Homework 12

This homework is due on Saturday, November 18, 2023 at 11:59PM. Self-grades and HW Resubmissions are due the following Saturday, December 2, 2023 at 11:59PM.

1. **SVD Proofs**

   (a) **Show, for a general matrix** $A \in \mathbb{R}^{m \times n}$ **with** $\operatorname{rank}(A) = r$ **and** $A = U\Sigma V^\top$**, that** $\operatorname{Null}(A) = \operatorname{Col}(V_{n-r})$**.** *(HINT: How do we show two sets are equal? Try and use that approach here. Consider the outer product summation form for the SVD. Also, consider using the rank-nullity theorem that* $\dim \operatorname{Col}(A) + \dim \operatorname{Null}(A) = n$*.)*

   (b) **Show, for a general matrix** $A \in \mathbb{R}^{m \times n}$ **with** $\operatorname{rank}(A) = r$ **and** $A = U\Sigma V^\top$**, that** $\operatorname{Col}(A) = \operatorname{Col}(U_r)$**.**

   (c) **Show, for a general matrix** $A \in \mathbb{R}^{m \times n}$ **with** $\operatorname{rank}(A) = r$ **and** $A = U\Sigma V^\top$**, that** $\operatorname{Null}(A^\top) = \operatorname{Col}(U_{m-r})$ **and** $\operatorname{Col}(A^\top) = \operatorname{Col}(V_r)$**. Then show:**

      i. $\dim \operatorname{Col}(A) + \dim \operatorname{Null}(A^\top) = m$**,**

      ii. and $\operatorname{Col}(A)$ **and** $\operatorname{Null}(A^\top)$ **are orthogonal.**

2. SVD System ID

Previously, we saw instances for how to solve system ID problems when $D \in \mathbb{R}^{m\times n}$ is full rank (for $m > n$). Now, let us consider more generally the following problem of estimating $\vec{p}$ in

$$D\vec{p} = \vec{s} \tag{1}$$

where $\vec{p} \in \mathbb{R}^n$, $\vec{s} \in \mathbb{R}^m$, and $D \in \mathbb{R}^{m\times n}$. We assume that $\text{rank}(D) = r < \min(m, n)$, and we do not make any further assumptions on the relationship between m and n. Let's assume that D has an SVD given by

$$D = U\Sigma V^\top \tag{2}$$

where $U \in \mathbb{R}^{m\times m}$ and $V \in \mathbb{R}^{n\times n}$ are orthonormal matrices. $\Sigma \in \mathbb{R}^{m\times n}$ has the following form:

$$\Sigma = \begin{bmatrix} \Sigma_r & 0_{r\times(n-r)} \\ 0_{(m-r)\times r} & 0_{(m-r)\times(n-r)} \end{bmatrix} \tag{3}$$

where $\Sigma_r = \begin{bmatrix} \sigma_1 & & & \\ & \sigma_2 & & \\ & & \ddots & \\ & & & \sigma_r \end{bmatrix}$ is a $r \times r$ diagonal matrix with nonzero elements along its diagonal.

Using this problem setup, we can rewrite our original system ID problem as

$$U\Sigma V^\top \vec{p} = \vec{s} \tag{4}$$

Our goal is to find $\vec{p}$ with smallest norm that best estimates $\vec{s}$.

For notational convenience, denote $U := \begin{bmatrix} U_r & U_{m-r} \end{bmatrix}$ where $U_r \in \mathbb{R}^{m\times r}$ is a matrix with the first r columns of U and $U_{m-r} \in \mathbb{R}^{m\times(m-r)}$ is a matrix with the last $m - r$ columns of U. Also, denote $V := \begin{bmatrix} V_r & V_{n-r} \end{bmatrix}$ where $V_r \in \mathbb{R}^{n\times r}$ is a matrix that has the first r columns of V, and $V_{n-r} \in \mathbb{R}^{n\times(n-r)}$ is a matrix that has the last $n - r$ columns of V. From SVD properties, we know that the columns of U_r form an orthonormal basis for $\text{Col}(D)$ and that the columns of V_{n-r} form an orthonormal basis for $\text{Null}(D)$.

(a) **Using the fact that U is orthonormal, show that $\Sigma V^\top \vec{p} = U^\top \vec{s}$.**

(b) **Show that we can write $\vec{p} = \begin{bmatrix} V_r & V_{n-r} \end{bmatrix} \begin{bmatrix} \vec{\alpha} \\ \vec{\beta} \end{bmatrix}$ for some vectors $\vec{\alpha}$ and $\vec{\beta}$ (i.e., find $\vec{\alpha} \in \mathbb{R}^r$ and $\vec{\beta} \in \mathbb{R}^{n-r}$). Show that changing $\vec{\beta}$ will not affect the result of $D\vec{p}$ and that we should set $\vec{\beta} = \vec{0}$ if we want to minimize $\|p\|$.** Since $V_{n-r}\vec{\beta}$ is the component of $\vec{p}$ that is in the nullspace of D, we can set $\vec{\beta}$ to be whatever we want. To choose the $\vec{p}$ with smallest norm, we will set $\vec{\beta} = \vec{0}$.

(c) From the previous part, we can rewrite $\vec{p} = V \begin{bmatrix} \vec{\alpha} \\ \vec{0} \end{bmatrix}$. This simplifies our system ID problem as follows:

$$\Sigma V^\top V \begin{bmatrix} \vec{\alpha} \\ \vec{0} \end{bmatrix} = U^\top \vec{s} \tag{5}$$

$$\Sigma \begin{bmatrix} \vec{\alpha} \\ \vec{0} \end{bmatrix} = U^\top \vec{s} \tag{6}$$

Simplify the left hand side of eq. (6) using eq. (3). Rewrite $U^\top \vec{s}$ **as** $\begin{bmatrix} U_r^\top \vec{s} \\ U_{m-r}^\top \vec{s} \end{bmatrix}$ **and find an expression for** $\vec{\alpha}$**.** *(HINT: Block matrix multiplication will work like normal matrix-vector multiplication here since* $\Sigma_r \in \mathbb{R}^{r \times r}$ *and* $\vec{\alpha} \in \mathbb{R}^r$*.)*

(d) **Use the previous part to come up with a solution for** $\vec{p}$**.**

(e) From the concept of projections, we know that the optimal solution for $\vec{p}$ satisfies the property that the projection error, namely $\vec{s} - D\vec{p}$, is orthogonal to the projection itself, namely $D\vec{p}$. Write $\vec{s} := \begin{bmatrix} U_r & U_{m-r} \end{bmatrix} \begin{bmatrix} \vec{w} \\ \vec{z} \end{bmatrix}$ for some vectors $\vec{w} \in \mathbb{R}^r$ and $\vec{z} \in \mathbb{R}^{m-r}$. **Find** $\vec{w}$ **and** $\vec{z}$**. Using this, show that our solution for** $\vec{p}$ **is optimal.**

3. SVD

(a) Consider the matrix

$$A = \begin{bmatrix} -1 & 1 & 5 \\ 3 & 1 & -1 \\ 2 & -1 & 4 \end{bmatrix}.$$

Observe that the columns of matrix A are mutually orthogonal with norms $\sqrt{14}$, $\sqrt{3}$, $\sqrt{42}$.

Verify numerically that columns $\begin{bmatrix} 1 \\ 1 \\ -1 \end{bmatrix}$ **and** $\begin{bmatrix} 5 \\ -1 \\ 4 \end{bmatrix}$ **are orthogonal to each other.**

(b) **Write $A = BD$, where B is an orthonormal matrix and D is a diagonal matrix.** What is B? What is D?

(c) **Write out a singular value decomposition of $A = U\Sigma V^\top$ using the previous part.** Note the ordering of the singular values in Σ should be from the largest to smallest. *(HINT: There is no need to compute the eigenvalues of anything. Use Theorem 14, Note 14.)*

(d) Given the matrix

$$A = \frac{1}{\sqrt{50}} \begin{bmatrix} 3 \\ 4 \end{bmatrix} \begin{bmatrix} 1 & -1 \end{bmatrix} + \frac{3}{\sqrt{50}} \begin{bmatrix} -4 \\ 3 \end{bmatrix} \begin{bmatrix} 1 & 1 \end{bmatrix}, \tag{7}$$

write out a singular value decomposition of matrix A in the form $U\Sigma V^\top$. Note the ordering of the singular values in Σ should be from the largest to smallest. *(HINT: You don't have to compute any eigenvalues for this. Some useful observations are that*

$$\begin{bmatrix} 3, 4 \end{bmatrix} \begin{bmatrix} -4 \\ 3 \end{bmatrix} = 0, \quad \begin{bmatrix} 1, -1 \end{bmatrix} \begin{bmatrix} 1 \\ 1 \end{bmatrix} = 0, \quad \| \begin{bmatrix} 3 \\ 4 \end{bmatrix} \| = \| \begin{bmatrix} -4 \\ 3 \end{bmatrix} \| = 5, \quad \| \begin{bmatrix} 1 \\ -1 \end{bmatrix} \| = \| \begin{bmatrix} 1 \\ 1 \end{bmatrix} \| = \sqrt{2}.$$

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(e) Define the matrix

$$A = \begin{bmatrix} -1 & 4 \\ 1 & 4 \end{bmatrix}.$$

Find the SVD of A by following the standard algorithm introduced in Note 14, i.e. by computing the eigendecomposition of $A^\top A$. Also find the eigenvectors and eigenvalues of A. Is there a relationship between the eigenvalues or eigenvectors of A with the SVD of A?

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