Homework 6

This homework is due on Saturday, October 7, 2023, at 11:59PM. Self-grades and HW Resubmissions are due on the following Saturday, October 14, 2022, at 11:59PM.

We provide you with the first 3 problems to help review some linear algebra pre-requisites from EECS16A. The remaining problems are specific to EECS16B.

1. Least Squares

(a) Consider the system of equations $\vec{a}x = \vec{b}$ where $\vec{a}, \vec{b} \in \mathbb{R}^2$ and $x \in \mathbb{R}$.

i. When applying least squares, we want to find the $\vec{v} \in \text{Span}(\vec{a})$ that is closest to $\vec{b}$ in Euclidean distance.

(HINT: It might be helpful to draw the vectors.)

(A) Projecting $\vec{b}$ onto $\vec{a}$

(B) Projecting $\vec{a}$ onto $\vec{b}$

(C) Subtracting $\vec{b}$ from $\vec{a}$

(D) Subtracting $\vec{a}$ from $\vec{b}$

(E) None of the above

Solution: Projecting $\vec{b}$ onto $\vec{a}$.

When we are finding $\vec{v}$, or the best approximation of $\vec{b}$ in the span of $\vec{a}$, we project $\vec{b}$ onto $\vec{a}$.

ii. The vector $\vec{v}$ can also be determined by minimizing the length of the error vector, represented as

(A) $\vec{v} = \text{argmin}_{\vec{b}} \left\|\vec{a} - \vec{b}\right\|$

(B) $\vec{v} = \text{argmin}_{\vec{v}} \|\vec{a} - \vec{v}\|$

(C) $\vec{v} = \text{argmin}_{\vec{b}} \left\|\vec{b} - \vec{v}\right\|$

(D) $\vec{v} = \text{argmin}_{\vec{v}} \left\|\vec{b} - \vec{v}\right\|$

Solution: $\vec{v} = \text{argmin}_{\vec{v}} \left\|\vec{b} - \vec{v}\right\|$.

In the least squares problem, we minimize the length of the error vector, $\vec{e}$, defined as the difference between the known vector $\vec{b}$ and the span of possible vectors $\vec{a}x = \vec{v}$. Thus the error vector is $\vec{e} = \vec{b} - \vec{v}$. And the vector $\vec{v}$ is the minimization argument.

(b) For the following systems of $A\vec{x} = \vec{b}$, determine if they have a unique least squares solution.

i. $A = \begin{bmatrix} 1 & 1 \\ 3 & 4 \\ 0 & 0 \end{bmatrix}, \quad \vec{b} = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}$

(A) Yes

(B) No

Solution: Yes. There is a unique least squares solution since A has linearly independent columns.

ii. $A = \begin{bmatrix} 1 & 4 \\ 3 & 12 \\ 2 & 8 \end{bmatrix}, \quad \vec{b} = \begin{bmatrix} 2 \\ 5 \\ 6 \end{bmatrix}$

(A) Yes

(B) No

Solution: No. There is not a unique least squares solution as A does not have linearly independent columns.

(c) For the following three questions, consider the system of $A\vec{x} = \vec{b}$ with $A = \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix}$ and $\vec{b} = \begin{bmatrix} 3 \\ 5 \end{bmatrix}$

i. Can we apply the least squares formula?

(A) Yes

(B) No

Solution: No. The fat matrix A does not have linearly independent columns. Additionally, $A^\top A$ is not invertible since its determinant is zero.

ii. What is the determinant of $A^\top A$? **Solution:**

$$A^\top A = \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 1 \end{bmatrix} \tag{1}$$

$$\det\left(A^\top A\right) = \det\left(\begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 1 \end{bmatrix}\right) = 0 \tag{2}$$

The zero determinant can be inspected since $A^\top A$ is not invertible (i.e., not full column rank, not linearly independent columns).

iii. Does $A\vec{x} = \vec{b}$ have zero, one, or more than one solution for $\vec{x}$?

(A) No solutions

(B) One unique solution

(C) More than one solution

Solution: More than one solution. There are less equations (rows) than unknowns (columns).

(d) Find the best approximation $x = \hat{x}$ to this system of equations:

$$a_1 x = b_1 \tag{3}$$

$$a_2 x = b_2 \tag{4}$$

i. Write the problem into $A\vec{x} \approx \vec{b}$ format and solve for $\hat{x}$ using least squares. Choose the correct $\hat{x}$.

(A) $\hat{x} = \dfrac{a_1 b_1 + a_2 b_2}{a_1^2 + a_2^2}$

(B) $\hat{x} = \dfrac{a_1 b_1 - a_2 b_2}{a_1^2 + a_2^2}$

(C) $\hat{x} = \frac{a_1 b_2 + a_2 b_1}{a_1^2 + a_2^2}$

(D) $\hat{x} = \frac{a_1 b_2 - a_2 b_1}{a_1^2 + a_2^2}$

(E) None of the above

Solution:

$$A x = \vec{b} \quad \rightarrow \quad \begin{bmatrix} a_1 \\ a_2 \end{bmatrix} x = \begin{bmatrix} b_1 \\ b_2 \end{bmatrix} \tag{5}$$

$$\hat{x} = \left(A^\top A\right)^{-1} A^\top \vec{b} = \left(\begin{bmatrix} a_1 & a_2 \end{bmatrix} \begin{bmatrix} a_1 \\ a_2 \end{bmatrix} \right)^{-1} \begin{bmatrix} a_1 & a_2 \end{bmatrix} \begin{bmatrix} b_1 \\ b_2 \end{bmatrix} \tag{6}$$

$$= \frac{a_1 b_1 + a_2 b_2}{a_1^2 + a_2^2} \tag{7}$$

ii. Suppose the inner product is defined instead as a non-Euclidean $\langle x, y \rangle = x^\top \begin{bmatrix} 2 & 0 \\ 0 & 1 \end{bmatrix} y$. Which of the following expressions must be true with respect to the minimized least squares error vector, $\vec{e}$?

(A) $\vec{e}^\top A = \vec{0}$

(B) $A^\top \vec{e} = \vec{0}$

(C) $A^\top \begin{bmatrix} 2 & 0 \\ 0 & 1 \end{bmatrix} \vec{e} = \vec{0}$

(D) $\left(A^\top \begin{bmatrix} 2 & 0 \\ 0 & 1 \end{bmatrix} A \right)^{-1} \vec{e} = \vec{0}$

(E) None of the above

Solution: The least squares error, $\vec{e}$, is minimized when it is orthogonal to every column of A (i.e., $\text{Col}(A)$). Orthogonality occurs when the inner product (in this case the non-Euclidean inner product) of two vectors is zero. Mathematically, $\langle \vec{a}_i, \vec{e} \rangle = 0$ for every column $\vec{a}_i$ of A. Thus, $A^\top \begin{bmatrix} 2 & 0 \\ 0 & 1 \end{bmatrix} \vec{e} = \vec{0}$.

2. Eigenstuff

(a) You are provided the matrix $\mathbf{A} = \begin{bmatrix} 1 & 0.4 & 0.7 \\ 0 & 0.6 & 0.2 \\ 0 & 0 & 0.1 \end{bmatrix}$, and matrix $\mathbf{B} = \begin{bmatrix} 1-\alpha & 0.4 & 0.7 \\ 0 & 0.6-\alpha & 0.2 \\ 0 & 0 & 0.1-\alpha \end{bmatrix}$ where $\alpha \in \mathbb{R}$. If there exists a vector $\vec{x} \in \mathbb{R}^3$ such that $\mathbf{B}\vec{x} = \vec{0}$ and $\vec{x} \neq \vec{0}$, which of the following are true? (**Select all that apply.**)

(A) $\text{rank}(\mathbf{A}) = 3$

(B) $\vec{x}$ is in the null space of $\mathbf{B}$

(C) $\vec{x}$ is in an eigenspace of $\mathbf{B}$

(D) $\vec{x}$ is in an eigenspace of $\mathbf{A}$

Solution:

(A) True. Notice that matrix $\mathbf{A}$ has three pivot columns, so that the rank of $\mathbf{A}$ is 3.

(B) True. $\mathbf{B}\vec{x} = \vec{0}$ follows the definition of the null space.

(C) True. Given the fact that there exists a vector $\vec{x}$ such that $\mathbf{B}\vec{x} = \vec{0}$ and $\vec{x} \neq \vec{0}$, we have $\mathbf{B}\vec{x} = \vec{0} = 0\vec{x}$ and $\vec{x} \neq \vec{0}$. Therefore, $\vec{x}$ is in the eigenspace of $\mathbf{B}$ that associated with eigenvalue $\lambda = 0$.

(D) True. Given $\mathbf{A}$ and $\mathbf{B}$, we have $\mathbf{B} = \begin{bmatrix} 1-\alpha & 0.4 & 0.7 \\ 0 & 0.6-\alpha & 0.2 \\ 0 & 0 & 0.1-\alpha \end{bmatrix} = \begin{bmatrix} 1 & 0.4 & 0.7 \\ 0 & 0.6 & 0.2 \\ 0 & 0 & 0.1 \end{bmatrix} - \begin{bmatrix} \alpha & 0 & 0 \\ 0 & \alpha & 0 \\ 0 & 0 & \alpha \end{bmatrix} = \mathbf{A} - \alpha\mathbf{I}$. Since $\mathbf{B}\vec{x} = (\mathbf{A} - \alpha\mathbf{I})\vec{x} = \vec{0}$, we have $\mathbf{A}\vec{x} = \alpha\vec{x}$. Therefore, $\vec{x}$ is in the eigenspace of $\mathbf{A}$ that associated with eigenvalue $\lambda = \alpha$.

(b) You are given that one of the eigenvalues of $\mathbf{A} = \begin{bmatrix} 1 & 0.4 & 0.7 \\ 0 & 0.6 & 0.2 \\ 0 & 0 & 0.1 \end{bmatrix}$ is $\lambda = 1$. Determine one possible eigenvector $\vec{v}$, corresponding to eigenvalue $\lambda = 1$.

(A) $\vec{v} = \begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix}$

(B) $\vec{v} = \begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix}$

(C) $\vec{v} = \begin{bmatrix} 2 \\ 0 \\ 0 \end{bmatrix}$

(D) $\vec{v} = \begin{bmatrix} 0 \\ -1 \\ 2 \end{bmatrix}$

Solution: Check if each vector $\vec{v}$ is an eigenvector associated with the eigenvalue $\lambda = 1$ by evaluating $A\vec{v} = \lambda\vec{v}$.

(A) $\vec{v}$ is an eigenvector, but associated with the eigenvalue $\lambda = 0.6$ and not the desired eigenvalue of $\lambda = 1$.

$$A\vec{v} = \begin{bmatrix} 1 & 0.4 & 0.7 \\ 0 & 0.6 & 0.2 \\ 0 & 0 & 0.1 \end{bmatrix} \begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix} = \begin{bmatrix} -0.6 \\ 0.6 \\ 0 \end{bmatrix} = 0.6 \cdot \vec{v} \neq 1 \cdot \vec{v} \tag{8}$$

(B) $\vec{v}$ is not an eigenvector.

$$A\vec{v} = \begin{bmatrix} 1 & 0.4 & 0.7 \\ 0 & 0.6 & 0.2 \\ 0 & 0 & 0.1 \end{bmatrix} \begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix} = \begin{bmatrix} 2.5 \\ 1.4 \\ 1 \end{bmatrix} \neq \lambda\vec{v} \tag{9}$$

(C) $\vec{v}$ is an eigenvector associated with the desired eigenvalue $\lambda = 1$.

$$A\vec{v} = \begin{bmatrix} 1 & 0.4 & 0.7 \\ 0 & 0.6 & 0.2 \\ 0 & 0 & 0.1 \end{bmatrix} \begin{bmatrix} 2 \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} 2 \\ 0 \\ 0 \end{bmatrix} = \lambda\vec{v} \tag{10}$$

(D) $\vec{v}$ is not an eigenvector.

$$A\vec{v} = \begin{bmatrix} 1 & 0.4 & 0.7 \\ 0 & 0.6 & 0.2 \\ 0 & 0 & 0.1 \end{bmatrix} \begin{bmatrix} 0 \\ -1 \\ 2 \end{bmatrix} = \begin{bmatrix} 1 \\ -0.2 \\ 2 \end{bmatrix} \neq \lambda\vec{v} \tag{11}$$

(c) Now you are provided a third matrix $\mathbf{C} = \begin{bmatrix} 0.2 & 0.8 & 0.2 \\ 0 & 0.4 & 0.2 \\ 0 & 0 & 0.8 \end{bmatrix}$ with eigenvectors $\vec{v}_1 = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$, $\vec{v}_2 = \begin{bmatrix} 4 \\ 1 \\ 0 \end{bmatrix}$, and $\vec{v}_3 = \begin{bmatrix} 2 \\ 1 \\ 2 \end{bmatrix}$. Matrix $\mathbf{C}$ is transition matrix where $\vec{x}[t+1] = \mathbf{C}\vec{x}[t]$. Additionally, the state vector at timestep $t = 1$ is $\vec{x}[1] = \begin{bmatrix} 8 \\ 1 \\ 4 \end{bmatrix}$. **After infinite timesteps, what is the value of the state vector $\vec{x}[t]$? That is, find $\lim_{t\to\infty} \vec{x}[t]$.** **Solution:** The matrix C is upper triangular, thus the eigenvalues can be determined from inspection as its diagonal elements: $\lambda_1 = 0.2$, $\lambda_2 = 0.4$, and $\lambda_3 = 0.8$. A quick check of $A\vec{v}_i = \lambda_i\vec{v}_i$ for each i assures these eigenvalues are correctly indexed with their corresponding eigenvector.

The initial state vector $\vec{x}[1]$ can be decomposed as a linear combination of the three eigenvectors as $\vec{x}[1] = \alpha\,\vec{v}_1 + \beta\,\vec{v}_2 + \gamma\,\vec{v}_3$ since the eigenvalues are distinct. Then the steady-state value of $\vec{x}[t]$ is evaluated using some eigenvector to eigenvalue simplifications.

$$\lim_{t\to\infty} \vec{x}[t] = \lim_{t\to\infty} C^t\vec{x}[1] = \lim_{t\to\infty} \left(C^t \cdot (\alpha\,\vec{v}_1 + \beta\,\vec{v}_2 + \gamma\,\vec{v}_3)\right) = \lim_{t\to\infty} \left(\alpha\lambda_1^t\vec{v}_1 + \beta\lambda_2^t\vec{v}_2 + \gamma\lambda_3^t\vec{v}_3\right) \tag{12}$$

$$= \lim_{t\to\infty} \left(\alpha\,(0.2)^t\vec{v}_1 + \beta\,(0.4)^t\vec{v}_2 + \gamma\,(0.8)^t\vec{v}_3\right) \tag{13}$$

$$\lim_{t\to\infty} \vec{x}[t] = \vec{0} \tag{14}$$

Finally, since all eigenvalues have magnitude less than one, the value of the state vector as t approaches infinity is $\lim_{t\to\infty} \vec{x}[t] = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$.

3. Orthogonal Space

Let $\vec{v}$ be a vector in $\mathbb{R}^2$, where $\mathbb{R}^2$ has an inner product. We define W to be the set of all vectors orthogonal to $\vec{v}$, i.e.

$$W = \{\vec{w} \mid \langle \vec{v}, \vec{w} \rangle = 0\} \tag{15}$$

(a) In the paragraph below, select the best choice for each blank to **complete the proof showing that W is a subspace**:

First, we need to show that the set contains the zero vector. We see that $\langle \vec{v}, \vec{0} \rangle = 0$, so this condition is fulfilled. Next, we need to show that the set (1)____________. Suppose we have $\vec{x}, \vec{y} \in W$, then (2)____________, so this condition is fulfilled. Finally, we need to show that the set (3)____________. Suppose we have $\alpha \in \mathbb{R}$ and $\vec{x} \in W$, then (4)____________, so this condition is fulfilled. Therefore the set is a valid subspace.

(1) (A) is closed under scalar multiplication
(B) is closed under vector addition
(C) is homogeneous
(D) is non-empty
(E) fulfills superposition

(2) (A) $\langle \vec{v}^\top \vec{x}, \vec{v}^\top \vec{y} \rangle = 0$
(B) $\langle \vec{v}, \vec{x} \rangle = \langle \vec{v}, \vec{y} \rangle$
(C) $\langle \vec{v} + \vec{x}, \vec{y} \rangle = \langle \vec{v}, \vec{x} \rangle + \langle \vec{v}, \vec{y} \rangle = 0$
(D) $\langle \vec{v}, \vec{x} + \vec{y} \rangle = \langle \vec{v}, \vec{x} \rangle + \langle \vec{v}, \vec{y} \rangle = 0$

(3) (A) is closed under scalar multiplication
(B) is closed under vector addition
(C) is homogeneous
(D) is non-empty
(E) fulfills superposition

(4) (A) $\langle \vec{v}, \alpha\vec{x} \rangle = \alpha\langle \vec{v}, \vec{x} \rangle = 0$
(B) $\langle \alpha\vec{v}, \alpha\vec{x} \rangle = \alpha\langle \vec{v}, \vec{x} \rangle = 0$
(C) $\langle \alpha\vec{v}^\top \vec{x}, \vec{0} \rangle = \alpha\langle \vec{v}^\top \vec{x}, \vec{0} \rangle = 0$
(D) $\alpha\langle \vec{v}, \vec{x} \rangle = \alpha \cdot 0$

Solution: In order, the correct choices are B, D, A, A.

First, we need to show that the set contains the zero vector. We see that $\langle \vec{v}, \vec{0} \rangle = 0$, so this condition is fulfilled. Next, we need to show that it is closed under addition or scalar multiplication. However, since the proof first assumes that suppose we have $\vec{x}, \vec{y} \in W$, this implies that we are doing closure under addition first, so we choose B for (1). Then $\langle \vec{v}, \vec{x} + \vec{y} \rangle = \langle \vec{v}, \vec{x} \rangle + \langle \vec{v}, \vec{y} \rangle = 0$ to show that it is closed under addition, so we choose D for (2). Then we need to show scalar multiplication for (3), so we choose A. For (4), we have $\langle \vec{v}, \alpha\vec{x} \rangle = \alpha\langle \vec{v}, \vec{x} \rangle = 0$, so we choose A.

(b) Now suppose the inner product is defined as $\langle \vec{x}, \vec{y} \rangle = \vec{x}^\top Q \vec{y}$ for $Q \in \mathbb{R}^{2\times 2}$.

i. If $\vec{v} = \begin{bmatrix} 1 \\ -1 \end{bmatrix}$ and we still define subspace W to be the set of all vectors that are orthogonal to $\vec{v}$ from part (a), which of the following options is a basis for W if the matrix $Q = \begin{bmatrix} 2 & 1 \\ 1 & 4 \end{bmatrix}$?

(A) $\begin{bmatrix} -2 \\ 3 \end{bmatrix}$

(B) $\begin{bmatrix} 1 \\ 3 \end{bmatrix}$

(C) $\begin{bmatrix} 3 \\ 1 \end{bmatrix}$

(D) $\begin{bmatrix} 1 \\ 1 \end{bmatrix}$

(E) $\begin{bmatrix} 2 \\ 4 \end{bmatrix}$

Solution: Plugging in $\vec{x} = \vec{v}$ into the inner product $\langle \vec{x}, \vec{y} \rangle$, we get:

$$\vec{x}^\top \begin{bmatrix} 2 & 1 \\ 1 & 4 \end{bmatrix} \vec{y} = \begin{bmatrix} 1 & -1 \end{bmatrix} \begin{bmatrix} 2 & 1 \\ 1 & 4 \end{bmatrix} \begin{bmatrix} y_1 \\ y_2 \end{bmatrix} = \begin{bmatrix} 1 & -3 \end{bmatrix} \begin{bmatrix} y_1 \\ y_2 \end{bmatrix} = y_1 - 3y_2 \tag{16}$$

Therefore, we just need to find all y_1, y_2 where $\langle \vec{v}, \vec{y} \rangle = y_1 - 3y_2 = 0$, which means that $\vec{y} = \alpha \begin{bmatrix} -3 \\ 1 \end{bmatrix}$ for some scalar $\alpha \in \mathbb{R}$. Therefore the basis for W is $\begin{bmatrix} 3 \\ 1 \end{bmatrix}$.

ii. What are the necessary properties for a valid, real inner product? **(Select all that apply.)**

(A) positive definiteness

(B) closed under scalar multiplication

(C) closed under vector addition

(D) quadratic

(E) linear

(F) non-empty

(G) symmetric

(H) contains the zero vector

Solution: A valid inner product is positive definite, linear (i.e., satisfies additivity and homogeneity), and symmetric (commutative).

A vector space is closed under scalar multiplication, closed under vector addition, and contains the zero vector. Although an inner product is an operator applied to a vector space, it is not a vector space itself, thus these properties necessary for a vector space are not correct choices.

iii. Which of the following choices of matrix Q results in a valid inner product $\langle \vec{x}, \vec{y} \rangle = \vec{x}^\top Q \vec{y}$? (**Select all that apply.**)

(A) $\begin{bmatrix} -1 & 0 \\ 0 & 3 \end{bmatrix}$

(B) $\begin{bmatrix} 1 & -2 \\ 0 & 1 \end{bmatrix}$

(C) $\begin{bmatrix} 15 & 0 \\ 0 & 0 \end{bmatrix}$

(D) $\begin{bmatrix} 1 & 0 \\ 1 & 0 \end{bmatrix}$

(E) None of the above

Solution: The general form of this inner product $\langle \vec{x}, \vec{y} \rangle = \vec{x}^{\top} Q \vec{y}$ is linear.

$$\langle \alpha \vec{x}_1 + \beta \vec{x}_2, \vec{y} \rangle = (\alpha \vec{x}_1 + \beta \vec{x}_2)^{\top} Q \vec{y} \tag{17}$$
$$= \alpha \left(\vec{x}_1^{\top} Q \vec{y} \right) + \beta \left(\vec{x}_2^{\top} Q \vec{y} \right) \tag{18}$$
$$= \alpha \langle \vec{x}_1, \vec{y} \rangle + \beta \langle \vec{x}_2, \vec{y} \rangle \tag{19}$$

thus we only need to verify each answer choice is both symmetric and positive definite. If the matrix Q is symmetric (or not), then the inner product is symmetric (or not). To test positive definiteness, we inspect if $\langle \vec{x}, \vec{x} \rangle \geq 0$ for vectors $\vec{x} \neq \vec{0}$.

(A) Not a valid inner product. Although $\begin{bmatrix} -1 & 0 \\ 0 & 3 \end{bmatrix}$ is symmetric, the inner product is not positive definite since

$$\langle \vec{x}, \vec{x} \rangle = \begin{bmatrix} x_1 & x_2 \end{bmatrix} \begin{bmatrix} -1 & 0 \\ 0 & 3 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = -x_1^2 + 3x_2^2 \tag{20}$$

which is negative when $x_1 > \sqrt{3} x_2$.

(B) Not a valid inner product. $\begin{bmatrix} 1 & -2 \\ 0 & 1 \end{bmatrix}$ is not symmetric.

(C) Not a valid inner product. Although $\begin{bmatrix} 15 & 0 \\ 0 & 0 \end{bmatrix}$ is symmetric, the inner product is positive *semi*-definite (but not positive definite) since

$$\langle \vec{x}, \vec{x} \rangle = \begin{bmatrix} x_1 & x_2 \end{bmatrix} \begin{bmatrix} 15 & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = 15x_1^2 \tag{21}$$

which is always positive for $x_1 \neq 0$ and $x_2 \in \mathbb{R}$. To be positive definite, the inner product $\langle \vec{x}, \vec{x} \rangle$ must be *strictly* positive and can only evaluate to 0 when $\vec{x} = \vec{0}$.

(D) Not a valid inner product. $\begin{bmatrix} 1 & 0 \\ 1 & 0 \end{bmatrix}$ is not symmetric.

4. Tracking Terry

Terry is a mischievous child, and his mother is interested in tracking him.

(a) Terry texts his current location as a vector $\vec{x}_v = \begin{bmatrix} 2 \\ 3 \end{bmatrix}$, but there is a problem! These coordinates are *not* in the standard basis, but rather in the basis $V = \begin{bmatrix} \vec{v}_1 & \vec{v}_2 \end{bmatrix}$. That is to say that the first number 2 above is how many multiples of $\vec{v}_1$ to use and the second number 3 is how many multiples of $\vec{v}_2$ to use in computing his actual location. Here, both $\vec{v}_1$ and $\vec{v}_2$ are vectors in the standard basis.

Let Terry's location in the standard basis be $\vec{x}$. Write $\vec{x}$ in terms of $\vec{v}_1$ and $\vec{v}_2$.

Solution: By definition, the first coordinate in the V basis is the coefficient of $\vec{v}_1$ and second coordinate in the V basis is the coefficient for $\vec{v}_2$. Hence

$$\vec{x} = V\vec{x}_v = \begin{bmatrix} \vec{v}_1 & \vec{v}_2 \end{bmatrix} \begin{bmatrix} 2 \\ 3 \end{bmatrix} = 2\vec{v}_1 + 3\vec{v}_2. \tag{22}$$

(b) Terry's friend tells you that Terry's location in the standard basis is $\vec{x} = \begin{bmatrix} 4 \\ 2 \end{bmatrix}$. Using this along with the previous info that Terry's location in the V basis is $\vec{x}_v = \begin{bmatrix} 2 \\ 3 \end{bmatrix}$, **is it possible to determine the basis vectors $\vec{v}_1, \vec{v}_2$ Terry is using. If it is impossible to do so, explain why.**

(HINT: How many unknowns do you have? How many equations?)

Solution: Solving for the basis vectors Terry is using (or in other words the axes in his coordinate space) is the same as solving for V in the change of basis equation:

$$V\vec{x}_v = \vec{x} \tag{23}$$

$$\begin{bmatrix} \vec{v}_1 & \vec{v}_2 \end{bmatrix} \begin{bmatrix} 2 \\ 3 \end{bmatrix} = \begin{bmatrix} v_{11} & v_{21} \\ v_{12} & v_{22} \end{bmatrix} \begin{bmatrix} 2 \\ 3 \end{bmatrix} = \begin{bmatrix} 4 \\ 2 \end{bmatrix} \tag{24}$$

There are four unknowns and only two equations, so this task is impossible.

(c) Terry's basis vectors $\vec{v}_1, \vec{v}_2$ get leaked to his mom on accident, so she knows they are

$$\vec{v}_1 = \begin{bmatrix} 1 \\ 1 \end{bmatrix} \quad \text{and} \quad \vec{v}_2 = \begin{bmatrix} 0 \\ 3 \end{bmatrix}. \tag{25}$$

To hide his location, Terry wants to switch to a new coordinate system with the basis vectors

$$\vec{p}_1 = \begin{bmatrix} 1 \\ 4 \end{bmatrix} \quad \text{and} \quad \vec{p}_2 = \begin{bmatrix} 0 \\ 1 \end{bmatrix} \tag{26}$$

In order to do this, he needs a way to convert coordinates from the V basis to the P basis. Thus, **find the matrix T such that if $\vec{x}_v$ is a location expressed in V coordinates and $\vec{x}_p$ is the same location expressed in P coordinates, then $\vec{x}_p = T\vec{x}_v$.**

Solution: The problem can be formulated as a change of basis problem. Since both $\vec{x}_v$ and $\vec{x}_p$ correspond to the same point, converting them to the standard basis gives us

$$V\vec{x}_v = P\vec{x}_p \tag{27}$$

Since we want to find T such that $\vec{x}_p = T\vec{x}_v$, we have:

$$\vec{x}_p = P^{-1}V\vec{x}_v \tag{28}$$

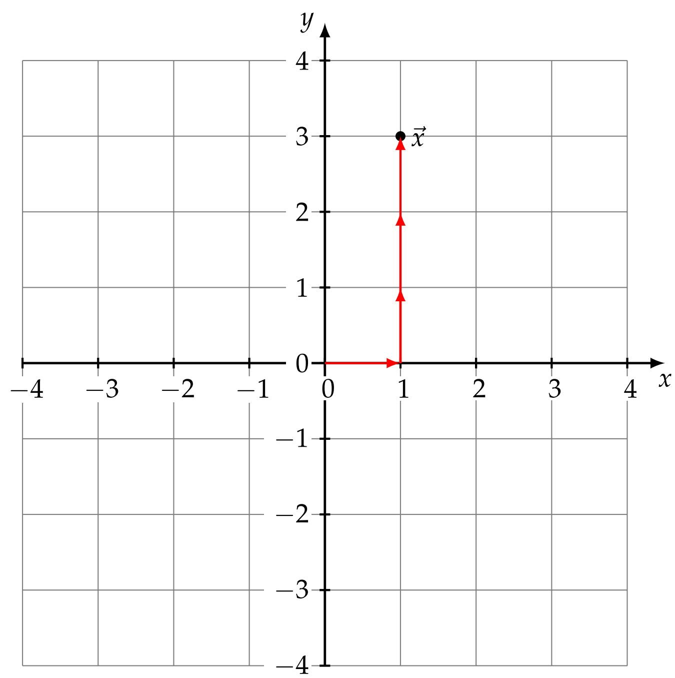
$$T = P^{-1}V \tag{29}$$

$$= \begin{bmatrix} 1 & 0 \\ 4 & 1 \end{bmatrix}^{-1} \begin{bmatrix} 1 & 0 \\ 1 & 3 \end{bmatrix} \tag{30}$$

$$= \begin{bmatrix} 1 & 0 \\ -4 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 1 & 3 \end{bmatrix} \tag{31}$$

$$= \begin{bmatrix} 1 & 0 \\ -3 & 3 \end{bmatrix} \tag{32}$$

(d) Terry now wants to make a map and route to where he currently is, $\vec{x} = \begin{bmatrix} 1 \\ 3 \end{bmatrix}$. **For both the P and V bases from part 4.c, illustrate the sum of scaled basis vectors that are necessary to go from the origin to $\vec{x}$.** An example is shown below when using the standard basis. This illustrates that the same location can be represented by many different coordinate systems/bases.



Solution:

Since we know $\vec{x} = \begin{bmatrix} 1 \\ 3 \end{bmatrix}$, $V = \begin{bmatrix} \vec{v}_1 & \vec{v}_2 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 1 & 2 \end{bmatrix}$, and $\vec{x} = V\vec{x}_v$, we can derive:

$$\vec{x}_v = V^{-1}\vec{x} \tag{33}$$

$$= \begin{bmatrix} 1 & 0 \\ 1 & 3 \end{bmatrix}^{-1} \begin{bmatrix} 1 \\ 3 \end{bmatrix} \tag{34}$$

$$= \begin{bmatrix} 1 & 0 \\ -\frac{1}{3} & \frac{1}{3} \end{bmatrix} \begin{bmatrix} 1 \\ 3 \end{bmatrix} \tag{35}$$

$$= \begin{bmatrix} 1 \\ \frac{2}{3} \end{bmatrix}. \tag{36}$$

Similarly, in order to compute $\vec{x}_p$, we have:

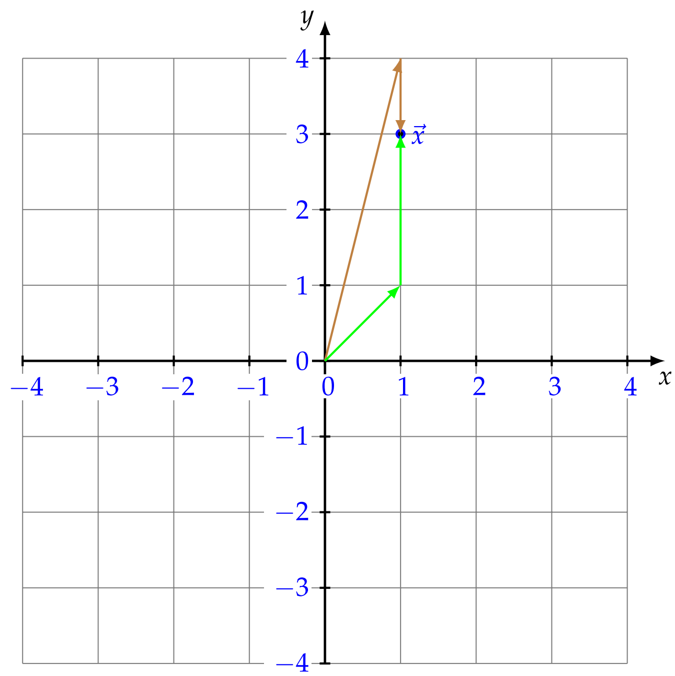
$$\vec{x}_p = P^{-1}\vec{x} \tag{37}$$

$$= \begin{bmatrix} 1 & 0 \\ 4 & 1 \end{bmatrix}^{-1} \begin{bmatrix} 1 \\ 3 \end{bmatrix} \tag{38}$$

$$= \begin{bmatrix} 1 & 0 \\ -4 & 1 \end{bmatrix} \begin{bmatrix} 1 \\ 3 \end{bmatrix} \tag{39}$$

$$= \begin{bmatrix} 1 \\ -1 \end{bmatrix}. \tag{40}$$

Therefore, we can illustrate the sum of scaled basis vectors according to $\vec{x}_v$ (green path), and $\vec{x}_p$ (brown path).



5. Eigenvectors and Diagonalization

(a) Let A be an $n \times n$ matrix with n linearly independent eigenvectors $\vec{v}_1, \vec{v}_2, \ldots, \vec{v}_n$, and corresponding eigenvalues $\lambda_1, \lambda_2, \ldots, \lambda_n$. Define V to be a matrix with $\vec{v}_1, \vec{v}_2, \ldots, \vec{v}_n$ as its columns, $V = \begin{bmatrix} \vec{v}_1 & \vec{v}_2 & \cdots & \vec{v}_n \end{bmatrix}$.

Show that $AV = V\Lambda$, where $\Lambda = \text{diag}(\lambda_1, \lambda_2, \ldots, \lambda_n)$, a diagonal matrix with the eigenvalues of A as its diagonal entries.

Solution:

$$
\begin{aligned}
AV &= A\begin{bmatrix} \vec{v}_1 & \vec{v}_2 & \cdots & \vec{v}_n \end{bmatrix} && (41)\\
&= \begin{bmatrix} A\vec{v}_1 & A\vec{v}_2 & \cdots, A\vec{v}_n \end{bmatrix} && (42)\\
&= \begin{bmatrix} \lambda_1\vec{v}_1 & \lambda_2\vec{v}_2 & \cdots & \lambda_n\vec{v}_n \end{bmatrix} && (43)\\
&= \begin{bmatrix} \vec{v}_1 & \vec{v}_2 & \cdots & \vec{v}_n \end{bmatrix} \begin{bmatrix} \lambda_1 & & & \\ & \lambda_2 & & \\ & & \ddots & \\ & & & \lambda_n \end{bmatrix} && (44)\\
&= V\Lambda && (45)
\end{aligned}
$$

(b) **Argue that V is invertible, and therefore**

$$A = V\Lambda V^{-1}. \quad (46)$$

(HINT: What condition on a matrix's columns means that it would be invertible? It is fine to cite the appropriate result from 16A.)

Solution: Columns of V are eigenvectors of A which are known to be linearly independent. Since V has linearly independent columns, it has full column rank, and therefore, is invertible.

$$
\begin{aligned}
AV &= V\Lambda && (47)\\
AVV^{-1} &= V\Lambda V^{-1} && (48)\\
A &= V\Lambda V^{-1} && (49)
\end{aligned}
$$

(c) **Write Λ in terms of the matrices A, V, and V^{-1}.**

Solution: We take $A = V\Lambda V^{-1}$ and apply invertible operations to both sides of the equality:

$$
\begin{aligned}
A &= V\Lambda V^{-1} && (50)\\
V^{-1}A &= V^{-1}V\Lambda V^{-1} && (51)\\
V^{-1}AV &= V^{-1}V\Lambda V^{-1}V && (52)\\
V^{-1}AV &= I\Lambda I && (53)\\
V^{-1}AV &= \Lambda. && (54)
\end{aligned}
$$

(d) A matrix A is deemed diagonalizable if there exists a square matrix U so that A can be written in the form $A = UDU^{-1}$ for the choice of an appropriate diagonal matrix D.

Show that the columns of U must be eigenvectors of the matrix A, and that the entries of D must be eigenvalues of A.

(HINT: Recall the definition of an eigenvector (i.e., $A\vec{v} = \lambda\vec{v}$). Then, recall what $U^{-1}U$ is. Lastly, consider how matrix multiplication works column-wise.)

Solution: We start with a calculation which is essentially the reverse of the calculation in part (b):

$$A = UDU^{-1} \tag{55}$$
$$AU = UDU^{-1}U \tag{56}$$
$$AU = UD. \tag{57}$$

Now let's expand the definitions of U as a square matrix and D as a diagonal matrix:

$$AU = A\begin{bmatrix}\vec{u}_1 & \dots & \vec{u}_n\end{bmatrix} \tag{58}$$
$$= \begin{bmatrix}A\vec{u}_1 & \dots & A\vec{u}_n\end{bmatrix} \tag{59}$$
$$UD = \begin{bmatrix}\vec{u}_1 & \dots & \vec{u}_n\end{bmatrix}\begin{bmatrix}d_1 & & \\ & \ddots & \\ & & d_n\end{bmatrix} \tag{60}$$
$$= \begin{bmatrix}d_1\vec{u}_1 & \dots & d_n\vec{u}_n\end{bmatrix}. \tag{61}$$

Comparing columns, we see that $A\vec{u}_i = d_i\vec{u}_i$. This is exactly the eigenvector-eigenvalue equation! In particular, this says that $\vec{u}_i$ is an eigenvector of A, with eigenvalue d_i.

The previous part shows that the *only* way to diagonalize A is using its eigenvalues/eigenvectors. Now we will explore a payoff for diagonalizing A – an operation that diagonalization makes *much* simpler.

(e) For a matrix A and a positive integer k, we define the exponent to be

$$A^k = \underbrace{A \cdot A \cdot \cdots \cdot A \cdot A}_{k \text{ times}} \tag{62}$$

Let's assume that matrix A is diagonalizable with eigenvalues $\lambda_1, \lambda_2, \ldots, \lambda_n$, and corresponding eigenvectors $\vec{v}_1, \vec{v}_2, \ldots, \vec{v}_n$ (i.e. the n eigenvectors are all linearly independent).

Show that A^k has eigenvalues $\lambda_1^k, \lambda_2^k, \ldots, \lambda_n^k$ and eigenvectors $\vec{v}_1, \vec{v}_2, \ldots, \vec{v}_n$. Conclude that A^k is diagonalizable.

Solution: Consider the i^{th} eigenvector of A, $\vec{v}_i$ and the corresponding eigenvalue λ_i.

$$A^k\vec{v}_i = A^{k-1} \cdot A\vec{v}_i \tag{63}$$
$$= A^{k-1}\lambda_i\vec{v}_i \tag{64}$$
$$= \lambda_i A^{k-2} \cdot A\vec{v}_i \tag{65}$$
$$= \lambda_i^2 A^{k-3} \cdot A\vec{v}_i \tag{66}$$

$$\vdots \tag{67}$$

$$= \lambda_i^k \vec{v}_i \tag{68}$$

Thus by definition, v_i is an eigenvector of A^k with corresponding eigenvalue λ_i^k.

Alternate solution: Since A is diagonalizable, we can express A as

$$A = V\Lambda V^{-1} \tag{69}$$

Substituting A as shown in Equation 69 in 62, we get

$$A^k = \underbrace{A \cdot A \cdot \cdots \cdot A \cdot A}_{k \text{ times}} \tag{70}$$

$$= \underbrace{V\Lambda V^{-1} \cdot V\Lambda V^{-1} \cdot \cdots \cdot V\Lambda V^{-1} \cdot V\Lambda V^{-1}}_{k \text{ times}} \tag{71}$$

$$= \underbrace{V\Lambda \left(V^{-1} \cdot V\right) \Lambda V^{-1} \cdot \cdots \cdot V\Lambda \left(V^{-1} \cdot V\right) \Lambda V^{-1}}_{k \text{ times}} \tag{72}$$

$$= V \underbrace{\Lambda \cdot \Lambda \cdot \cdots \cdot \Lambda \cdot \Lambda}_{k \text{ times}} V^{-1} \tag{73}$$

$$= V\Lambda^k V^{-1} \tag{74}$$

Since Λ is a diagonal matrix,

$$\Lambda^k = \begin{bmatrix} \lambda_1^k & & & \\ & \lambda_2^k & & \\ & & \ddots & \\ & & & \lambda_n^k \end{bmatrix} \tag{75}$$

Thus, A^k is clearly diagonalizable, where the eigenvectors of A^k are just the eigenvectors of A, and the eigenvalues of A^k are $\lambda_1^k, \ldots, \lambda_n^k$.

6. Vector Differential Equations

Note: it's recommended to finish the previous question (Eigenvectors and Diagonalization) before this problem.

Consider a system of ordinary differential equations that can be written in the form

$$\frac{\mathrm{d}}{\mathrm{d}t}\vec{x}(t) := \begin{bmatrix} \frac{\mathrm{d}}{\mathrm{d}t}x_1(t) \\ \frac{\mathrm{d}}{\mathrm{d}t}x_2(t) \end{bmatrix} = A \begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix} = A\vec{x}(t) \tag{76}$$

where $x_1, x_2 : \mathbb{R} \to \mathbb{R}$ are scalar functions of time t, and $A \in \mathbb{R}^{2\times 2}$ is a 2×2 matrix with constant coefficients. We call eq. (76) a vector differential equation.

(a) It turns out that we can actually turn all higher-order differential equations with constant coefficients into vector differential equations of the style of eq. (76).

Consider a second-order ordinary differential equation

$$\frac{\mathrm{d}^2 y(t)}{\mathrm{d}t^2} + a\frac{\mathrm{d}y(t)}{\mathrm{d}t} + by(t) = 0, \tag{77}$$

where $a, b \in \mathbb{R}$.

Write this differential equation in the form of (eq. (76)), by choosing appropriate variables $x_1(t)$ and $x_2(t)$.

(HINT: Your original unknown function $y(t)$ has to be one of those variables. The heart of the question is to figure out what additional variable can you use so that you can express eq. (77) without having to take a second derivative, and instead just taking the first derivative of something.)

Solution: If we set $x_1(t) = y(t)$, $x_2(t) = \frac{\mathrm{d}y(t)}{\mathrm{d}t}$, then we have

$$\frac{\mathrm{d}x_1(t)}{\mathrm{d}t} = \frac{\mathrm{d}y(t)}{\mathrm{d}t} = x_2(t) \tag{78}$$

$$\frac{\mathrm{d}x_2(t)}{\mathrm{d}t} = \frac{\mathrm{d}^2 y(t)}{\mathrm{d}t^2} = -a\frac{\mathrm{d}y(t)}{\mathrm{d}t} - by(t) = -ax_2(t) - bx_1(t) \tag{79}$$

We can write this in the form of eq. (76) as follows

$$\frac{\mathrm{d}}{\mathrm{d}t}\vec{x} = \frac{\mathrm{d}}{\mathrm{d}t}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ -b & -a \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \tag{80}$$

(b) It turns out that all two-dimensional vector linear differential equations with distinct eigenvalues have a solution in the general form

$$\begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix} = \begin{bmatrix} c_0 e^{\lambda_1 t} + c_1 e^{\lambda_2 t} \\ c_2 e^{\lambda_1 t} + c_3 e^{\lambda_2 t} \end{bmatrix} \tag{81}$$

where c_0, c_1, c_2, c_3 are constants, and λ_1, λ_2 are the eigenvalues of A (this can be proven by just repeating the same steps in the previous parts and using the fact that distinct eigenvalues implies linearly independent eigenvectors). Thus, an alternate way of solving this type of differential equation in the future is to now use your knowledge that the solution is of this form and just solve for the constants c_i.

Now let $a = -1$ and $b = -2$ in eq. (77), i.e.

$$\frac{\mathrm{d}^2 y(t)}{\mathrm{d}t^2} - \frac{\mathrm{d}y(t)}{\mathrm{d}t} - 2y(t) = 0, \tag{82}$$

Solve eq. (82) with the initial conditions $y(0) = 1$, $\frac{dy}{dt}(0) = 1$, using the general form in eq. (81). *(HINT: You get two equations using the initial conditions above. How many unknowns are here?) (HINT: Given your specific choice of x_1 and x_2 in part (a), how many unknowns are there really?)*

Solution: We have

$$\begin{bmatrix} 0 & 1 \\ -b & -a \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ 2 & 1 \end{bmatrix} \tag{83}$$

First, we calculate the eigenvalues of this matrix. The characteristic polynomial is

$$\det\left(\begin{bmatrix} -\lambda & 1 \\ 2 & 1-\lambda \end{bmatrix}\right) = \lambda^2 - \lambda - 2 = (\lambda - 2)(\lambda + 1) \tag{84}$$

Thus the eigenvalues are $\lambda_1 = -1, \lambda_2 = 2$. Since they are distinct, we can proceed with this method.

We know the solution for $x_1(t), x_2(t)$ is of the form

$$\begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix} = \begin{bmatrix} c_0 e^{-t} + c_1 e^{2t} \\ c_2 e^{-t} + c_3 e^{2t} \end{bmatrix} \tag{85}$$

At $t = 0$, we have $y(0) = 1$ and $\frac{dy}{dt}(0) = 1$. Using our differential equation (eq. (82)), we can get $\frac{d^2y}{dt^2}(0) = \frac{dy}{dt}(0) + 2y(0) = 3$. Plugging these in,

$$x_1(0) = y(0) = 1 = c_0 + c_1 \tag{86}$$

$$x_2(0) = \frac{dy}{dt}(0) = 1 = c_2 + c_3 \tag{87}$$

$$\frac{dx_1}{dt}(0) = \frac{dy}{dt}(0) = 1 = -c_0 + 2c_1 \tag{88}$$

$$\frac{dx_2}{dt}(0) = \frac{d^2y}{dt^2}(0) = 3 = -c_2 + 2c_3 \tag{89}$$

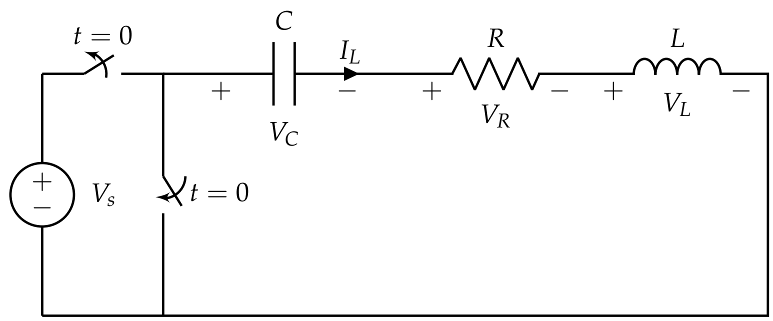
This gives $c_0 = \frac{1}{3}, c_1 = \frac{2}{3}, c_2 = -\frac{1}{3}, c_3 = \frac{4}{3}$. Alternatively, you could've seen that $c_2 = -c_0$ and $c_3 = 2c_1$ since $x_2(t)$ is the derivative of $x_1(t)$ which makes it solvable with just the first 2 equations. Thus we have

$$x_1(t) = y(t) = \frac{1}{3}e^{-t} + \frac{2}{3}e^{2t} \tag{90}$$

$$x_2(t) = \frac{dy(t)}{dt} = -\frac{1}{3}e^{-t} + \frac{4}{3}e^{2t} \tag{91}$$

7. RLC Responses

Consider the following circuit:



Assume the circuit above has reached steady state for $t < 0$. At time $t = 0$, the switch changes state and disconnects the voltage source, replacing it with a short.

The sequence of problems 2 - 6 combined will try to show you the various RLC system responses and how they relate to changing circuit properties.

(a) We first need to construct our state space system. Our state variables are the current through the inductor $x_1(t) = I_L(t)$ and the voltage across the capacitor $x_2(t) = V_C(t)$ since these are the quantities whose derivatives show up in the system of equations governing our circuit. Now, **show that the system of differential equations in terms of our state variables that describes this circuit for $t \geq 0$ is**

$$\frac{\mathrm{d}}{\mathrm{d}t}x_1(t) = -\frac{R}{L}x_1(t) - \frac{1}{L}x_2(t) \tag{92}$$
$$\frac{\mathrm{d}}{\mathrm{d}t}x_2(t) = \frac{1}{C}x_1(t). \tag{93}$$

Solution: For this part, we need to find two differential equations, each including a derivative of one of the state variables.

First, let's consider the capacitor equation $I_C(t) = C\frac{\mathrm{d}}{\mathrm{d}t}V_C(t)$. In this circuit, $I_C(t) = I_L(t)$, so we can write

$$I_C(t) = C\frac{\mathrm{d}}{\mathrm{d}t}V_C(t) = I_L(t) \tag{94}$$
$$\frac{\mathrm{d}}{\mathrm{d}t}V_C(t) = \frac{1}{C}I_L(t). \tag{95}$$

If we use the state variable names, we can write this as

$$\frac{\mathrm{d}}{\mathrm{d}t}x_2(t) = \frac{1}{C}x_1(t), \tag{96}$$

so now we have one differential equation.

For the other differential equation, consider the voltage drop across the capacitor, resistor and inductor. At $t \geq 0$, the voltage difference between the positive '+' terminal of C and the negative '-' terminal of L is given by

$$V_C(t) + V_R(t) + V_L(t) = 0. \tag{97}$$

Using Ohm's Law $V_R(t) = RI_L(t)$ and the inductor equation $V_L(t) = L\frac{\mathrm{d}}{\mathrm{d}t}I_L(t)$, we can write this as

$$V_C(t) + RI_L(t) + L\frac{\mathrm{d}}{\mathrm{d}t}I_L(t) = 0, \tag{98}$$

which we can rewrite as

$$\frac{\mathrm{d}}{\mathrm{d}t}I_L(t) = -\frac{R}{L}I_L(t) - \frac{1}{L}V_C(t). \tag{99}$$

If we use the state variable names, this becomes

$$\frac{\mathrm{d}}{\mathrm{d}t}x_1(t) = -\frac{R}{L}x_1(t) - \frac{1}{L}x_2(t), \tag{100}$$

and we have a second differential equation.

To summarize, the final system is

$$\frac{\mathrm{d}}{\mathrm{d}t}x_1(t) = -\frac{R}{L}x_1(t) - \frac{1}{L}x_2(t) \tag{101}$$

$$\frac{\mathrm{d}}{\mathrm{d}t}x_2(t) = \frac{1}{C}x_1(t). \tag{102}$$

(b) **Write the system of equations in vector/matrix form with the vector state variable $\vec{x}(t) = \begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix} = \begin{bmatrix} I_L(t) \\ V_C(t) \end{bmatrix}$. This should be in the form $\frac{\mathrm{d}}{\mathrm{d}t}\vec{x}(t) = A\vec{x}(t)$ with a 2×2 matrix A.**

Solution: By inspection from the previous part, we have

$$\begin{bmatrix} \frac{\mathrm{d}}{\mathrm{d}t}x_1(t) \\ \frac{\mathrm{d}}{\mathrm{d}t}x_2(t) \end{bmatrix} = \begin{bmatrix} -\frac{R}{L} & -\frac{1}{L} \\ \frac{1}{C} & 0 \end{bmatrix} \begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix}, \tag{103}$$

which is in the form $\frac{\mathrm{d}}{\mathrm{d}t}\vec{x}(t) = A\vec{x}(t)$, with

$$A = \begin{bmatrix} -\frac{R}{L} & -\frac{1}{L} \\ \frac{1}{C} & 0 \end{bmatrix}. \tag{104}$$

(c) **Show that, for the 2×2 matrix A, the two eigenvalues of A are**

$$\lambda_1 = -\frac{1}{2}\frac{R}{L} + \frac{1}{2}\sqrt{\left(\frac{R}{L}\right)^2 - \frac{4}{LC}} \tag{105}$$

$$\lambda_2 = -\frac{1}{2}\frac{R}{L} - \frac{1}{2}\sqrt{\left(\frac{R}{L}\right)^2 - \frac{4}{LC}}. \tag{106}$$

(HINT: The quadratic formula will be involved.)

Solution: To find the eigenvalues, we'll solve $\det(A - \lambda I) = 0$. In other words, we want to find λ such that

$$\det(A - \lambda I) = \det\left(\begin{bmatrix} -\frac{R}{L} - \lambda & -\frac{1}{L} \\ \frac{1}{C} & -\lambda \end{bmatrix}\right) \tag{107}$$

$$= -\lambda\left(-\frac{R}{L} - \lambda\right) + \frac{1}{LC} \tag{108}$$

$$= \lambda^2 + \frac{R}{L}\lambda + \frac{1}{LC} = 0. \tag{109}$$

The quadratic formula gives

$$\lambda = -\frac{1}{2}\frac{R}{L} \pm \frac{1}{2}\sqrt{\left(\frac{R}{L}\right)^2 - \frac{4}{LC}} \tag{110}$$

as desired.

(d) **Under what condition on the circuit parameters R, L, C will A have two distinct real eigenvalues?**

Solution: For both eigenvalues to be real and distinct, we need the quantity inside the square root to be positive. In other words, we need

$$\frac{R^2}{L^2} - \frac{4}{LC} > 0, \tag{111}$$

or, equivalently,

$$R > 2\sqrt{\frac{L}{C}}. \tag{112}$$

(e) **Under what condition on the circuit parameters R, L, C will A have two imaginary eigenvalues? What will the eigenvalues be in this case?** **Solution:** The only way for both eigenvalues to be purely imaginary is to have $R = 0$. In this case, the eigenvalues would be

$$\lambda = \pm \mathrm{j}\sqrt{\frac{1}{LC}}. \tag{113}$$

(f) Assuming that the circuit parameters are such that there are a pair of (potentially complex) eigenvalues λ_1, λ_2 so that $\lambda_1 \neq \lambda_2$, **show that the corresponding eigenvectors $\vec{v}_{\lambda_1}, \vec{v}_{\lambda_2}$ are**

$$\vec{v}_{\lambda_1} = \begin{bmatrix} 1 \\ \frac{1}{\lambda_1 C} \end{bmatrix} \quad \text{and} \quad \vec{v}_{\lambda_2} = \begin{bmatrix} 1 \\ \frac{1}{\lambda_2 C} \end{bmatrix}. \tag{114}$$

Solution: We use the definition of an eigenvector and eigenvalue. We want $A\vec{v}_{\lambda_i} = \lambda_i \vec{v}_{\lambda_i}$. Note that, for any y,

$$A\begin{bmatrix} 0 \\ y \end{bmatrix} = \begin{bmatrix} -\frac{R}{L} & -\frac{1}{L} \\ \frac{1}{C} & 0 \end{bmatrix}\begin{bmatrix} 0 \\ y \end{bmatrix} = \begin{bmatrix} -\frac{y}{L} \\ 0 \end{bmatrix} \tag{115}$$

is not a scalar multiple of $\begin{bmatrix} 0 \\ y \end{bmatrix}$, so no eigenvector is of the form $\begin{bmatrix} 0 \\ y \end{bmatrix}$. Thus they must all be of the form $\begin{bmatrix} y_1 \\ y_2 \end{bmatrix}$ with $y_1 \neq 0$, and we can divide through by y_1 to show that every eigenvector is of the form $\begin{bmatrix} 1 \\ y \end{bmatrix}$ for some y.

Thus,

$$\begin{bmatrix} -\frac{R}{L} & -\frac{1}{L} \\ \frac{1}{C} & 0 \end{bmatrix}\begin{bmatrix} 1 \\ y \end{bmatrix} = \lambda_i \begin{bmatrix} 1 \\ y \end{bmatrix} = \begin{bmatrix} \lambda_i \\ \lambda_i \cdot y \end{bmatrix} \tag{116}$$

We also know that:

$$\begin{bmatrix} -\frac{R}{L} & -\frac{1}{L} \\ \frac{1}{C} & 0 \end{bmatrix} \begin{bmatrix} 1 \\ y \end{bmatrix} = \begin{bmatrix} -\frac{R}{L} - \frac{y}{L} \\ \frac{1}{C} \end{bmatrix} \tag{117}$$

Equating the two equations from above gives:

$$\begin{bmatrix} \lambda_i \\ \lambda_i \cdot y \end{bmatrix} = \begin{bmatrix} -\frac{R}{L} - \frac{y}{L} \\ \frac{1}{C} \end{bmatrix}. \tag{118}$$

From the second row we see that $y = \frac{1}{\lambda_i C}$. Now we find the eigenvectors as:

$$\vec{v}_{\lambda_1} = \begin{bmatrix} 1 \\ \frac{1}{\lambda_1 C} \end{bmatrix} \tag{119}$$

$$\vec{v}_{\lambda_2} = \begin{bmatrix} 1 \\ \frac{1}{\lambda_2 C} \end{bmatrix} \tag{120}$$

Alternatively, you can try to use the standard approach of finding the nullspace of $A - \lambda_i I$ to arrive at the same answer as above.

(g) Assuming circuit parameters such that the two eigenvalues of A are distinct, let $V = \begin{bmatrix} \vec{v}_{\lambda_1} & \vec{v}_{\lambda_2} \end{bmatrix}$ be a specific eigenbasis. Consider a coordinate system for which we can write $\vec{x}(t) = V\vec{\widetilde{x}}(t)$. **Show that the $\widetilde{A}$ so that $\frac{\mathrm{d}}{\mathrm{d}t}\vec{\widetilde{x}}(t) = \widetilde{A}\vec{\widetilde{x}}(t)$ is**

$$\widetilde{A} = \begin{bmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{bmatrix}. \tag{121}$$

(HINT: Write out the original differential equation $\frac{\mathrm{d}}{\mathrm{d}t}\vec{x}(t) = A\vec{x}(t)$, then use the given change of coordinates to write everything in terms of $\vec{\widetilde{x}}(t)$.)

Solution: V is given by:

$$V = \begin{bmatrix} 1 & 1 \\ \frac{1}{\lambda_1 C} & \frac{1}{\lambda_2 C} \end{bmatrix} \tag{122}$$

We know that V transforms from the $\widetilde{x}$ coordinate frame to the x coordinate frame, V^{-1} transforms back, and A takes gives the relationship from x to $\frac{\mathrm{d}}{\mathrm{d}t}x$.

Therefore to go from $\widetilde{x}$ to $\frac{\mathrm{d}}{\mathrm{d}t}\widetilde{x}$:

$$\widetilde{A} = V^{-1}AV \tag{123}$$

$$= \begin{bmatrix} 1 & 1 \\ \frac{1}{\lambda_1 C} & \frac{1}{\lambda_2 C} \end{bmatrix}^{-1} \begin{bmatrix} -\frac{R}{L} & -\frac{1}{L} \\ \frac{1}{C} & 0 \end{bmatrix} \begin{bmatrix} 1 & 1 \\ \frac{1}{\lambda_1 C} & \frac{1}{\lambda_2 C} \end{bmatrix} \tag{124}$$

$$= \frac{\lambda_1 \lambda_2 C}{\lambda_1 - \lambda_2} \begin{bmatrix} \frac{1}{\lambda_2 C} & -1 \\ -\frac{1}{\lambda_1 C} & 1 \end{bmatrix} \begin{bmatrix} -\frac{R}{L} & -\frac{1}{L} \\ \frac{1}{C} & 0 \end{bmatrix} \begin{bmatrix} 1 & 1 \\ \frac{1}{\lambda_1 C} & \frac{1}{\lambda_2 C} \end{bmatrix} \tag{125}$$

$$= \begin{bmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{bmatrix} \tag{126}$$

You didn't have to multiply things out explicitly. You could have just noticed that the eigenvector matrix will diagonalize the A matrix such that $AV = V\Lambda$ or $V^{-1}AV = \Lambda$, as per one of the problems on the last homework.

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