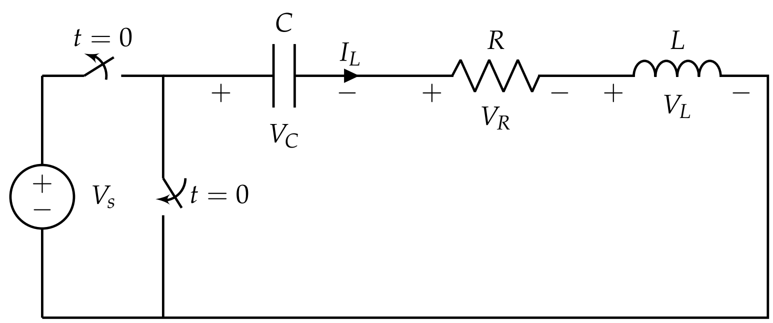
Homework 7

This homework is due on Saturday, October 14, 2023, at 11:59PM. Self-grades and HW Resubmissions are due on Saturday, October 21, 2023, at 11:59PM.

1. **Alternative "second order" perspective on solving the RLC circuit**

In Homework 6, we solved an RLC circuit by setting state variables $x_1(t) = V_C(t)$ and $x_2(t) = I_L(t)$, and using these to build a linear first-order vector differential equation. In this problem, we will see how to solve the same system by picking *different* state variables $x_1(t) = V_C(t)$ and $x_2(t) = \frac{\mathrm{d}}{\mathrm{d}t}V_C(t)$, getting a linear *second order scalar* differential equation, and solving that differential equation.

Consider the following circuit like you saw in lecture, discussion, and previous homeworks:



As before, assume that the system has reached steady-state for $t < 0$. At time $t = 0$, the switch changes state and disconnects the voltage source, replacing it with a short.

Suppose now we insisted on expressing everything in terms of one waveform $V_C(t)$ instead of two of them (voltage across the capacitor and current through the inductor). This is called the "second-order" point of view, because we will end up using second derivatives.

For this problem, use R for the resistor, L for the inductor, and C for the capacitor in all the expressions.

(a) **Write the current $I_L(t)$ through the inductor in terms of the voltage $V_C(t)$ across the capacitor.**

(b) Now, notice that the voltage drop across the inductor involves $\frac{\mathrm{d}}{\mathrm{d}t}I_L(t)$. **Write the voltage drop across the inductor, $V_L(t)$, in terms of the second derivative of $V_C(t)$.**

(c) **Show that a differential equation governing $V_C(t)$ is**

$$\frac{\mathrm{d}^2}{\mathrm{d}t^2}V_C(t) + \frac{R}{L}\frac{\mathrm{d}}{\mathrm{d}t}V_C(t) + \frac{1}{LC}V_C(t) = 0. \tag{1}$$

(d) Recall that previously in class, we solved a second-order differential equation of the form

$$\frac{\mathrm{d}^2 y(t)}{\mathrm{d}t^2} + a\frac{\mathrm{d}y(t)}{\mathrm{d}t} + by(t) = 0 \tag{2}$$

by converting it into a matrix differential equation

$$\frac{\mathrm{d}}{\mathrm{d}t}\begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix} = \underbrace{\begin{bmatrix} 0 & 1 \\ -b & -a \end{bmatrix}}_{A}\begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix} \tag{3}$$

where $x_1(t) := y(t)$ and $x_2(t) := \frac{\mathrm{d}y(t)}{\mathrm{d}t}$. It turned out that, if A has two distinct eigenvalues, the solution to this homogeneous differential equation have the form

$$\begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix} = \begin{bmatrix} c_1 \mathrm{e}^{\lambda_1 t} + c_2 \mathrm{e}^{\lambda_2 t} \\ c_3 \mathrm{e}^{\lambda_1 t} + c_4 \mathrm{e}^{\lambda_2 t} \end{bmatrix}. \tag{4}$$

where λ_1, λ_2 are the eigenvalues of A, and c_1, c_2, c_3, c_4 are constants determined by the initial conditions and the coefficients a, b in the differential equation. We would like to use this to construct a solution for $V_C(t)$.

Show that, if we identify $y(t) := V_C(t)$, then

$$x_1(t) = V_C(t) \qquad x_2(t) = \frac{\mathrm{d}}{\mathrm{d}t}V_C(t), \tag{5}$$

and that the matrix A for our RLC circuit is

$$A = \begin{bmatrix} 0 & 1 \\ -\frac{1}{LC} & -\frac{R}{L} \end{bmatrix}. \tag{6}$$

Then, show that the two eigenvalues of A are

$$\lambda_1 = -\frac{R}{2L} + \frac{1}{2}\sqrt{\frac{R^2}{L^2} - \frac{4}{LC}}, \qquad \lambda_2 = -\frac{R}{2L} - \frac{1}{2}\sqrt{\frac{R^2}{L^2} - \frac{4}{LC}}. \tag{7}$$

NOTE: From this part onwards, we will assume that the circuit parameters R, L, C are chosen so that the eigenvalues of A are distinct.

(e) Now, we solve for $x_1(t) = V_c(t)$ by determining c_1 and c_2 and plugging those, along with λ_1 and λ_2, into eq. (4). Note that determining c_3 and c_4 isn't necessary to find $x_1(t)$, but we need them to set up a system of equations to solve for c_1 and c_2. **Show that**

$$c_3 = \lambda_1 c_1, \qquad c_4 = \lambda_2 c_2. \tag{8}$$

Then use the initial conditions of the RLC circuit to show that

$$c_1 = \frac{\lambda_2}{\lambda_2 - \lambda_1}V_s, \qquad c_2 = -\frac{\lambda_1}{\lambda_2 - \lambda_1}V_s. \tag{9}$$

(HINT: This part is complicated, so here's some scaffolding to start you off. First, differentiate the expression we have $x_1(t) = c_1\mathrm{e}^{\lambda_1 t} + c_2\mathrm{e}^{\lambda_2 t}$ to get a form for $x_2(t)$, and match coefficients of $\mathrm{e}^{\lambda_1 t}$ and $\mathrm{e}^{\lambda_2 t}$ to get the desired expressions for c_3 and c_4. Next, use the initial conditions for RLC to see what $V_C(0)$ are and $\frac{\mathrm{d}}{\mathrm{d}t}V_C(t)\Big|_{t=0}$ are. This corresponds to $x_1(0)$ and $x_2(0)$. Plug $t = 0$ into the "sum of exponentials" form for x_1 and x_2. This will get you two equations, one for each x_i, for c_1 and c_2, which you can then solve.)

(HINT: The following matrix inverse formula may be useful:

$$\begin{bmatrix} 1 & 1 \\ \lambda_1 & \lambda_2 \end{bmatrix}^{-1} = \frac{1}{\lambda_2 - \lambda_1}\begin{bmatrix} \lambda_2 & -1 \\ -\lambda_1 & 1 \end{bmatrix} \tag{10}$$

)

We have found $\lambda_1, \lambda_2, c_1, c_2$, so by substituting into eq. (4) we have solved for $x_1(t) = V_C(t)$!

2. **Solving the Differential Equation with Input**

Recall that in Discussion 2A we tried to solve the differential equation with input:

$$\frac{\mathrm{d}}{\mathrm{d}t}x(t) = \lambda x(t) + bu_c(t) \tag{11}$$

$$x(0) = x_0. \tag{12}$$

for some continuous input $u_c(t)$.

The general strategy we employ is:

- First we replace our continuous input $u_c(t)$ with an input $u(t)$ which is piecewise constant on the intervals $[i\Delta, (i+1)\Delta)$, that is,

$$u(t) = u(i\Delta) = u[i] \qquad t \in [i\Delta, (i+1)\Delta) \qquad i \in \{0, 1, 2, \dots\} := \mathbb{N}. \tag{13}$$

  Using this assumption, in discussion we:

  – solved the differential equation on each interval $[i\Delta, (i+1)\Delta)$ and got a solution expressing $x(t)$ in terms of $x_d[i] := x(i\Delta)$ and $u[i]$, for $t \in [i\Delta, (i+1)\Delta)$;
  – arrived at a formula for $x_d[i+1]$ in terms of $x_d[i]$ and $u[i]$;
  – used this to get a formula for $x_d[i]$ in terms of x_0 and the inputs $u[0], u[1], \dots, u[i-1]$;
  – approximated $x(t) \approx x_d[\lfloor \frac{t}{\Delta} \rfloor]$ to recover an approximate value for $x(t)$, that is,

$$x(t) \approx \left(\mathrm{e}^{\lambda\Delta}\right)^{\lfloor \frac{t}{\Delta} \rfloor} x_0 + b\frac{\mathrm{e}^{\lambda\Delta} - 1}{\lambda} \sum_{k=0}^{\lfloor \frac{t}{\Delta} \rfloor - 1} \left(\mathrm{e}^{\lambda\Delta}\right)^{(\lfloor \frac{t}{\Delta} \rfloor - 1) - k} u[k]. \tag{14}$$

- In this homework, we will take the limit $\Delta \to 0$. This transfers back from u to u_c – we saw in discussion that piecewise constant functions on very small intervals, i.e., our u, approximate general continuous functions u_c arbitrarily well. Using Riemann sums and calculus, we will turn the sum into an integral and show that, if u approximates u_c as $\Delta \to 0$, then

$$x(t) = \mathrm{e}^{\lambda t} x_0 + b \int_0^t \mathrm{e}^{\lambda(t-\tau)} u_c(\tau)\,\mathrm{d}\tau\,. \tag{15}$$

(a) We first need to relate $u[i]$ to u_c. Suppose that the $u[i]$ is a sample of $u_c(t)$, namely,

$$u[i] = u_c(i\Delta). \tag{16}$$

To clarify where this fits in with the earlier notation:

- $u(t)$ is a piecewise constant function;
- $u[i]$ is the discrete input that constructs $u(t)$ based on eq. (13);
- and $u_c(t)$ is the underlying input $u[i]$ is sampled from based on eq. (16).

This is one good way to get a piecewise constant approximator of a continuous function.

Substitute an appropriate value of u_c for $u[k]$ in eq. (14) from the discussion.

NOTE: Don't take any limits in this part of the problem; just do the substitution.

(b) To simplify our (discrete-time) eq. (14) so we can take $\Delta \to 0$, we would like to make some approximations which are valid for small Δ.

By using the following two estimates for small Δ:[1]

[1] Both these approximations become equalities in the limit $\Delta \to 0$.

i. $\lfloor \frac{t}{\Delta} \rfloor \approx \frac{t}{\Delta}$;
ii. $\frac{e^{\lambda\Delta}-1}{\lambda} \approx \Delta$;[2]

show that

$$x(t) \approx e^{\lambda t} x_0 + b e^{-\lambda\Delta} \sum_{k=0}^{\frac{t}{\Delta}-1} e^{\lambda(t-k\Delta)} u_c(k\Delta)\Delta. \tag{17}$$

(c) **Take the limit of $x(t)$ as $\Delta \to 0$, and show that $x(t)$ is given by eq. (15).**

Recall that the definite integral is defined from Riemann sums as

$$\int_0^t f(\tau)\,\mathrm{d}\tau = \lim_{n\to\infty} \sum_{k=0}^{n-1} f(\tau_k^*)\Delta_k \tag{18}$$

where $0 = \tau_0 < \tau_1 < \cdots < \tau_n = t$, $\tau_k^* \in [\tau_k, \tau_{k+1}]$, and $\Delta_k = \tau_{k+1} - \tau_k$. The Δ_k is the length of the base of the rectangles and the $f(\tau_k^*)$ are the heights. As n goes to infinity, the rectangles get skinnier and skinnier, but there are more and more of them.

(HINT: Start with eq. (17) and take limits on both sides. What is n? What is τ_k and τ_k^*? What is Δ_k? What is f?)

(HINT: We chose the form of eq. (17) carefully; it turns out that Δ_k is one particular term involving Δ that goes to 0 as $\Delta \to 0$, and also that it is independent of k.)

This problem brings together many different concepts and uses a lot of notation. As such, it may be difficult to fully comprehend everything the first time. Being able to grind through complex mathematical problems like this is part of the vaunted "mathematical maturity" that this class helps you foster. As the semester continues, you will find that these kinds of problems will seem progressively easier, both to understand quickly and to solve. But it won't happen without practice.

Contributors:

- Kuan-Yun Lee.
- Anant Sahai.
- Druv Pai.
- Nikhil Shinde.
- Sanjit Batra.
- Aditya Arun.
- Sidney Buchbinder.

[2] We can see this approximation using Taylor's theorem from calculus.

