# Homework 11

**This homework is due on Saturday, November 11, 2023 at 11:59PM. Self-grades and HW Resubmissions are due the following Saturday, November 18, 2023 at 11:59PM.**

1. **Spectral Theorem for Real Symmetric Matrices**

We want to show that every real symmetric matrix can be diagonalized by a matrix of its orthonormal eigenvectors. In other words, a symmetric matrix $S \in \mathbb{R}^{n\times n}$, i.e., a matrix $S$ such that $S = S^\top$, can be written as $S = V\Lambda V^\top$, where $V \in \mathbb{R}^{n\times n}$ is an orthonormal matrix of eigenvectors of $S$ and $\Lambda \in \mathbb{R}^{n\times n}$ is a diagonal matrix of corresponding real eigenvalues of $S$. This is called the Spectral Theorem for real symmetric matrices.

To do this, we will use a proof which is similar to the proof of existence of the Schur decomposition. Along the way, we will practice block matrix manipulation and the induction proof technique.

(a) One part of the spectral theorem can be proved without any further delay. **Prove that the eigenvalues $\lambda$ of a real, symmetric matrix $S$ are real.**

*(HINT: Let $\lambda$ be an eigenvalue of $S$ with corresponding nonzero eigenvector $\vec{v}$. Evaluate $\overline{\vec{v}}^\top S\vec{v}$ in two different ways: $\overline{\vec{v}}^\top (S\vec{v})$ and $(\overline{\vec{v}}^\top S)\vec{v}$. What does this show about $\lambda$?)*

(b) For the main proof that every real symmetric matrix is diagonalized by a matrix of its orthonormal real eigenvectors, we will proceed by *induction*.

Recall that an inductive proof trying to prove a statement that depends on $n$, say $P_n$[1], is true for all positive integers $n$, has two steps:

- A base case – prove that $P_1$ is true.
- An inductive step – for every $n \geq 2$, given that $P_{n-1}$ is true, prove that $P_n$ is true.[2]

By doing these two steps, we show $P_n$ is true for all $n$.

In our case, the statement $P_n$ is "every $n \times n$ symmetric matrix $S$ can be diagonalized as $S = V\Lambda V^\top$, where $V$ is the real orthonormal matrix of eigenvectors of $S$, and $\Lambda$ is the real diagonal matrix of corresponding eigenvalues of $S$."

**Show the base case: every $1 \times 1$ symmetric matrix $S$ can be written as $S = V\Lambda V^\top$, where $V$ is a real and orthonormal matrix of eigenvectors of $S$, and $\Lambda$ is a real and diagonal matrix of corresponding eigenvalues of $S$.**

*(HINT: Every $1 \times 1$ matrix is symmetric, and also diagonal, by definition; the only real orthonormal $1 \times 1$ matrices are $\begin{bmatrix}1\end{bmatrix}$ and $\begin{bmatrix}-1\end{bmatrix}$.)*

[1]Lecture used $S_n$, but $S$ is already being used for symmetric matrix here.

[2]This is the so-called *weak induction* paradigm; it contrasts with *strong induction*, which you can learn in future classes like CS70.

(c) With the base case done, we are now in the inductive step. Let $S$ be an arbitrary $n \times n$ symmetric matrix; ultimately, we want to show that $S = V\Lambda V^\top$, where $V$ is a real and orthonormal matrix of eigenvectors of $S$, and $\Lambda$ is a real and diagonal matrix of corresponding eigenvalues of $S$.

To start, let $\lambda$ be an eigenvalue of $S$, and let $\vec{q}$ be any normalized eigenvector of $S$ corresponding to eigenvalue $\lambda$. Let $\widetilde{Q} \in \mathbb{R}^{n\times(n-1)}$ be a set of orthonormal vectors chosen so that $Q := \begin{bmatrix} \vec{q} & \widetilde{Q} \end{bmatrix} \in \mathbb{R}^{n\times n}$ is an orthonormal matrix.[3] **Show the following equality:**

$$Q^\top S Q = \begin{bmatrix} \lambda & \vec{0}_{n-1}^\top \\ \vec{0}_{n-1} & S_0 \end{bmatrix} \qquad \textbf{where} \qquad S_0 := \widetilde{Q}^\top S \widetilde{Q}. \tag{1}$$

*(HINT: Expand $Q$ as a block matrix $\begin{bmatrix} \vec{q} & \widetilde{Q} \end{bmatrix}$ and multiply $Q^\top S Q = \begin{bmatrix} \vec{q} & \widetilde{Q} \end{bmatrix}^\top S \begin{bmatrix} \vec{q} & \widetilde{Q} \end{bmatrix}$.)*
*(HINT: Since $Q$ is orthonormal, we have $Q^\top Q = I_n$. What does this mean for the values of $\vec{q}^\top \vec{q}$ and $\widetilde{Q}^\top \vec{q}$? Use block matrix multiplication on $Q^\top Q = \begin{bmatrix} \vec{q} & \widetilde{Q} \end{bmatrix}^\top \begin{bmatrix} \vec{q} & \widetilde{Q} \end{bmatrix}$ again.)*

(d) **Show that the matrix $S_0$ is a real symmetric matrix.**

(e) Since $S_0$ is a real symmetric $(n-1)\times(n-1)$ matrix, by our inductive assumption, $S_0$ can be orthonormally diagonalized as $S_0 = V_0 \Lambda_0 V_0^\top$, where $\Lambda_0$ is a real diagonal matrix of eigenvalues of $S_0$ and $V_0 \in \mathbb{R}^{(n-1)\times(n-1)}$ is a real orthonormal matrix of corresponding eigenvectors of $S_0$.

Define

$$V := Q \begin{bmatrix} 1 & \vec{0}_{n-1}^\top \\ \vec{0}_{n-1} & V_0 \end{bmatrix} \qquad \textbf{and} \qquad \Lambda := V^\top S V. \tag{2}$$

i. **Show that $V$ is orthonormal.**

ii. **Show that $\Lambda$ is diagonal.**

iii. **Show that $S = V\Lambda V^\top$.**

*(HINT: Use block matrix multiplication again.)*

Thus, we have found a real orthonormal $V$ and real diagonal $\Lambda$ such that $S = V\Lambda V^\top = V\Lambda V^{-1}$. We have seen in a previous homework that if $A = V\Lambda V^{-1}$, then $\Lambda$ are the eigenvalues of $A$, and $V$ are the corresponding eigenvectors. Thus, given $P_{n-1}$ – the fact that we can orthonormally diagonalize $(n-1)\times(n-1)$ real symmetric matrices – we have proven $P_n$ – the fact that we can orthonormally diagonalize $n \times n$ real symmetric matrices. Thus, we've proved the Spectral Theorem for real symmetric matrices by induction!

[3] This matrix $\widetilde{Q}$ can be generated via Gram-Schmidt, for example.

 

**2. SVD and the fundamental subspaces**

Consider a matrix $A \in \mathbb{R}^{m\times n}$ with $\text{rank}(A) = r$. The compact SVD of A is given by $A = U_r\Sigma_r V_r^\top$ where

$$U_r = \begin{bmatrix}\vec{u}_1 \cdots \vec{u}_r\end{bmatrix} \in \mathbb{R}^{m\times r}, \quad \Sigma_r = \begin{bmatrix}\sigma_1 & & \\ & \ddots & \\ & & \sigma_r\end{bmatrix} \in \mathbb{R}^{r\times r}, \quad V_r = \begin{bmatrix}\vec{v}_1 \cdots \vec{v}_r\end{bmatrix} \in \mathbb{R}^{n\times r}$$

with $\sigma_1 \geq \cdots \geq \sigma_r > 0$ being the singular values of $A$.

(a) **Which one of the following sets is always guaranteed to form an orthonormal basis for** $\text{Col}(A)$**?**
*(Please fill in one of the circles for the options below. You will only be graded on your final answer.)*

i. $\{\vec{u}_1, \cdots, \vec{u}_r\}$
ii. $\{\sigma_1\vec{u}_1, \ldots, \sigma_r\vec{u}_r\}$
iii. $\{\vec{v}_1, \ldots, \vec{v}_r\}$
iv. $\{\sigma_1\vec{v}_1, \ldots, \sigma_r\vec{v}_r\}$

| Option | i | ii | iii | iv |
|---|---|---|---|---|
| Answer | ○ | ○ | ○ | ○ |

(b) **Which one of the following sets is always guaranteed to form an orthonormal basis for** $\text{Col}(A^\top)$**?**
*(Please fill in one of the circles for the options below. You will only be graded on your final answer.)*

i. $\{\vec{u}_1, \cdots, \vec{u}_r\}$
ii. $\{\sigma_1\vec{u}_1, \ldots, \sigma_r\vec{u}_r\}$
iii. $\{\vec{v}_1, \ldots, \vec{v}_r\}$
iv. $\{\sigma_1\vec{v}_1, \ldots, \sigma_r\vec{v}_r\}$

| Option | i | ii | iii | iv |
|---|---|---|---|---|
| Answer | ○ | ○ | ○ | ○ |

Now suppose that the considered $A$ matrix has the following compact SVD components:

$$U_r = \begin{bmatrix}0 & \frac{1}{\sqrt{2}} \\ 1 & 0 \\ 0 & \frac{1}{\sqrt{2}}\end{bmatrix}, \quad \Sigma_r = \begin{bmatrix}2 & 0 \\ 0 & 1\end{bmatrix}, \quad V_r = \begin{bmatrix}1 & 0 \\ 0 & 1 \\ 0 & 0\end{bmatrix}.$$

(c) **Using the given compact SVD, state** $\alpha$**, where** $\alpha$ **is the tightest upper bound** $\|A\vec{x}\| \leq \alpha$ **for any** $\vec{x}$ **such that** $\|\vec{x}\| \leq 1$**.**

(d) **Given the compact SVD, which of the following provides a valid full SVD for** $A = U\Sigma V^\top$**?**
*(Please fill in one of the circles for the options below. You will only be graded on your final answer.)*

i. $U = \begin{bmatrix}0 & \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ 1 & 0 & 0 \\ 0 & \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}}\end{bmatrix}, \quad \Sigma = \begin{bmatrix}1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 0\end{bmatrix}, \quad V = \begin{bmatrix}1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1\end{bmatrix}$

ii. $U = \begin{bmatrix} 0 & \frac{1}{\sqrt{2}} & 0 \\ 1 & 0 & -1 \\ 0 & \frac{1}{\sqrt{2}} & 0 \end{bmatrix}, \quad \Sigma = \begin{bmatrix} 2 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \quad V = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$

iii. $U = \begin{bmatrix} 0 & \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ 1 & 0 & 0 \\ 0 & \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} \end{bmatrix}, \quad \Sigma = \begin{bmatrix} 2 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \quad V = \begin{bmatrix} 1 & 0 & -1 \\ 0 & 1 & 1 \\ 0 & 0 & 0 \end{bmatrix}$

iv. $U = \begin{bmatrix} 0 & \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ 1 & 0 & 0 \\ 0 & \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} \end{bmatrix}, \quad \Sigma = \begin{bmatrix} 2 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \quad V = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$

| Option | i | ii | iii | iv |
|---|---|---|---|---|
| Answer | ◯ | ◯ | ◯ | ◯ |

**3. SVD of a matrix with orthogonal columns**

Let $A = \begin{bmatrix} \vec{a}_1 & \cdots & \vec{a}_n \end{bmatrix} \in \mathbb{R}^{m \times n}$ where $\vec{a}_i^\top \vec{a}_j = 0$ for all $1 \leq i, j \leq n$ such that $i \neq j$, and $\vec{a}_i^\top \vec{a}_i \neq 0$ for all $i = 1, \ldots, n$. **What is the set of singular values of** $A$ ***for all such matrices*** $A$**?**

*(Please fill in one of the circles for the options below. You will only be graded on your final answer.)*

(a) $\{0\}$ (all zero)

(b) $\{\sqrt{\|\vec{a}_1\|}, \ldots, \sqrt{\|\vec{a}_n\|}\}$

(c) $\{\|\vec{a}_1\|, \ldots, \|\vec{a}_n\|\}$

(d) $\{\|\vec{a}_1\|^2, \ldots, \|\vec{a}_n\|^2\}$

(e) $\{1\}$ (all one)

| Option | a | b | c | d | e |
|---|---|---|---|---|---|
| Answer | ◯ | ◯ | ◯ | ◯ | ◯ |

**4. Frobenius Norm**

In this problem we will investigate the basic properties of the Frobenius norm.

Similar to how the norm of vector $\vec{x} \in \mathbb{R}^n$ is defined as $\|x\| = \sqrt{\sum_{i=1}^n x_i^2}$, the Frobenius norm of a matrix $A \in \mathbb{R}^{m \times n}$ is defined as

$$\|A\|_F = \sqrt{\sum_{i=1}^m \sum_{j=1}^n |A_{ij}|^2}. \tag{3}$$

$A_{ij}$ is the entry in the $i^{\text{th}}$ row and the $j^{\text{th}}$ column. This is basically the norm that comes from treating a matrix like a big vector filled with numbers.

(a) With the above definitions, **show that for a** $2 \times 2$ **matrix** $A$**:**

$$\|A\|_F = \sqrt{\operatorname{tr}(A^\top A)}. \tag{4}$$

*Note:* The trace of a matrix is the sum of its diagonal entries. For example, let $A \in \mathbb{R}^{m \times n}$, then,

$$\operatorname{tr}(A) = \sum_{i=1}^{\min(n,m)} A_{ii} \tag{5}$$

Think about how/whether this expression eq. (4) generalizes to general $m \times n$ matrices.

(b) **Show for any matrix** $A \in \mathbb{R}^{m \times n}$**:**

$$\|A\|_F = \left\|A^\top\right\|_F \tag{6}$$

A purely written or mathematical solution will be sufficient for this problem.

*(HINT: For the mathematical solution, use the trace interpretation from eq. (3).)*

(c) **Show that if** $U$ **and** $V$ **are square orthonormal matrices, then**

$$\|UA\|_F = \|AV\|_F = \|A\|_F. \tag{7}$$

*(HINT: Use the trace interpretation from part (a) and the equation from part (b).)*

(d) **Use the SVD decomposition to show that** $\|A\|_F = \sqrt{\sum_{i=1}^n \sigma_i^2}$**, where** $\sigma_1, \ldots, \sigma_n$ **are the singular values of** $A$**.**

*(HINT: The previous part might be quite useful.)*

(e) **Show that for any matrix** $A \in \mathbb{R}^{m \times n}$ **and any vector** $\vec{x} \in \mathbb{R}^n$:

$$\|A\vec{x}\|^2 \leq \|A\|_F^2 \|\vec{x}\|^2 \tag{8}$$

*(HINT: Use the summation form of matrix multiplication to find an expression for each element of $A\vec{x}$ and use this to find the expression for $\|A\vec{x}\|^2$. Then, use the fact that $|\sum ab|^2 \leq \left(\sum |a|^2\right)\left(|\sum |b|^2\right)$ (called the Cauchy-Schwarz inequality).)*

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