

Homework 5

This homework is due on Friday, February 25, 2022, at 11:59PM. Self-grades and HW Resubmissions are due on the following Friday, March 4, 2022, at 11:59PM.

1. Reading Lecture Notes

Staying up to date with lectures is an important part of the learning process in this course. Here are links to the notes that you need to read for this week: [Note 5](#) and [Note 6](#).

- (a) Consider an RC circuit with a sinusoidal voltage input $V(t) = A \cos(\omega t)$. We are interested in finding the voltage on the capacitor in steady state (after a long time has passed). **Can we solve this using our standard differential equation techniques? Can we solve this with phasors? Which one is more concise and why?**

Solution: Standard Differential Equations: We can solve this using our standard differential equation techniques. Recall that if the D.E. takes the form $\frac{d}{dt}x(t) = \lambda x(t) + u(t)$, we can solve using $x(t) = x_0 e^{\lambda t} + \int_0^t e^{\lambda(t-\theta)} u(\theta) d\theta$. This works, but requires us to perform an integral.

Phasor Analysis: Using phasor analysis, we can solve this by using circuit analysis techniques. Phasor analysis will be more concise because fundamentally, phasor analysis is built for sinusoidal inputs (since they are complex exponentials) at steady state. Phasor analysis only requires one to solve a simple RC circuit using standard circuit analysis techniques. **Phasor analysis is much more concise than solving the D.E..**

- (b) There are two ways to make a low pass filter (discussed in the notes). **What are they?**

Solution: The first method is to construct a series R-C circuit (measuring the voltage across C). The second method is to construct a series L-R circuit (measuring the voltage across R). These are extremely similar. In fact, the transfer function for both cases has the same form: $H(j\omega) = \frac{1}{1+j\frac{\omega}{\omega_c}}$.

The only difference is that $\omega_c = \frac{1}{RC}$ for R-C circuits and $\omega_c = \frac{R}{L}$ for L-R circuits.

- (c) **Draw the voltage sources between terminals a and b in figure 1 as a single equivalent voltage source between terminals a and b , and label its voltage value. How does this equivalence relate to filtering and phasor analysis?**

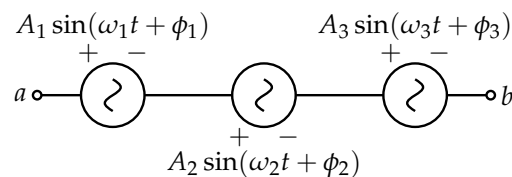
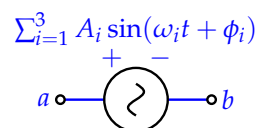


Figure 1: Three voltage sources in series.

Solution:



The equivalence between a single voltage source that is the sum of many different sinusoids of different frequencies and multiple voltage sources in series allows us to perform phasor analysis

separately through superposition for each frequency to predict how much of each frequency component will appear in some output voltage or current. Each of the different frequencies will be affected independently, so that certain frequency signals can be affected more than others, achieving filtering of signals as a function of frequency.

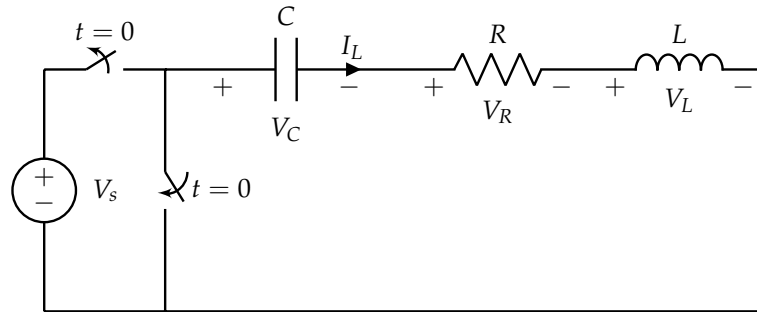
(d) **How we can address filter loading?**

Solution: One way to address loading is to pass a voltage signal from the output of a filter to the input of the next filter by using a unity gain buffer. The second method of addressing filter loading is by choosing the component values in such a way that the impedance of the following filters at the frequencies of interest are large relative to the component from which the output is taken.

2. Alternative “second order” perspective on solving the RLC circuit

In **Homework 4**, we solved an RLC circuit by setting state variables $x_1(t) = V_C(t)$ and $x_2(t) = I_L(t)$, and using these to build a linear first-order vector differential equation. In this problem, we will see how to solve the same system by picking *different* state variables $x_1(t) = V_C(t)$ and $x_2(t) = \frac{d}{dt} V_C(t)$, getting a linear *second order scalar* differential equation, and solving that differential equation.

Consider the following circuit like you saw in lecture, discussion, and the previous homework:



As before, assume that the system has reached steady-state for $t < 0$. At time $t = 0$, the switch changes state and disconnects the voltage source, replacing it with a short.

Suppose now we insisted on expressing everything in terms of one waveform $V_C(t)$ instead of two of them (voltage across the capacitor and current through the inductor). This is called the “second-order” point of view, because we will end up using second derivatives.

For this problem, use R for the resistor, L for the inductor, and C for the capacitor in all the expressions.

- (a) **Write the current $I_L(t)$ through the inductor in terms of the voltage $V_C(t)$ across the capacitor.**

Solution: The current $I_L(t)$ through the inductor L must be the same as the current $I_C(t)$ through C , which is $C \frac{d}{dt} V_C(t)$. Hence, we can write

$$I_L(t) = C \frac{d}{dt} V_C(t). \quad (1)$$

- (b) Now, notice that the voltage drop across the inductor involves $\frac{d}{dt} I_L(t)$. **Write the voltage drop across the inductor, $V_L(t)$, in terms of the second derivative of $V_C(t)$.**

Solution: The voltage drop is

$$V_L(t) = L \frac{d}{dt} I_L(t) = LC \frac{d}{dt} \left(\frac{d}{dt} V_C(t) \right) = LC \frac{d^2}{dt^2} V_C(t). \quad (2)$$

- (c) **Show that a differential equation governing $V_C(t)$ is**

$$\frac{d^2}{dt^2} V_C(t) + \frac{R}{L} \frac{d}{dt} V_C(t) + \frac{1}{LC} V_C(t) = 0. \quad (3)$$

Solution: Note that the current passing through the resistor is

$$I_R(t) = -\frac{V_C(t) + V_L(t)}{R} = C \frac{d}{dt} V_C(t). \quad (4)$$

or equivalently,

$$V_L(t) + RC \frac{d}{dt} V_C(t) + V_C(t) = 0. \quad (5)$$

Plugging in $V_L(t)$, we have

$$LC \frac{d^2}{dt^2} V_C(t) + RC \frac{d}{dt} V_C(t) + V_C(t) = 0. \quad (6)$$

Finally, dividing by LC ,

$$\frac{d^2}{dt^2} V_C(t) + \frac{R}{L} \frac{d}{dt} V_C(t) + \frac{1}{LC} V_C(t) = 0. \quad (7)$$

(d) Recall that in **Homework 3** Problem 5, we solved a second-order differential equation of the form

$$\frac{d^2 y(t)}{dt^2} + a \frac{dy(t)}{dt} + by(t) = 0 \quad (8)$$

by converting it into a matrix differential equation

$$\frac{d}{dt} \begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix} = \underbrace{\begin{bmatrix} 0 & 1 \\ -b & -a \end{bmatrix}}_A \begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix} \quad (9)$$

where $x_1(t) := y(t)$ and $x_2(t) := \frac{dy(t)}{dt}$. It turned out that, if A has two distinct eigenvalues, the solution to this homogeneous differential equation have the form

$$\begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix} = \begin{bmatrix} c_1 e^{\lambda_1 t} + c_2 e^{\lambda_2 t} \\ c_3 e^{\lambda_1 t} + c_4 e^{\lambda_2 t} \end{bmatrix}. \quad (10)$$

where λ_1, λ_2 are the eigenvalues of A , and c_1, c_2, c_3, c_4 are constants determined by the initial conditions and the coefficients a, b in the differential equation. We would like to use this to construct a solution for $V_C(t)$.

Show that, if we identify $y(t) := V_C(t)$, then

$$x_1(t) = V_C(t) \quad x_2(t) = \frac{d}{dt} V_C(t), \quad (11)$$

and that the matrix A for our RLC circuit is

$$A = \begin{bmatrix} 0 & 1 \\ -\frac{1}{LC} & -\frac{R}{L} \end{bmatrix}. \quad (12)$$

Then, show that the two eigenvalues of A are

$$\lambda_1 = -\frac{R}{2L} + \frac{1}{2} \sqrt{\frac{R^2}{L^2} - \frac{4}{LC}}, \quad \lambda_2 = -\frac{R}{2L} - \frac{1}{2} \sqrt{\frac{R^2}{L^2} - \frac{4}{LC}}. \quad (13)$$

NOTE: From this part onwards, we will assume that the circuit parameters R, L, C are chosen so that the eigenvalues of A are distinct.

Solution: We have

$$x_1(t) = y(t) = V_C(t) \quad (14)$$

$$x_2(t) = \frac{d}{dt} y(t) = \frac{d}{dt} V_C(t). \quad (15)$$

Examining the coefficients in eq. (3), we see

$$a = \frac{R}{L} \quad (16)$$

$$b = \frac{1}{LC}. \quad (17)$$

Thus

$$A = \begin{bmatrix} 0 & 1 \\ -\frac{1}{LC} & -\frac{R}{L} \end{bmatrix}. \quad (18)$$

The characteristic polynomial of A is¹

$$p_A(\lambda) := \det(A - \lambda I) \quad (19)$$

$$= \det\left(\begin{bmatrix} -\lambda & 1 \\ -\frac{1}{LC} & -\frac{R}{L} - \lambda \end{bmatrix}\right) \quad (20)$$

$$= (-\lambda)\left(-\frac{R}{L} - \lambda\right) - 1 \cdot \left(-\frac{1}{LC}\right) \quad (21)$$

$$= \lambda^2 + \frac{R}{L}\lambda + \frac{1}{LC}. \quad (22)$$

The solutions to $p_A(\lambda) = 0$ are obtained by the quadratic formula to be

$$\lambda_1 = -\frac{R}{2L} + \frac{1}{2}\sqrt{\frac{R^2}{L^2} - \frac{4}{LC}} \quad \lambda_2 = -\frac{R}{2L} - \frac{1}{2}\sqrt{\frac{R^2}{L^2} - \frac{4}{LC}}. \quad (23)$$

- (e) Now, we solve for $x_1(t) = V_C(t)$ by determining c_1 and c_2 and plugging those, along with λ_1 and λ_2 , into eq. (10). Note that determining c_3 and c_4 isn't necessary to find $x_1(t)$, but we need them to set up a system of equations to solve for c_1 and c_2 . **Show that**

$$c_3 = \lambda_1 c_1, \quad c_4 = \lambda_2 c_2. \quad (24)$$

Then use the initial conditions of the RLC circuit to show that

$$c_1 = \frac{\lambda_2}{\lambda_2 - \lambda_1} V_s, \quad c_2 = -\frac{\lambda_1}{\lambda_2 - \lambda_1} V_s. \quad (25)$$

(HINT: This part is complicated, so here's some scaffolding to start you off. First, differentiate the expression we have $x_1(t) = c_1 e^{\lambda_1 t} + c_2 e^{\lambda_2 t}$ to get a form for $x_2(t)$, and match coefficients of $e^{\lambda_1 t}$ and $e^{\lambda_2 t}$ to get the desired expressions for c_3 and c_4 . Next, use the initial conditions for RLC to see what $V_C(0)$ are and $\left.\frac{d}{dt}V_C(t)\right|_{t=0}$ are. This corresponds to $x_1(0)$ and $x_2(0)$. Plug $t = 0$ into the "sum of exponentials" form for x_1 and x_2 . This will get you two equations, one for each x_i , for c_1 and c_2 , which you can then solve.)

(HINT: The following matrix inverse formula may be useful:

$$\begin{bmatrix} 1 & 1 \\ \lambda_1 & \lambda_2 \end{bmatrix}^{-1} = \frac{1}{\lambda_2 - \lambda_1} \begin{bmatrix} \lambda_2 & -1 \\ -\lambda_1 & 1 \end{bmatrix} \quad (26)$$

)

Solution: By definition,

$$x_2(t) = \frac{d}{dt}V_C(t) = \frac{d}{dt}x_1(t) \quad (27)$$

so if

$$x_1(t) = c_1 e^{\lambda_1 t} + c_2 e^{\lambda_2 t} \quad (28)$$

then

$$x_2(t) = \frac{d}{dt}x_1(t) \quad (29)$$

¹Notice that it looks very similar to the original differential equation. This is not an accident, and holds more generally, but that is outside the scope of this problem.

$$= \frac{d}{dt} (c_1 e^{\lambda_1 t} + c_2 e^{\lambda_2 t}) \quad (30)$$

$$= \lambda_1 c_1 e^{\lambda_1 t} + \lambda_2 c_2 e^{\lambda_2 t}. \quad (31)$$

But we know that

$$x_2(t) = c_3 e^{\lambda_1 t} + c_4 e^{\lambda_2 t}. \quad (32)$$

Thus by pattern matching the coefficients of $e^{\lambda_1 t}$ and $e^{\lambda_2 t}$, we get

$$c_3 = \lambda_1 c_1 \quad c_4 = \lambda_2 c_2. \quad (33)$$

Now to solve for c_1 and c_2 . Recall that in steady state, a capacitor looks like an open circuit, so $V_C(0) = V_s$. By definition, $V_C(t) = x_1(t)$, so $x_1(0) = V_s$. Plugging in, we have

$$V_s = x_1(0) = c_1 e^{\lambda_1 \cdot 0} + c_2 e^{\lambda_2 \cdot 0} = c_1 + c_2. \quad (34)$$

Now we have one equation in the variables c_1 and c_2 . To solve the system we need two equations. This motivates looking at

$$x_2(0) = \lambda_1 c_1 e^{\lambda_1 \cdot 0} + \lambda_2 c_2 e^{\lambda_2 \cdot 0} = \lambda_1 c_1 + \lambda_2 c_2. \quad (35)$$

To find the physical value of $x_2(0) = \left. \frac{d}{dt} V_C(t) \right|_{t=0}$, note that in steady state there is no change in any state variable by definition, so $\left. \frac{d}{dt} V_C(t) \right|_{t=0} = 0$. (An alternate physically motivated argument is to note that inductor current in steady state is $I_L = 0$, and it cannot change infinitely fast, so at time 0 we have $I_L(0) = 0$. Since $I_L(t) \propto \frac{dV_C(t)}{dt}$, we also have $\left. \frac{dV_C(t)}{dt} \right|_{t=0} = 0$.) Hence $x_2(0) = 0$. This sets up the system of equations

$$c_1 + c_2 = V_s \quad (36)$$

$$\lambda_1 c_1 + \lambda_2 c_2 = 0. \quad (37)$$

There are several ways we can solve this system, and one way is to note that this is a matrix-vector equation of the form

$$\begin{bmatrix} 1 & 1 \\ \lambda_1 & \lambda_2 \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \end{bmatrix} = \begin{bmatrix} V_s \\ 0 \end{bmatrix}. \quad (38)$$

To solve it, we can use the matrix inverse that was provided by the hint to get

$$\begin{bmatrix} c_1 \\ c_2 \end{bmatrix} = \begin{bmatrix} 1 & 1 \\ \lambda_1 & \lambda_2 \end{bmatrix}^{-1} \begin{bmatrix} V_s \\ 0 \end{bmatrix} \quad (39)$$

$$= \frac{1}{\lambda_2 - \lambda_1} \begin{bmatrix} \lambda_2 & -1 \\ -\lambda_1 & 1 \end{bmatrix} \begin{bmatrix} V_s \\ 0 \end{bmatrix} \quad (40)$$

$$= \frac{1}{\lambda_2 - \lambda_1} \begin{bmatrix} \lambda_2 \\ -\lambda_1 \end{bmatrix} V_s. \quad (41)$$

Thus we have

$$c_1 = \frac{\lambda_2}{\lambda_2 - \lambda_1} V_s, \quad c_2 = -\frac{\lambda_1}{\lambda_2 - \lambda_1} V_s. \quad (42)$$

We have found $\lambda_1, \lambda_2, c_1, c_2$, so by substituting into eq. (10) we have solved for $x_1(t) = V_C(t)$!

3. Designing Filters

In the lab, we will design various filter circuits using low-pass, high-pass, and band-pass filter elements. In this problem, we will walk through the use cases of these filter elements.

- (a) First, you remember that you saw in lecture that you can build simple filters using a resistor and a capacitor. **Design a simple first-order passive low-pass filter with the following specification using a $1\ \mu\text{F}$ capacitor.** ("Passive" means that the filter does not require any power supply to operate on the input signal. Passive components include resistors, capacitors, inductors, diodes, etc., while an example of an active component would be an op-amp).

- Low-pass filter: cut-off frequency $f_c = 2400\ \text{Hz}$, $\omega_c = 2\pi \cdot 2400\ \frac{\text{rad}}{\text{s}}$. Hz can be interpreted as "cycles/sec", and $\frac{\text{rad}}{\text{s}}$ can be interpreted as " 2π radians/cycle".

Recall that the cutoff-frequency of such a filter is just where the magnitude of the filter is $\frac{1}{\sqrt{2}}$ of its peak value.

Show your work to find the resistor value that creates this low-pass filter. Draw the schematic-level representation of your design. Please mark V_{in} , V_{out} , and the ground node(s) in your schematic. Round your results to two significant figures.

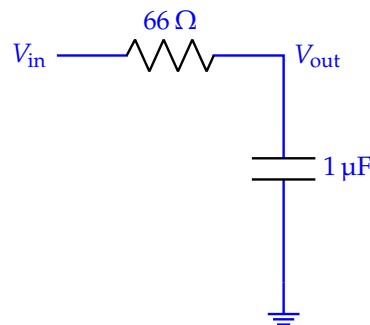
Solution: Low-pass filter

$$\omega_c = 2\pi f_c = \frac{1}{RC} \quad (43)$$

$$f_c = \frac{1}{2\pi RC} = 2400\ \text{Hz} \quad (44)$$

$$R = \frac{1}{2\pi \cdot 1\ \mu\text{F} \cdot 2400\ \text{Hz}} = 66\ \Omega \quad (45)$$

Therefore, we need a $66\ \Omega$ resistor.



- (b) Now design a simple first-order passive high-pass filter with the following specification using a $1\ \mu\text{F}$ capacitor.

- High-pass filter: cut-off frequency $f_c = 100\ \text{Hz}$, $\omega_c = 2\pi \cdot 100\ \frac{\text{rad}}{\text{s}}$

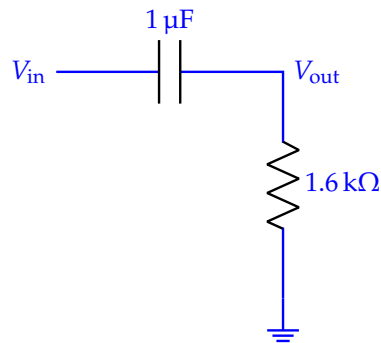
Show your work to find the resistor value that creates this high-pass filter. Draw the schematic-level representation of your design. Please mark V_{in} , V_{out} , and the ground node(s) in your schematic. Round your results to two significant figures.

Solution: High-pass filter

$$f_c = \frac{1}{2\pi RC} = 100\ \text{Hz} \quad (46)$$

$$R = \frac{1}{2\pi \cdot 1\ \mu\text{F} \cdot 100\ \text{Hz}} = 1.6\ \text{k}\Omega \quad (47)$$

Therefore, we need a $1.6\ \text{k}\Omega$ resistor. Note that we want a 24 times lower frequency, which means a 24 times higher time constant, which means a 24 times higher resistor.

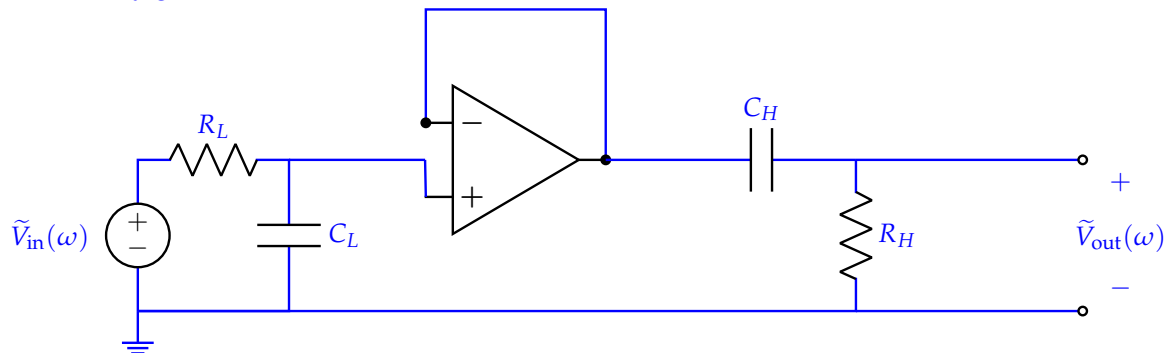


- (c) You can try to build a bandpass filter by cascading the first-order low-pass and high-pass filters you designed in parts (a) and (b). To do this, you might be tempted to connect the V_{out} node of your low-pass filter directly to the V_{in} node of your high-pass filter. However, if you did this, just as you saw in 16A for voltage dividers, the purported high-pass filter would “load” the low-pass filter and you might get some potentially complicated mess instead of what you wanted.

Show how you can use an ideal op-amp configured as a unity gain buffer to eliminate this loading effect to cascade the low-pass and high-pass filters, and write the resulting transfer function of the combined circuit. Draw the magnitude and phase transfer functions of the combined circuit. What kind of filter is this? You can optionally use the included Jupyter notebook `plot_tf.ipynb`. (HINT: Read Section 2.1 in [Note 7](#).)

(NOTE: In Python, use `1j` when your transfer function has a `j`.)

Solution: Consider the circuit given below, which is the low pass and the high pass, connected with a unity gain buffer:



We know that when we cascade circuits, the combined transfer function is the multiplication of the individual elements. For the Low Pass Filter $H_L(j\omega)$, Unity Gain Buffer $H_{unity}(j\omega)$, and High Pass Filter $H_H(j\omega)$.

$$H(j\omega) = H_L(j\omega) \cdot H_{unity}(j\omega) \cdot H_H(j\omega) \quad (48)$$

And we know that:

$$H_L(j\omega) = \frac{1}{1 + j\omega R_L C_L}, \quad H_{unity}(j\omega) = 1, \quad H_H(j\omega) = \frac{j\omega R_H C_H}{1 + j\omega R_H C_H} \quad (49)$$

Combining the transfer functions, we get:

$$H(j\omega) = \frac{1}{(1 + j\omega R_L C_L)} \cdot \frac{j\omega R_H C_H}{(1 + j\omega R_H C_H)} \quad (50)$$

The magnitude and phase transfer functions are shown below. We can see that this is a band pass filter.

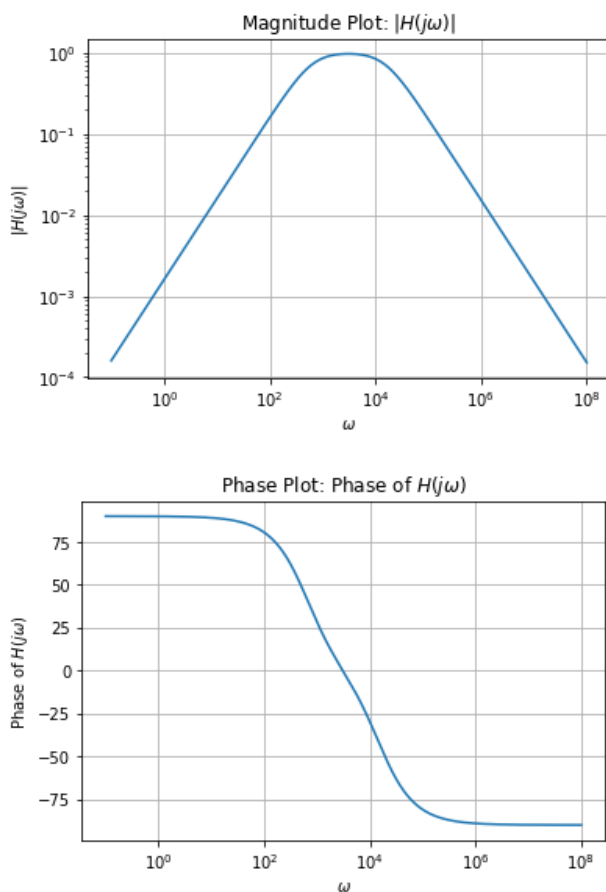


Figure 2: Magnitude and Phase transfer functions

- (d) Write down an expression for the time-domain output waveform $V_{\text{out}}(t)$ of this filter if the input voltage is $V_{\text{in}}(t) = 1 \sin(1000t)$ V. Round your answer to 2 significant digits.

Solution: We can find the transfer function at this point:

$$\left| H(j\omega = j10^3) \right| = 0.85 \quad (51)$$

$$\angle H(j\omega = j10^3) = 0.49 \text{ rad} = 28.23^\circ \quad (52)$$

Therefore the output will be:

$$V_{\text{out}}(t) = 0.85 \sin(1000t + 0.49) \text{ V.} \quad (53)$$

4. Phasors and Eigenvalues

Suppose that we have the two-dimensional system of differential equations expressed in matrix/vector form:

$$\frac{d}{dt}\vec{x}(t) = A\vec{x}(t) + \vec{b}u(t) \quad (54)$$

where for this problem, the matrix A and the vector \vec{b} are both real.

- (a) Give a necessary condition on the eigenvalues λ_k of A such that any impact of an initial condition will eventually completely die out. (i.e. the system will reach steady-state.)

You don't have to prove this. The idea here is to make sure that you understand what kind of thing is required. (*HINT: Read Section 2 in Note 5.*)

Solution: (Recall how the diagonalization we have done in the past takes us to an coordinate system where the matrix representing the differential equation has only diagonal entries being the eigenvalues, corresponding to differential equations of the form $\frac{d}{dt}z(t) = \lambda z(t)$.)

The condition is that all eigenvalues must have real parts that are less than zero. In equations

$$\forall k, \operatorname{Re}(\lambda_k) < 0 \quad (55)$$

This condition derives from the fact that the solutions to differential equations in the eigenspace contain terms that look like $e^{\lambda_k t}$. So, if all the eigenvalues are have strictly negative real parts, then all such exponential terms will die out.

If any of the eigenvalues have strictly positive real parts, then the exponential terms corresponding to them will blow up as growing exponentials.

The case of $\lambda = 0$ or having a zero real part in general (purely imaginary eigenvalues) is a little more ambiguous in feeling. This suggests that some constant offset (for the case of $\lambda = 0$) or some steady oscillation at a natural frequency of the system can persist throughout all time. But persisting isn't dying out and so we really want the eigenvalues to have strictly negative real parts for us to be able to ignore the initial conditions.

The argument above implicitly assumes that we can find enough linearly independent eigenvectors to get a basis. But what if we can't? We will explicitly address that case later in the course, but so far, we have seen in the cases that we have explored that what seems to happen is that even in the new basis, we seem to get a copy of an existing eigenvalue showing up again. This gives us some confidence that the condition that we are expressing is probably the right one, but we aren't fully sure yet since we have no proof that covers not having enough eigenvectors. We also know that these kinds of "not enough eigenvectors" cases can occur in physical circuits, since we saw the critically damped case in a previous homework.

- (b) Now assume that $u(t)$ has a phasor representation \tilde{U} . In other words, $u(t) = \tilde{U}e^{j\omega t} + \overline{\tilde{U}}e^{-j\omega t}$. Assume that the vector solution $\vec{x}(t)$ to the system of differential equations (54) can also be written in phasor form as

$$\vec{x}(t) = \tilde{\vec{X}}e^{j\omega t} + \overline{\tilde{\vec{X}}}e^{-j\omega t}. \quad (56)$$

Derive an expression for $\tilde{\vec{X}}$ involving $A, \vec{b}, j\omega, \tilde{U}$, and the identity matrix I . (Here, we assume that $j\omega$, and $-j\omega$ are not eigenvalues of A , which indicates that $\det(j\omega I - A)$ and $\det(-j\omega I - A)$ are non-zero.)

Solution: As the hint suggests, plugging back (56) into (54) we get the following:

$$\frac{d}{dt} \left(\tilde{\vec{X}}e^{j\omega t} + \overline{\tilde{\vec{X}}}e^{-j\omega t} \right) = A(\tilde{\vec{X}}e^{j\omega t} + \overline{\tilde{\vec{X}}}e^{-j\omega t}) + \vec{b}(\tilde{U}e^{j\omega t} + \overline{\tilde{U}}e^{-j\omega t}) \quad (57)$$

$$(j\omega \tilde{\vec{X}}e^{j\omega t} - j\omega \overline{\tilde{\vec{X}}}e^{-j\omega t}) = (A\tilde{\vec{X}} + \vec{b}\tilde{U})e^{j\omega t} + (A\overline{\tilde{\vec{X}}} + \vec{b}\overline{\tilde{U}})e^{-j\omega t} \quad (58)$$

$$(59)$$

Note that \vec{X} and \tilde{U} do not depend on time since they are phasors. Next, we can group the coefficients with the same exponential terms,

$$j\omega\vec{X} = A\vec{X} + \vec{b}\tilde{U} \quad (60)$$

$$-j\omega\vec{X} = A\vec{X} + \vec{b}\tilde{U} \quad (61)$$

$$\Rightarrow \overline{(j\omega)}\vec{X} = \overline{(A\vec{X} + \vec{b}\tilde{U})} \quad (62)$$

$$\Rightarrow (j\omega)\vec{X} = (A\vec{X} + \vec{b}\tilde{U}) \quad (63)$$

We see that equations (60) and (63) match, which is good. Note that, here we are assuming A and \vec{b} are real. Next, we can solve (60) to get \vec{X} :

$$j\omega\vec{X} = A\vec{X} + \vec{b}\tilde{U} \quad (64)$$

$$\Rightarrow (j\omega I - A)\vec{X} = \vec{b}\tilde{U} \quad (65)$$

$$\Rightarrow \vec{X} = (j\omega I - A)^{-1}\vec{b}\tilde{U}. \quad (66)$$

Notice that we didn't need to explicitly deal with the conjugate terms. We know that their solution is just going to be the conjugate of what we computed here, because of the properties of complex arithmetic.

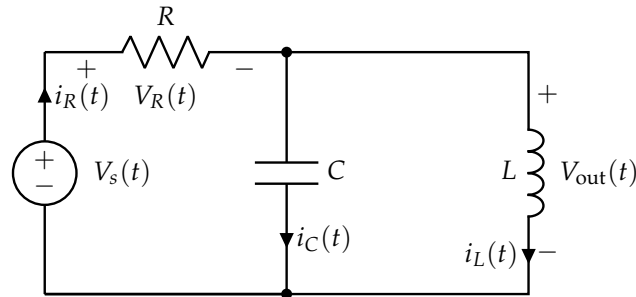
It turns out that it is possible to invert a general matrix M by writing it as some matrix M_c (that depends on M) divided by the determinant of M . (This is a fact related to something called the adjoints of matrices that are studied when one considers a combinatorial perspective on determinants, and thinks about things that are sometimes called "cofactors".) This is not something that is covered in 16AB because it cannot be proved at the level of mathematical maturity that is fair to assume for courses at this level.

However, the above linear-algebraic fact has a consequence for transfer functions. It tells you that all the polynomial terms in the denominators of the transfer functions are going to have the eigenvalues of the system as their roots. Why? Because the roots of $\det(j\omega I - A)$ tell you the eigenvalues of A . In later courses like 120, 105, 140, and beyond, you will see these roots of the denominators referred to as "poles" based on terminology from complex analysis. When you see them, understand that they are just the eigenvalues of the system in disguise. When you see conversations in later courses (or in your job or research) about understanding the placement of poles, understand that what is being talked about is where the relevant eigenvalues of the system are.

5. Phasor-Domain Circuit Analysis

The analysis techniques you learned previously in 16A for resistive circuits are equally applicable for analyzing circuits driven by sinusoidal inputs in the phasor domain. In this problem, we will walk you through the steps with a concrete example.

Consider the following circuit where the input voltage is sinusoidal. The end goal of our analysis is to find an equation for $V_{\text{out}}(t)$.



The components in this circuit are given by:

$$V_s(t) = 10\sqrt{2} \cos\left(100t - \frac{\pi}{4}\right) \quad (67)$$

$$R = 5 \Omega \quad (68)$$

$$L = 50 \text{ mH} \quad (69)$$

$$C = 2 \text{ mF} \quad (70)$$

- (a) Give the amplitude V_0 , input frequency ω , and phase ϕ of the input voltage V_s .

Solution: A sinusoid takes the form $v(t) = V_0 \cos(\omega t + \phi)$. Given $V_s(t)$, we find:

$$V_0 = 10\sqrt{2} \text{ volt} \quad (71)$$

$$\omega = 100 \text{ rad/sec} \quad (72)$$

$$\phi = -\frac{\pi}{4} \text{ rad} \quad (73)$$

- (b) Transform the circuit into the phasor domain. What are the impedances of the resistor, capacitor, and inductor? What is the phasor \tilde{V}_s of the input voltage $V_s(t)$?

Solution:

$$Z_L = j\omega L = j5\Omega \quad (74)$$

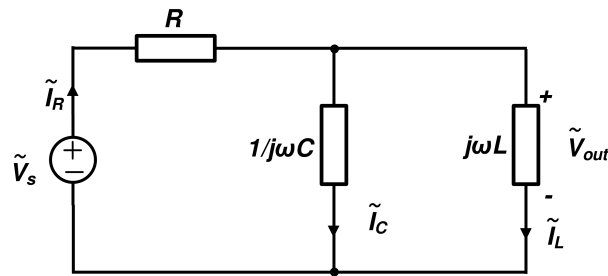
$$Z_C = \frac{1}{j\omega C} = -j5\Omega \quad (75)$$

$$Z_R = R = 5\Omega \quad (76)$$

$$\tilde{V}_s = \frac{|V_s|}{2} e^{j\angle V_s} = 5\sqrt{2} e^{-j\frac{\pi}{4}} \quad (77)$$

- (c) Use the circuit equations to **solve for** \tilde{V}_{out} , the phasor representing the output voltage.

Solution: The phasor representation of the circuit is shown below:



Where

$$\tilde{I}_R = \frac{\tilde{V}_S - \tilde{V}_{out}}{R} \quad (78)$$

$$\tilde{I}_L = \frac{\tilde{V}_{out}}{j\omega L} \quad (79)$$

$$\tilde{I}_C = \tilde{V}_{out} \cdot j\omega C \quad (80)$$

Rewriting the current relation in terms of voltage phasors gives:

$$\frac{\tilde{V}_S - \tilde{V}_{out}}{R} = \frac{\tilde{V}_{out}}{j\omega L} + \tilde{V}_{out} \cdot j\omega C \quad (81)$$

Substituting the component values in the above equation we get

$$\frac{\tilde{V}_S - \tilde{V}_{out}}{5} = \frac{\tilde{V}_{out}}{5j} + \frac{\tilde{V}_{out} \cdot j}{5} \quad (82)$$

$$= \frac{\tilde{V}_{out}}{5j} - \frac{\tilde{V}_{out}}{5j} \quad (83)$$

$$= 0 \quad (84)$$

Which gives:

$$\tilde{V}_{out} = \tilde{V}_S \quad (85)$$

We found that $\tilde{V}_{out} = \tilde{V}_S$ because this circuit is in resonance; i.e., the capacitor and inductor have the exact values that cause current and voltage to endlessly oscillate between them at this frequency. If we chose a different value for ω with these same component values, the circuit would not be in resonance and \tilde{V}_{out} and \tilde{V}_S would no longer be equal.

One may think that this answer seems weird. For \tilde{V}_{out} to equal \tilde{V}_S means that no current is flowing through the resistor. This means that somehow, the impedance of the parallel L and C combination would have to be infinity. Let's check what that is:

$$Z_L \parallel Z_C = \frac{(j5) \cdot (-j5)}{j5 + (-j5)} = +\infty \quad (86)$$

Wow! Indeed it is infinity. This shows something counterintuitive that can occur with phasors and impedances. For resistors, one may think that parallel connections always lower the resistance. However, since imaginary impedances can be positive imaginary and negative imaginary, a parallel connection can make the impedance bigger or smaller. The same kind of counterintuitive behavior is also possible for series combinations. Resistors in series always increase the

resistance. But the same L and C in series can combine to have a zero impedance at the natural frequency.

If one wants to know why **something divided by 0** is ∞ in the complex plane, read this Wiki article: [Riemann Sphere](#). This is another facet of complex analysis, and why engineers were drawn to it when modeling physical systems for design purposes.

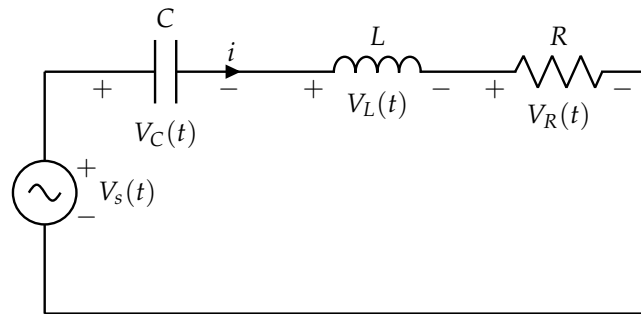
- (d) **Convert the phasor \tilde{V}_{out} back to get the time-domain signal $V_{\text{out}}(t)$.**

Solution: Since $\tilde{V}_{\text{out}} = \tilde{V}_S$,

$$v_{\text{out}}(t) = 10\sqrt{2} \cos\left(100t - \frac{\pi}{4}\right) \quad (87)$$

6. RLC filter

Consider the following RLC circuit:



- (a) Write down the impedance of a series RLC circuit in the form $Z_{RLC}(j\omega) = X(\omega) + jY(\omega)$, where $X(\omega)$ and $Y(\omega)$ are real-valued functions of ω . (HINT: Rationalize denominators before simplifying by collecting real and imaginary like terms.)

Solution: Since the capacitor, resistor, and inductor are in series, the equivalent impedance is given by

$$Z_{RLC}(j\omega) = R + Z_L(j\omega) + Z_C(j\omega) \quad (88)$$

$$\implies Z_{RLC}(j\omega) = R + j\omega L + \frac{1}{j\omega C} \quad (89)$$

Rationalizing the denominator,

$$\frac{1}{j} = -j \quad (90)$$

$$Z_{RLC}(j\omega) = R + j(\omega L - \frac{1}{\omega C}) \quad (91)$$

Hence,

$$X(\omega) = R \quad (92)$$

and

$$Y(\omega) = \omega L - \frac{1}{\omega C} \quad (93)$$

- (b) Write the transfer function from V_S to V_R — the voltage drop across the resistor.

Solution: For an impedance divider, we know that:

$$\tilde{V}_{out}(j\omega) = \frac{R}{Z_{RLC}(j\omega)} \tilde{V}_{in}(j\omega) = \frac{R}{R + j(\omega L - \frac{1}{\omega C})} \tilde{V}_{in}(j\omega) \quad (94)$$

Let us define the transfer function $H(j\omega)$ as:

$$H(j\omega) = \frac{\tilde{V}_{out}(j\omega)}{\tilde{V}_{in}(j\omega)} \quad (95)$$

We then get

$$H(j\omega) = \frac{R}{R + j(\omega L - \frac{1}{\omega C})} \quad (96)$$

To simplify, we can divide through by R to get:

$$H(j\omega) = \frac{1}{1 + j(\omega \frac{L}{R} - \frac{1}{\omega RC})} \quad (97)$$

(c) Find the magnitude $|H(j\omega)|$ and phase $\angle H(j\omega)$ of the transfer function in terms of ω, R, L, C .

Solution: From the last part, our transfer function is:

$$H(j\omega) = \frac{1}{1 + j(\omega \frac{L}{R} - \frac{1}{\omega RC})} \quad (98)$$

We can find the magnitude $|H(j\omega)|$ and phase $\angle H(j\omega)$ by pattern matching our $H(j\omega)$ with the results we found in lecture. For completeness, let us solve for them here.

For the magnitude $|H(j\omega)|$:

$$\begin{aligned} |H(j\omega)| &= \left| \frac{1}{1 + j(\omega \frac{L}{R} - \frac{1}{\omega RC})} \right| \\ &= \frac{|1|}{\left| 1 + j(\omega \frac{L}{R} - \frac{1}{\omega RC}) \right|} \\ &= \frac{1}{\sqrt{(1 + j(\omega \frac{L}{R} - \frac{1}{\omega RC}))(1 - j(\omega \frac{L}{R} - \frac{1}{\omega RC}))}} \\ &= \frac{1}{\sqrt{1 + (\omega \frac{L}{R} - \frac{1}{\omega RC})^2}} \end{aligned} \quad (99)$$

For the phase $\angle H(j\omega)$, we can rationalize the denominator of $H(j\omega)$ to re-write it in a form $H(j\omega) = X(\omega) + jY(\omega)$, where $X(\omega)$ and $Y(\omega)$ are real-valued functions representing the real and imaginary parts of $H(j\omega)$:

$$\begin{aligned} H(j\omega) &= \frac{1}{1 + j(\omega \frac{L}{R} - \frac{1}{\omega RC})} \\ &= \frac{1}{1 + j(\omega \frac{L}{R} - \frac{1}{\omega RC})} \times \frac{1 - j(\omega \frac{L}{R} - \frac{1}{\omega RC})}{1 - j(\omega \frac{L}{R} - \frac{1}{\omega RC})} \\ &= \frac{1 - j(\omega \frac{L}{R} - \frac{1}{\omega RC})}{1 + (\omega \frac{L}{R} - \frac{1}{\omega RC})^2} \\ &= \frac{1}{1 + (\omega \frac{L}{R} - \frac{1}{\omega RC})^2} + j \frac{-(\omega \frac{L}{R} - \frac{1}{\omega RC})}{1 + (\omega \frac{L}{R} - \frac{1}{\omega RC})^2} \end{aligned} \quad (100)$$

We recognize that we can write $H(j\omega) = X(\omega) + jY(\omega)$ where:

$$X(\omega) = \frac{1}{1 + (\omega \frac{L}{R} - \frac{1}{\omega RC})^2} \quad (101)$$

$$Y(\omega) = -\frac{\omega \frac{L}{R} - \frac{1}{\omega RC}}{1 + (\omega \frac{L}{R} - \frac{1}{\omega RC})^2} \quad (102)$$

We know that $\angle H(j\omega) = \text{atan2}(Y(\omega), X(\omega))$, or more simply, $\angle H(j\omega) = \text{atan2}(Y(\omega)/X(\omega), 1)$. Hence:

$$\begin{aligned} \angle H(j\omega) &= \text{atan2}\left(\frac{Y(\omega)}{X(\omega)}, 1\right) \\ &= \text{atan2}\left(\frac{-\frac{\omega \frac{L}{R} - \frac{1}{\omega RC}}{1 + (\omega \frac{L}{R} - \frac{1}{\omega RC})^2}}{\frac{1}{1 + (\omega \frac{L}{R} - \frac{1}{\omega RC})^2}}, 1\right) \end{aligned}$$

$$\begin{aligned}
 &= \text{atan2}\left(\frac{-(\omega\frac{L}{R} - \frac{1}{\omega RC})}{1}, 1\right) \\
 &= -\text{atan2}\left(\omega\frac{L}{R} - \frac{1}{\omega RC}, 1\right)
 \end{aligned} \tag{103}$$

Alternatively, we can find the phase $\angle H(j\omega)$ by recognizing that if $H(j\omega) = \frac{A(j\omega)}{B(j\omega)}$, then:

$$\angle H(j\omega) = \angle A(j\omega) - \angle B(j\omega) \tag{104}$$

(Can you prove this to yourself? Try representing $H(j\omega)$, $A(j\omega)$, and $B(j\omega)$ as phasors.)

Thus, we can simply solve for $\angle H(j\omega)$ as:

$$\begin{aligned}
 \angle H(j\omega) &= \angle 1 - \angle\left(1 + j\left(\omega\frac{L}{R} - \frac{1}{\omega RC}\right)\right) \\
 &= 0 - \text{atan2}\left(\omega\frac{L}{R} - \frac{1}{\omega RC}, 1\right) \\
 &= -\text{atan2}\left(\omega\frac{L}{R} - \frac{1}{\omega RC}, 1\right)
 \end{aligned} \tag{105}$$

Both approaches to solve for $\angle H(j\omega)$ are valid here and arrive to the same answer.

- (d) For the specific values for R, L, C given by overdamped and underdamped cases in the previous HW, use a computer to sketch plots of the magnitude and phase of the transfer function above. You can optionally use the included Jupyter notebook `plot_tf.ipynb`. Recall that the overdamped case had $R = 1\text{ k}\Omega, C = 10\text{ nF}, L = 25\text{ }\mu\text{H}$; underdamped had $R = 1\text{ }\Omega, C = 10\text{ nF}, L = 25\text{ }\mu\text{H}$. (NOTE: In Python, use `1j` when your transfer function has a `j`.)

Solution:

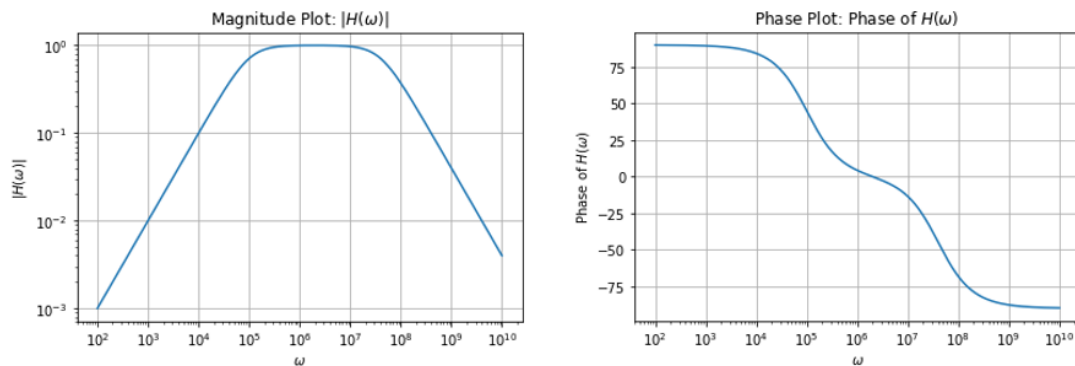


Figure 3: Magnitude and Phase transfer functions for overdamped case

$$\text{Overdamped } R = 1\text{ k}\Omega, C = 10\text{ nF}, L = 25\text{ }\mu\text{H} \tag{106}$$

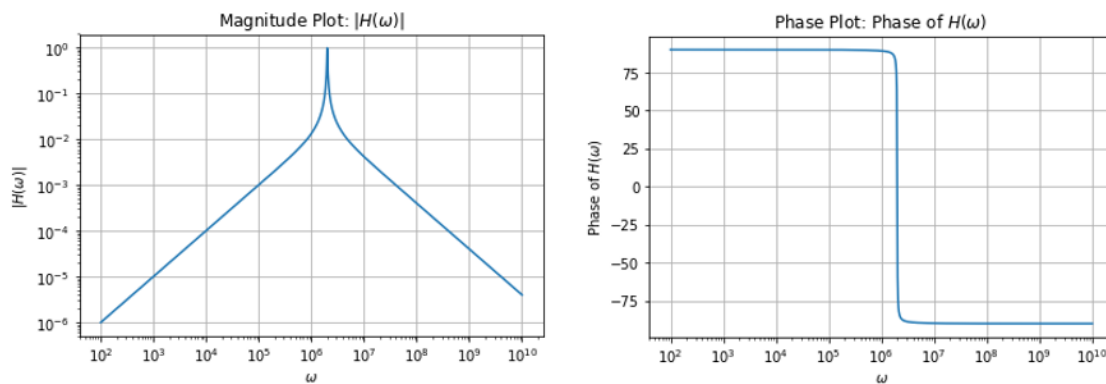


Figure 4: Magnitude and Phase transfer functions for underdamped case

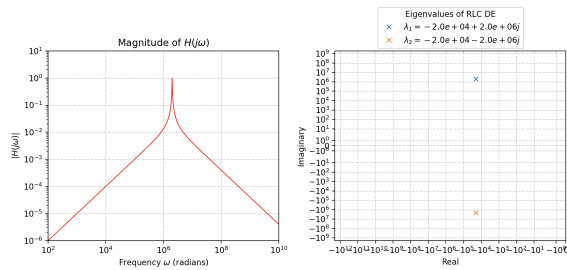
$$\text{Underdamped } R = 1 \Omega, C = 10 \text{ nF}, L = 25 \mu\text{H} \quad (107)$$

- (e) To see how the values of R, L, C impact the impedance at different frequencies, run the included Jupyter notebook `hw5rlc_transfer.ipynb`. The script will generate two plots, the transfer function of the circuit as a function of frequency and the location of the eigenvalues in the imaginary, real plane. **Explain qualitatively how the peak (resonant) frequency, the width of the peak, and the location of the eigenvalues depend on the resistance, inductance, and capacitance:**

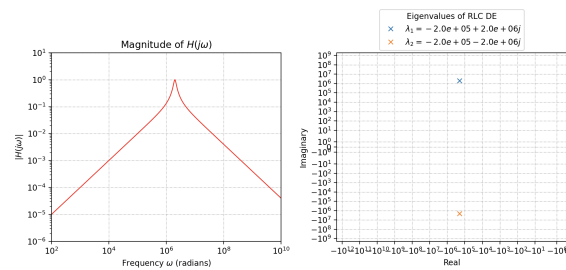
	R	L	C
I	1	2.5E-5	1E-8
II	10	2.5E-5	1E-8
III	10	2.5E-5	2E-9
IV	500	1.0E-4	2E-8

Table 1: Values for RLC Bandwidth problem, part e

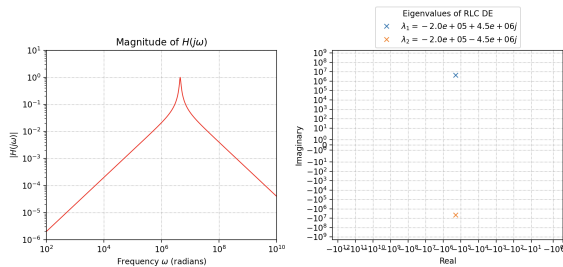
Solution: In this part, the values for I are default, then parts II, III, IV show how the impedance peak can change location and magnitude. Please see the figure below, then explanatory text in the caption.



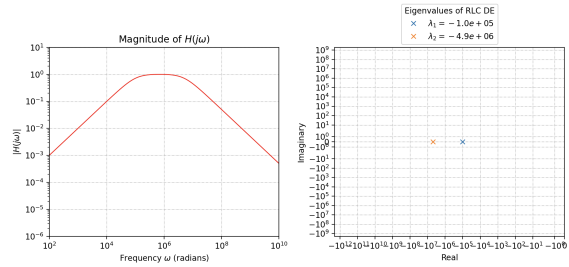
(a) $R = 1, L = 2.5e-5, C = 1e-8$



(b) $R = 10, L = 2.5e-5, C = 1e-8$



(c) $R = 10, L = 2.5e-5, C = 2e-8$



(d) $R = 500, L = 0.0001, C = 2e-8$

Figure 5: Here you can see how changing L, C changes the resonant frequency which determines where the peak is, and increasing R increases the width of the resonant peak. The eigenvalues location on the imaginary axis tells you where the peak is going to be, and the distance to the imaginary axis (real part) tells you how wide the peak is going to be.

7. Uniqueness justification for solutions to matrix/vector differential equations

In general, we have seen that we need to justify our methods of solving differential equations with a uniqueness proof. This is important as it allows us to trust our solution as being the only one for the problem at hand.

Consider matrix-vector differential equations of the form:

$$\frac{d}{dt}\vec{x}(t) = A\vec{x}(t) + \vec{b}u(t) \quad (108)$$

with some initial condition $\vec{x}(0) = \vec{x}_0$.

All the uniqueness proofs that you have done for yourself have been concerned with scalar differential equations, and scalar differential equations driven by inputs. So, why can we trust the solutions that we are getting for such matrix-vector differential equations?

This question takes us part of the way to the answer.

- (a) Suppose that the $n \times n$ matrix A has n distinct eigenvalues and corresponding eigenvectors $\vec{v}_1, \vec{v}_2, \dots, \vec{v}_n$, so that the matrix $V = [\vec{v}_1 \ \vec{v}_2 \ \dots \ \vec{v}_n]$ has linearly independent columns.

Find the diagonalized system corresponding to (108). Show that if you are given any valid solution for the original system (108), you can change coordinates to the eigenbasis and also get a valid solution for the diagonalized system. List the new initial conditions that are satisfied in the diagonalized system.

Solution: Because the eigenvector matrix V has linearly independent columns, we know it is invertible. We can therefore change the coordinates into the eigenbasis coordinates \vec{y} with the following conversions:

$$\vec{y} = V^{-1}\vec{x} \quad \vec{x} = V\vec{y} \quad (109)$$

We transform the original differential equation by substituting $\vec{x} = V\vec{y}$:

$$\frac{d}{dt}V\vec{y}(t) = AV\vec{y}(t) + \vec{b}u(t) \quad (110)$$

$$\frac{d}{dt}\vec{y}(t) = V^{-1}AV\vec{y}(t) + V^{-1}\vec{b}u(t) \quad (111)$$

$$\frac{d}{dt}\vec{y}(t) = \Lambda\vec{y}(t) + \vec{W}u(t) \quad (112)$$

where $\vec{W} = V^{-1}\vec{b}$ and $\Lambda = V^{-1}AV$.

We also transform the initial conditions into the eigenbasis:

$$\vec{y}(0) = \vec{y}_0 = V^{-1}\vec{x}_0 \quad (113)$$

Because the matrix V is composed of the eigenvectors of A , and is invertible, we know that $\Lambda = V^{-1}AV$ is the diagonal eigenvalue matrix, with each diagonal entry $\Lambda_{i,i}$ equal to the eigenvalue associated with the i^{th} eigenvector of V .

Because the Λ matrix is diagonal, the above matrix equation yields a collection of uncoupled scalar differential equations with initial conditions:

$$\frac{d}{dt}\vec{y}_k = \lambda_k\vec{y}_k + \vec{W}_k u(t), \quad \vec{y}_k(0) = \vec{y}_0[k] \quad (114)$$

for $k \in [1, 2, \dots, n]$ where the subscript indicates indexing into the vector.

Finally, we must show that if we are given a valid solution for (108), then it remains a valid solution for the collection of differential equations in the transformed coordinates (114). Let $\vec{x}_{\text{sol}}(t)$ satisfy (108) and the initial condition $\vec{x}_{\text{sol}}(0) = \vec{x}_0$.

We can transform this solution into the eigenbasis: $\vec{y}_{\text{sol}}(t) = V^{-1}\vec{x}_{\text{sol}}(t)$.

At $t = 0$, $\vec{y}_{\text{sol}}(0) = V^{-1}\vec{x}_{\text{sol}}(0) = V^{-1}\vec{x}_0$, so we see that the initial condition is satisfied in the eigenbasis.

We also check if the transformed solution $\vec{y}_{\text{sol}}(t)$ satisfies the transformed differential equation:

$$\frac{d}{dt}\vec{y}_{\text{sol}}(t) = \frac{d}{dt}V^{-1}\vec{x}_{\text{sol}}(t) = V^{-1}\frac{d}{dt}\vec{x}_{\text{sol}}(t) \quad (115)$$

$$= V^{-1}\left(A\vec{x}_{\text{sol}}(t) + \vec{b}u(t)\right) \quad (116)$$

$$= V^{-1}\left(AV\vec{y}_{\text{sol}}(t) + \vec{b}u(t)\right) \quad (117)$$

$$= V^{-1}AV\vec{y}_{\text{sol}}(t) + V^{-1}\vec{b}u(t) \quad (118)$$

$$= \Lambda\vec{y}_{\text{sol}}(t) + \vec{W}u(t) \quad (119)$$

We see that indeed $y_{\text{sol}}(t)$ satisfied the transformed differential equations. Thus a valid solution in the original basis produces a valid solution in the changed basis.

- (b) You have already proved the uniqueness of solutions for any scalar differential equation of the form $\frac{d}{dt}x(t) = \lambda x(t) + u(t)$ with specified initial condition $x(0) = x_0$. How can you use this fact and the result of the previous part to argue that the solution must be unique for the matrix/vector differential equation?

(HINT: (Start by assuming that you have two solutions to the original problem. Use the result of part a) to formulate solutions to the transformed, diagonal problem. Apply the uniqueness results for scalar ODEs to the solutions of the diagonalized problem. Finally conclude by considering what the invertibility of the transformation matrix V implies about the two solutions of the original problem?))

Solution: Two common approaches to show uniqueness (in general) are directly, and via contradiction. To show uniqueness directly here, we would consider two (not necessarily distinct) solutions that satisfy (108), and show that they are equal. This means that any two solutions of the differential equation are equal to each other, so there is only one possible solution. To show uniqueness by contradiction, we would consider two *distinct* solutions to (108), and show that they are equal, which is a contradiction. The end result is the same, but the overall logic is slightly different.

If we wanted to show uniqueness by contradiction, we could do the following: let us assume that the solution is not unique, so we have two distinct solutions in the original basis $\vec{x}_1(t)$ and $\vec{x}_2(t)$ which satisfy (108) and the initial condition. Because V has linearly independent columns, the matrix V is invertible. Thus, we can transform the two distinct original solutions into two distinct solutions in the eigenbasis:

$$\vec{y}_1(t) = V^{-1}\vec{x}_1(t) \quad \vec{y}_2(t) = V^{-1}\vec{x}_2(t) \quad \vec{y}_1(t) \neq \vec{y}_2(t) \quad (120)$$

We have already proved that the solutions for scalar equations of the form $\frac{d}{dt}x(t) = \lambda x(t) + u(t)$ with specified initial condition $x(0) = x_0$ are unique. In the previous part of the problem, we saw that we could transform the original matrix-vector differential equations into a collection of scalar equations of this form. Therefore, the solutions to these scalar equations must be unique.

This now leads to the contradiction: We know there can only be a single unique solution in the eigenbasis, but our initial assumption of two unique solutions in the original basis leads to two distinct solutions in the eigenbasis. Thus our assumption is not valid, and so there must be a unique solution.

If we wanted to show uniqueness directly, we would begin by assuming that we have two, not necessarily distinct, solutions in the original basis $\vec{x}_1(t)$ and $\vec{x}_2(t)$. We can again transform them into two solutions $\vec{y}_1(t) = V^{-1}\vec{x}_1(t)$ and $\vec{y}_2(t) = V^{-1}\vec{x}_2(t)$. We know that $\vec{y}_1(t)$ and $\vec{y}_2(t)$ satisfy the same diagonalized differential equation, which means each of their elements satisfy the same collection of scalar equations of the form $\frac{d}{dt}x(t) = \lambda x(t) + u(t)$. We know that the solutions to these differential equations are unique, so each element of $\vec{y}_1(t)$ and $\vec{y}_2(t)$ are the same. Then

$\vec{y}_1(t) = \vec{y}_2(t)$ as desired, so $\vec{x}_1(t) = V\vec{y}_1(t) = V\vec{y}_2(t) = \vec{x}_2(t)$. We conclude that the matrix differential equation (108) admits a unique solution.

Side note: what both of these approaches are getting at is that since V is invertible, the mapping from y to x via $x = Vy$ is bijective. That is, each x in the standard basis corresponds to exactly one y in the new basis. Then, since there is only one solution in the eigenbasis, there must be also exactly one solution in the standard basis.

We will see later in the course how the assumption we made on the eigenvectors of A is not actually needed for this proof to hold. But for now, it is important to understand this case first.

8. (OPTIONAL) Make Your Own Problem.

Write your own problem about content covered in the course thus far, and provide a thorough solution to it.

NOTE: This can be a totally new problem, a modification on an existing problem, or a Jupyter part for a problem that previously didn't have one. Please cite all sources for anything (including course material) that you used as inspiration.

NOTE: High-quality problems may be used as inspiration for the problems we choose to put on future homeworks or exams.

9. Homework Process and Study Group

Citing sources and collaborators are an important part of life, including being a student!

We also want to understand what resources you find helpful and how much time homework is taking, so we can change things in the future if possible.

- (a) **What sources (if any) did you use as you worked through the homework?**
- (b) **If you worked with someone on this homework, who did you work with?**
List names and student ID's. (In case of homework party, you can also just describe the group.)
- (c) **Roughly how many total hours did you work on this homework? Write it down here where you'll need to remember it for the self-grade form.**

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