a) Consider the matrix

$$A = \begin{bmatrix} -1 & 1 & 5 \\ 3 & 1 & -1 \\ 2 & -1 & 4 \end{bmatrix}.$$

Observe that the columns of matrix $A$ are mutually orthogonal with norms $\sqrt{14}, \sqrt{3}, \sqrt{42}$.

**Verify numerically that columns** $\begin{bmatrix} 1 \\ 1 \\ -1 \end{bmatrix}$ **and** $\begin{bmatrix} 5 \\ -1 \\ 4 \end{bmatrix}$ **are orthogonal to each other.**

**Solution:** Taking the inner product of the two vectors, we have

$$\left\langle \begin{bmatrix} 1 \\ 1 \\ -1 \end{bmatrix}, \begin{bmatrix} 5 \\ -1 \\ 4 \end{bmatrix} \right\rangle = \begin{bmatrix} 1 \\ 1 \\ -1 \end{bmatrix}^\top \begin{bmatrix} 5 \\ -1 \\ 4 \end{bmatrix} = 5 - 1 - 4 = 0. \tag{51}$$

So the two columns are orthogonal to each other.

(b) **Write $A = BD$, where $B$ is an orthonormal matrix and $D$ is a diagonal matrix.** What is $B$? What is $D$?

**Solution:** We compute the norm for each column and divide each column by its norm to obtain matrix $B$. Matrix $D$ is formed by placing the norms on the diagonal.

$$B = \begin{bmatrix} -\frac{1}{\sqrt{14}} & \frac{1}{\sqrt{3}} & \frac{5}{\sqrt{42}} \\ \frac{3}{\sqrt{14}} & \frac{1}{\sqrt{3}} & -\frac{1}{\sqrt{42}} \\ \frac{2}{\sqrt{14}} & -\frac{1}{\sqrt{3}} & \frac{4}{\sqrt{42}} \end{bmatrix} \tag{52}$$

$$D = \begin{bmatrix} \sqrt{14} & 0 & 0 \\ 0 & \sqrt{3} & 0 \\ 0 & 0 & \sqrt{42} \end{bmatrix} \tag{53}$$

(c) **Write out a singular value decomposition of $A = U\Sigma V^\top$ using the previous part.** Note the ordering of the singular values in $\Sigma$ should be from the largest to smallest. *(HINT: There is no need to compute the eigenvalues of anything. Use Theorem 14, Note 15.)*

**Solution:** Using part b, we can write

$$A = BD = BDI = \begin{bmatrix} -\frac{1}{\sqrt{14}} & \frac{1}{\sqrt{3}} & \frac{5}{\sqrt{42}} \\ \frac{3}{\sqrt{14}} & \frac{1}{\sqrt{3}} & -\frac{1}{\sqrt{42}} \\ \frac{2}{\sqrt{14}} & -\frac{1}{\sqrt{3}} & \frac{4}{\sqrt{42}} \end{bmatrix} \begin{bmatrix} \sqrt{14} & 0 & 0 \\ 0 & \sqrt{3} & 0 \\ 0 & 0 & \sqrt{42} \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}. \tag{54}$$

Reordering the columns and rows of $B$ and $I$ so that the diagonal entries of $D$ are in non-decreasing order, we have

$$\begin{bmatrix} \frac{5}{\sqrt{42}} & -\frac{1}{\sqrt{14}} & \frac{1}{\sqrt{3}} \\ -\frac{1}{\sqrt{42}} & \frac{3}{\sqrt{14}} & \frac{1}{\sqrt{3}} \\ \frac{4}{\sqrt{42}} & \frac{2}{\sqrt{14}} & -\frac{1}{\sqrt{3}} \end{bmatrix} \begin{bmatrix} \sqrt{42} & 0 & 0 \\ 0 & \sqrt{14} & 0 \\ 0 & 0 & \sqrt{3} \end{bmatrix} \begin{bmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}. \tag{55}$$

Then by Note 16, Theorem 14, this is an SVD of $A$.

 

(d) Given the matrix

$$A = \frac{1}{\sqrt{50}}\begin{bmatrix}3\\4\end{bmatrix}\begin{bmatrix}1 & -1\end{bmatrix} + \frac{3}{\sqrt{50}}\begin{bmatrix}-4\\3\end{bmatrix}\begin{bmatrix}1 & 1\end{bmatrix}, \tag{56}$$

**write out a singular value decomposition of matrix $A$ in the form $U\Sigma V^\top$.** Note the ordering of the singular values in $\Sigma$ should be from the largest to smallest. *(HINT: You don't have to compute any eigenvalues for this. Some useful observations are that*

$$\begin{bmatrix}3, 4\end{bmatrix}\begin{bmatrix}-4\\3\end{bmatrix} = 0, \quad \begin{bmatrix}1, -1\end{bmatrix}\begin{bmatrix}1\\1\end{bmatrix} = 0, \quad \left\|\begin{bmatrix}3\\4\end{bmatrix}\right\| = \left\|\begin{bmatrix}-4\\3\end{bmatrix}\right\| = 5, \quad \left\|\begin{bmatrix}1\\-1\end{bmatrix}\right\| = \left\|\begin{bmatrix}1\\1\end{bmatrix}\right\| = \sqrt{2}.$$

*)*

**Solution:**

The singular value decomposition can be written in the form

$$A = \sum_{i=1}^{2} \sigma_i \vec{u}_i \vec{v}_i^\top, \tag{57}$$

with unit orthonormal vectors $\{\vec{u}_i\}$ and $\{\vec{v}_i\}$. From the given observations, we can see that the vectors we were provided are orthogonal, so we can just normalize them to get the desired answer. Taking it step by step:

$$A = \frac{1}{\sqrt{50}}\begin{bmatrix}3\\4\end{bmatrix}\begin{bmatrix}1 & -1\end{bmatrix} + \frac{3}{\sqrt{50}}\begin{bmatrix}-4\\3\end{bmatrix}\begin{bmatrix}1 & 1\end{bmatrix} \tag{58}$$

$$= \frac{5}{\sqrt{50}}\begin{bmatrix}\frac{3}{5}\\\frac{4}{5}\end{bmatrix}\begin{bmatrix}1 & -1\end{bmatrix} + \frac{3 \cdot 5}{\sqrt{50}}\begin{bmatrix}-\frac{4}{5}\\\frac{3}{5}\end{bmatrix}\begin{bmatrix}1 & 1\end{bmatrix} \tag{59}$$

$$= \frac{5\sqrt{2}}{\sqrt{50}}\begin{bmatrix}\frac{3}{5}\\\frac{4}{5}\end{bmatrix}\begin{bmatrix}\frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}}\end{bmatrix} + \frac{3 \cdot 5\sqrt{2}}{\sqrt{50}}\begin{bmatrix}-\frac{4}{5}\\\frac{3}{5}\end{bmatrix}\begin{bmatrix}\frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}}\end{bmatrix} \tag{60}$$

$$= 1\begin{bmatrix}\frac{3}{5}\\\frac{4}{5}\end{bmatrix}\begin{bmatrix}\frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}}\end{bmatrix} + 3\begin{bmatrix}-\frac{4}{5}\\\frac{3}{5}\end{bmatrix}\begin{bmatrix}\frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}}\end{bmatrix}. \tag{61}$$

From this, we can derive

$$\vec{u}_1 = \begin{bmatrix}-\frac{4}{5}\\\frac{3}{5}\end{bmatrix}, \vec{u}_2 = \begin{bmatrix}\frac{3}{5}\\\frac{4}{5}\end{bmatrix}, \vec{v}_1 = \begin{bmatrix}\frac{1}{\sqrt{2}}\\\frac{1}{\sqrt{2}}\end{bmatrix}, \vec{v}_2 = \begin{bmatrix}\frac{1}{\sqrt{2}}\\-\frac{1}{\sqrt{2}}\end{bmatrix} \tag{62}$$

and corresponding singular values $\sigma_1 = 3, \sigma_2 = 1$ because we need to order them by size in decreasing order. This gives the singular value decomposition

$$A = \begin{bmatrix}-\frac{4}{5} & \frac{3}{5}\\\frac{3}{5} & \frac{4}{5}\end{bmatrix}\begin{bmatrix}3 & 0\\0 & 1\end{bmatrix}\begin{bmatrix}\frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}}\\\frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}}\end{bmatrix}^\top \tag{63}$$

(e) Define the matrix

$$A = \begin{bmatrix}-1 & 4\\1 & 4\end{bmatrix}.$$

**Find the SVD of $A$ by following the standard algorithm introduced in Note 14, i.e. by computing the eigendecomposition of $A^\top A$. Also find the eigenvectors and eigenvalues of $A$. Is there a relationship between the eigenvalues or eigenvectors of $A$ with the SVD of $A$?**

**Solution:** Since we have a square matrix, we will arbitrarily use $A^\top A$ for our SVD:

$$A^\top A = \begin{bmatrix} -1 & 1 \\ 4 & 4 \end{bmatrix} \begin{bmatrix} -1 & 4 \\ 1 & 4 \end{bmatrix} \tag{64}$$

$$= \begin{bmatrix} 2 & 0 \\ 0 & 32 \end{bmatrix} \tag{65}$$

Next, we find the eigenvalues of the above matrix.

$$\det(A^\top A - \lambda I) = (2 - \lambda)(32 - \lambda) = 0 \tag{66}$$

Hence, the eigenvalues are $\lambda_1 = 32$ and $\lambda_2 = 2$, and the singular values are $\sigma_1 = \sqrt{32} = 4\sqrt{2}$ and $\sigma_2 = \sqrt{2}$.

Next, we find the right singular vectors (i.e. the columns of V). Finding $\text{null}(A^\top A - \lambda_1 I)$ and $\text{null}(A^\top A - \lambda_2 I)$ will give us $\vec{v}_1$ and $\vec{v}_2$ respectively.

Hence, $\vec{v}_1 = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$ and $\vec{v}_2 = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$ (the eigenvectors are already normalized here).

Lastly, we find the right singular vectors (the columns of U)

$$\vec{u}_1 = \frac{1}{\sigma_1} A v_1 \tag{67}$$

$$= \frac{1}{4\sqrt{2}} \begin{bmatrix} -1 & 4 \\ 1 & 4 \end{bmatrix} \begin{bmatrix} 0 \\ 1 \end{bmatrix} \tag{68}$$

$$= \begin{bmatrix} \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} \end{bmatrix} \tag{69}$$

Similarly, we get $\vec{u}_2 = \begin{bmatrix} -\frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} \end{bmatrix}$.

So the full SVD representation of $A$ is given below

$$A = \begin{bmatrix} \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{bmatrix} \begin{bmatrix} 4\sqrt{2} & 0 \\ 0 & \sqrt{2} \end{bmatrix} \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \tag{70}$$

Now that we have found the SVD of $A$, we will find the eigenvalues and eigenvectors of $A$. Let us start with the eigenvalues:

$$\det(A - \lambda I) = (-1 - \lambda)(4 - \lambda) - 4 = \lambda^2 - 3\lambda - 8 = 0 \tag{71}$$

Using the quadratic formula, the eigenvalues are $\lambda_1 = \frac{3+\sqrt{41}}{2} \approx 4.7$ and $\lambda_2 = \frac{3-\sqrt{41}}{2} \approx -1.7$.

Since we already used $\vec{v}_1, \vec{v}_2$ for the SVD, let us denote the eigenvectors of $A$ as $\vec{r}_1, \vec{r}_2$.
Finding $\text{null}(A - \lambda_1 I)$ and $\text{null}(A - \lambda_2 I)$ will give us $\vec{r}_1$ and $\vec{r}_2$ respectively.

 

Hence, the normalized eigenvectors of $A$ are $\vec{r}_1 \approx \begin{bmatrix} -0.98 \\ 0.17 \end{bmatrix}$ and $\vec{r}_2 \approx \begin{bmatrix} -0.57 \\ -0.82 \end{bmatrix}$.

We notice that there is no relationship between the eigenvalues or eigenvectors of $A$ with the SVD of $A$.

**Contributors:**

- Anish Muthali.
- Elena Jia.
- Tanmay Gautam.