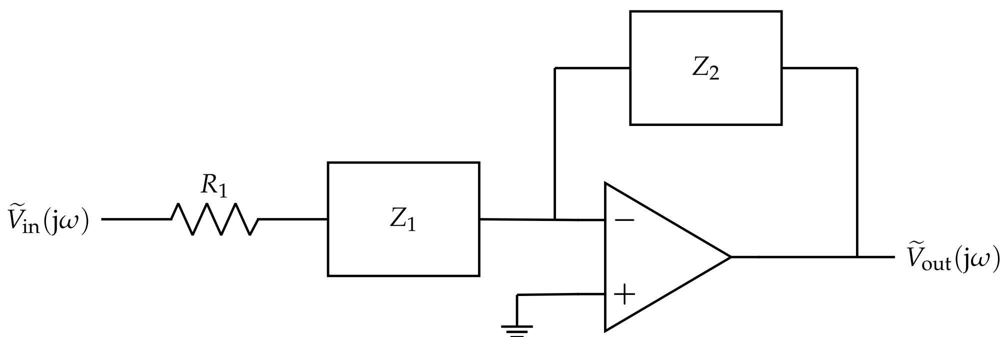
Homework 5

This homework is due on Saturday, September 30th, 2023, at 11:59PM. Self-grades and HW Resubmissions are due on the following Saturday, October 7th, 2023, at 11:59PM.

1. Circuit Design Part 2

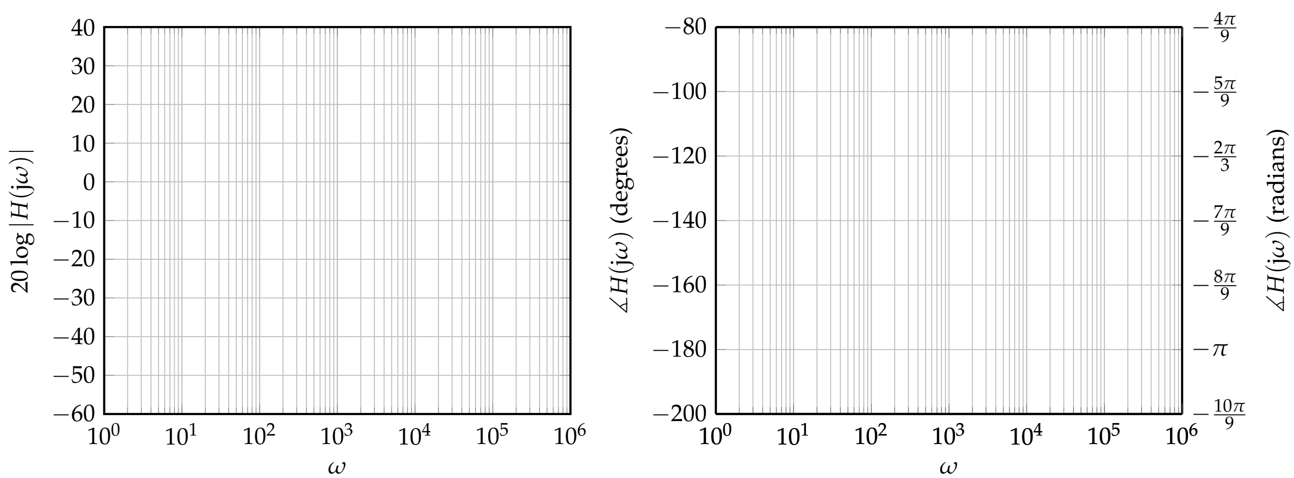
In the previous homework, you analyzed the following circuit in phasor domain:



You (hopefully) determined that $Z_2 = R = 10\,\text{k}\Omega$ and $Z_1 = \frac{1}{\text{j}\omega C}$ with $C = 1\,\mu\text{F}$ and $R_1 = 1\,\text{k}\Omega$. This gave you the following transfer function:

$$H(\text{j}\omega) = -\frac{R}{R_1} \cdot \frac{1}{1 - \frac{\text{j}}{\omega C R_1}} \tag{1}$$

(a) **Draw the magnitude and phase Bode plots (straight-line approximations to the transfer function) of this transfer function.** Blank plots are provided here for you to use.



Solution: The transfer function is

$$H(\text{j}\omega) = -10\frac{1}{1 - \frac{\text{j}\cdot 1000}{\omega}} \tag{2}$$

This is a first-order-high-pass filter with cut-off frequency $\omega_o = 1000\,\frac{\text{rad}}{\text{s}}$.

The magnitude is given by

$$|H(j\omega)| = 10\frac{1}{\sqrt{1+\frac{10^6}{\omega^2}}} \tag{3}$$

Hence we have the following properties for the magnitude Bode plot approximation:

i. At $\omega > \omega_o$, the magnitude is $20\log(10) = 20\,\text{dB}$.
ii. At $\omega < \omega_o$, the magnitude drops by 20 dB per decade of ω.

The phase is given by

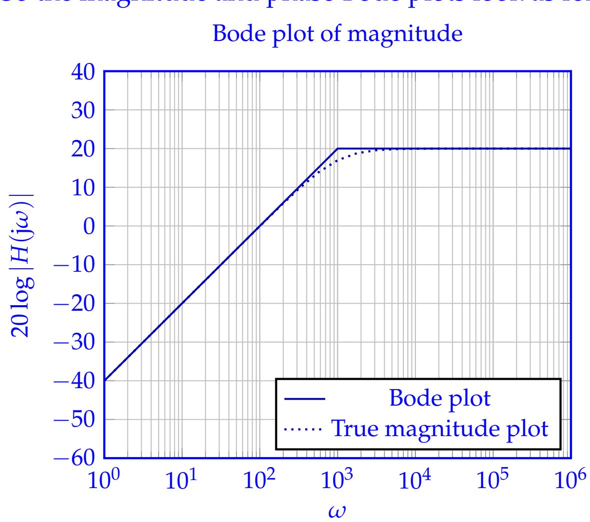
$$\angle(H(\omega)) = \angle(-10) - \angle\left(1 - \frac{j\cdot 1000}{\omega}\right) \tag{4}$$

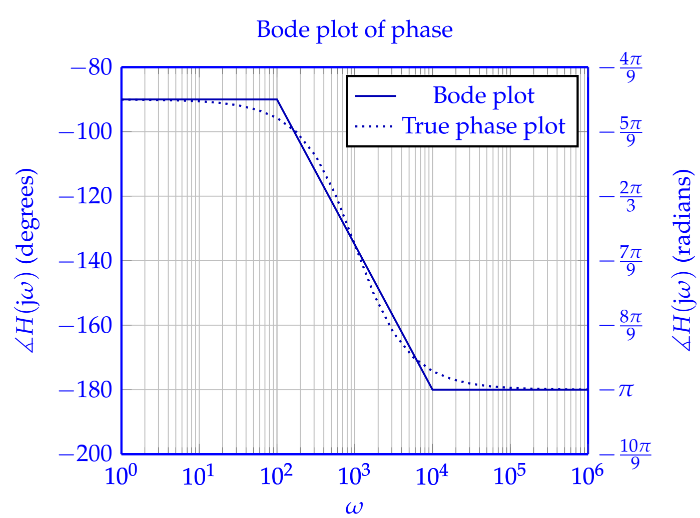
$$= -180^\circ - \text{atan2}\left(-\frac{1000}{\omega}, 1\right) \tag{5}$$

Hence we have the following properties for the phase Bode plot:

i. At $\omega < \frac{\omega_o}{10}$, the phase is $-180^\circ - \text{atan2}(-\infty, 1) = -90^\circ$.
ii. At $\omega > 10\omega_o$, the phase is $-180^\circ - \text{atan2}(-0, 1) = -180^\circ$.
iii. At $\omega = \omega_o$, the phase is $-180^\circ - \text{atan2}(-1, 1) = -135^\circ$.

So the magnitude and phase Bode plots look as follows:





Note that plotting the true magnitude/phase using a computer is not necessary for credit – they are just there to illustrate to you how our Bode plot approximations are close to the true plots at many frequencies. But you can see that there are some frequencies where the approximation is not so good. At these frequencies, if they are relevant to the problem you are doing, you should check with the true plot instead of the approximated one.

2. Hambley P6.53

A transfer function is given by

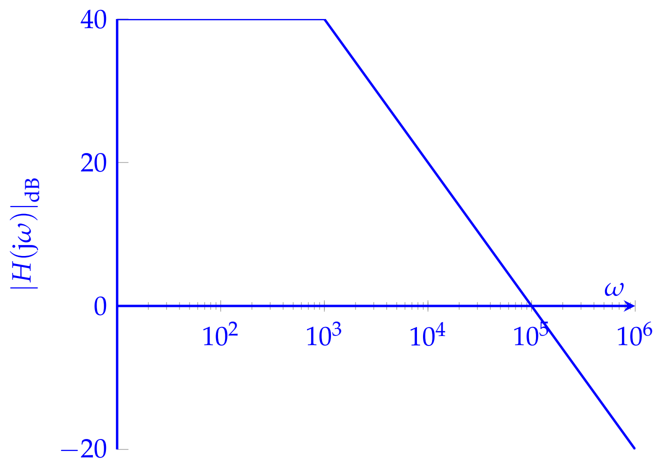
$$H(\mathrm{j}\omega) = \frac{100}{1 + \mathrm{j}\frac{\omega}{1000}} \tag{6}$$

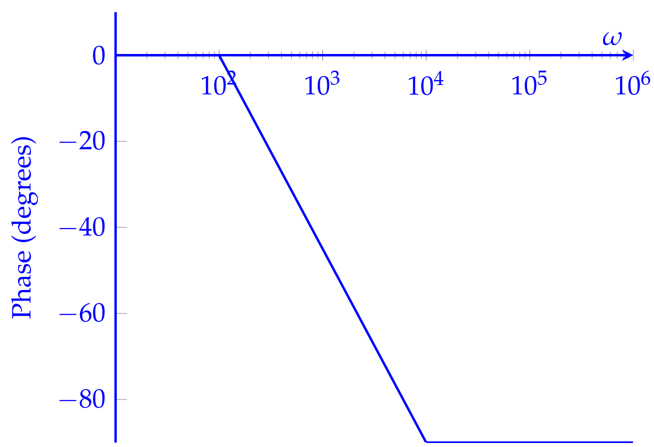
Sketch the asymptotic magnitude and phase Bode plots to scale. What is the value of the half-power frequency?

Solution: We have that

$$|H(\mathrm{j}\omega)| = \frac{100}{\sqrt{1 + \left(\frac{\omega}{1000}\right)^2}} \tag{7}$$

so $|H(\mathrm{j}\omega)|_{\mathrm{dB}} = 20\log(100) - 20\log\left(\sqrt{1+\left(\frac{\omega}{1000}\right)^2}\right) = 40 - 20\log\left(\sqrt{1+\left(\frac{\omega}{1000}\right)^2}\right)$. The half-power point is $\omega = 1000$. The asymptotic Bode plots are:





3. Hambley P6.55

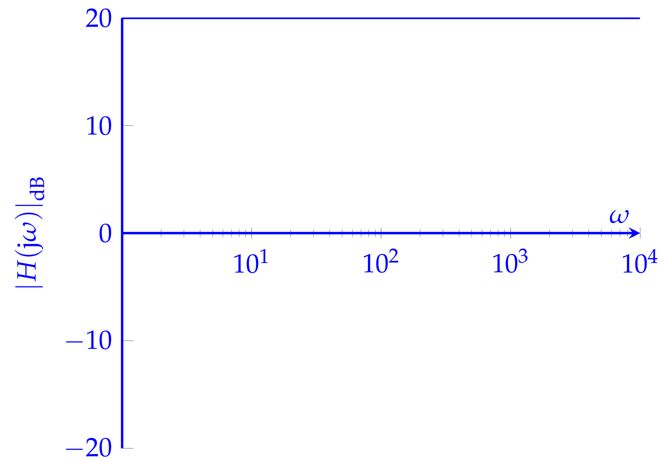
Sketch the asymptotic magnitude and phase Bode plots to scale for the transfer function

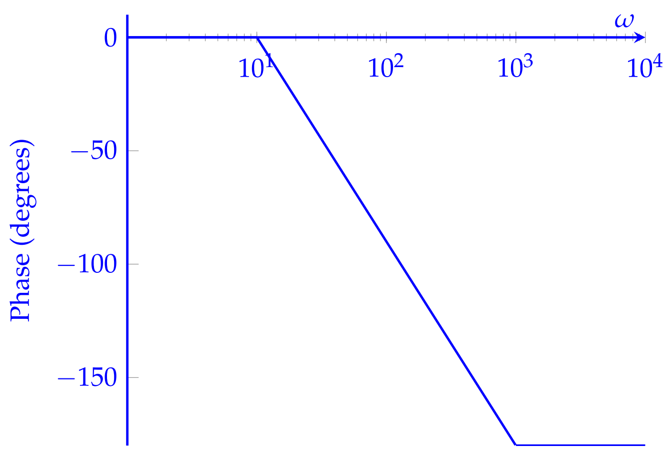
$$H(\mathrm{j}\omega) = 10\frac{1 - \mathrm{j}\frac{\omega}{100}}{1 + \mathrm{j}\frac{\omega}{100}} \tag{8}$$

Solution: We have that

$$|H(\mathrm{j}\omega)| = 10\frac{\sqrt{1 + \left(\frac{\omega}{100}\right)^2}}{\sqrt{1 + \left(\frac{\omega}{100}\right)^2}} = 10 \tag{9}$$

so $|H(\mathrm{j}\omega)|_{\mathrm{dB}} = 20\log(10) = 20$. The phase is $-2\arctan\left(\frac{\omega}{100}\right)$. The asymptotic Bode plots are:





4. Bandpass Filter: Lowpass and Highpass Cascade

Consider an input signal that is composed of the superposition of:

- $A_p := 20\,\text{mV}$ level pure tone at frequency $f_p := 60\,\text{Hz}$ and phase ϕ_p corresponding to power line noise.
- $A_v := 1\,\text{mV}$ level pure tone at frequency $f_v := 600\,\text{Hz}$ and phase ϕ_v corresponding to a voice signal.
- $A_f := 10\,\text{mV}$ level pure tone at frequency $f_f := 60\,\text{kHz}$ and phase ϕ_f corresponding to fluorescent light control electronics noise.

We would like to keep the 600 Hz tone, which could correspond to a voice signal.

NOTE: The phases ϕ are symbolic – we do not provide numerical values – but the amplitudes A are not symbolic.

(a) **Write the $V_{\text{in}}(t)$ that describes the above input in time domain, in the following format:**.

$$V_{\text{in}}(t) = A_p \cos(2\pi f_p t + \phi_p) + A_v \cos(2\pi f_v t + \phi_v) + A_f \cos\left(2\pi f_f t + \phi_f\right) \tag{10}$$

Solution: Each pure tone in the input signal corresponds to a cosine wave. All of these tones can be added up:

$$\begin{aligned} V_{\text{in}}(t) &= (20 \times 10^{-3}) \cos((2\pi \cdot 60)t + \phi_p) && (11)\\ &+ (1 \times 10^{-3}) \cos((2\pi \cdot 600)t + \phi_v) && (12)\\ &+ (10 \times 10^{-3}) \cos\left((2\pi \cdot 60000)t + \phi_f\right) && (13) \end{aligned}$$

(b) **What are the angular frequencies (i.e., $\omega_p, \omega_v, \omega_f$) involved and the phasors associated with each tone?** Remember that the frequencies of the tones are provided in Hz. To convert these frequencies to angular frequencies, we use $\omega = 2\pi f$.

NOTE: This scenario is common in applications; usually, the data collected is in "regular" frequencies, but the analysis requires angular frequencies.

Solution:

$$\begin{aligned} \text{Power line:} &\quad 376.99\,\tfrac{\text{rad}}{\text{s}}\\ \text{Voice:} &\quad 3769.9\,\tfrac{\text{rad}}{\text{s}}\\ \text{Fluorescent light:} &\quad 376\,991.1\,\tfrac{\text{rad}}{\text{s}} \end{aligned}$$

Recall that the phasor is defined as the term in front of $e^{j\omega t}$ in the complex exponential expansion, and that $V\cos(\omega t) = \frac{V}{2}e^{j\omega t} + \frac{V}{2}e^{-j\omega t}$.

$$\begin{aligned} \text{Power line:} &\quad \widetilde{V}_{\text{in},p} = (20 \times 10^{-3})e^{j\phi_p} = 20e^{j\phi_p} \times 10^{-3}\text{V}\\ \text{Voice:} &\quad \widetilde{V}_{\text{in},v} = (1 \times 10^{-3})e^{j\phi_v} = e^{j\phi_v} \times 10^{-3}\text{V}\\ \text{Fluorescent light:} &\quad \widetilde{V}_{\text{in},f} = (10 \times 10^{-3})e^{j\phi_f} = e^{j\phi_f} \times 10^{-2}\text{V} \end{aligned}$$

Note that from this part and beyond, we will be computing many values for $\widetilde{V}_{\text{in}}$, $H(\mathrm{j}\omega)$, ω_p, ω_v, ω_f, etc.. When self grading, as long as your calculations are correct, do not mark yourself down for how you round if it is different from what is here in the solutions, as long as your solutions are consistent with your previous answers.

(c) To achieve your goal of keeping the voice tone but rejecting the noise from the power-lines and fluorescent lights, **at what frequency do you want to have the cutoff frequency for the lowpass filters**?

(HINT: To arrive at a unique solution consider computing the geometric mean (the analogous quantity to the arithmetic mean on a log scale) of the two frequencies of interest.)

Solution: There are a wide range of possible cutoff-frequencies you can choose for the lowpass filter. You can choose any value in between the voice frequency and the fluorescent light frequency and get some filtering (and get credit). Your specific choice depends on the tradeoffs you may want to take. For example, if you choose a cutoff frequency close to the voice frequency, you will suppress the fluorescent light frequency well, but will also start to attenuate the voice frequency a little. If you choose a cutoff frequency close to the fluorescent light frequency, you will minimize attenuation of the voice frequency, but also have very little suppression of the fluorescent light frequency. Choosing this value in a principled way depends on more contextual information.

One reasonable solution (as was mentioned by the hint) is to choose a frequency that is the geometric mean of the voice and fluorescent frequencies. The geometric mean puts the cutoff frequency half way between the voice and fluorescent frequencies on a log scale:

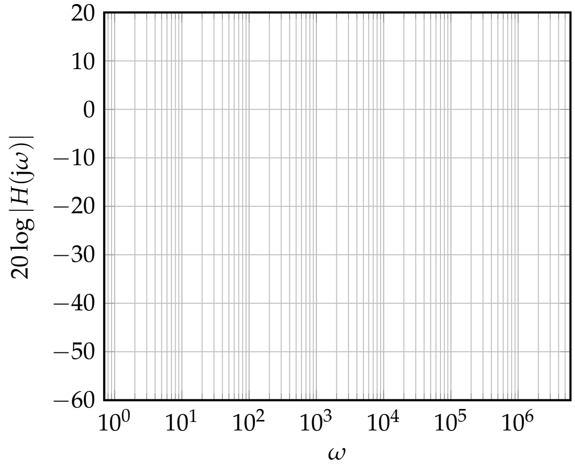
$$f_c = \sqrt{f_v \cdot f_f} = \sqrt{600 \cdot 60000}\,\mathrm{Hz} = 6000\,\mathrm{Hz} \qquad \omega_c = 2\pi f_c = 2\pi \cdot 6000\,\frac{\mathrm{rad}}{\mathrm{s}} = 37\,699.1\,\frac{\mathrm{rad}}{\mathrm{s}} \tag{14}$$

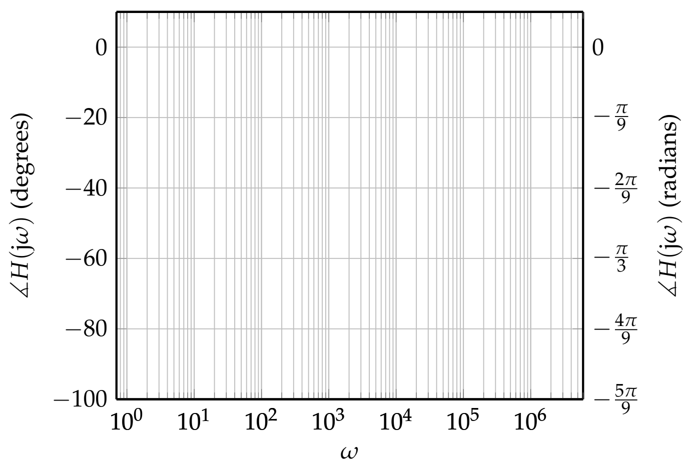
A second reasonable way to select a cutoff frequency is to note that since 600 Hz and 60 kHz are fairly far apart, we can simply choose a frequency 10x the voice frequency. This will minimize attenuation of the voice tone while still suppressing the fluorescent light tone:

$$f_c = f_v \cdot 10 = 6\,\mathrm{kHz} \qquad \omega_c = 2\pi \cdot f_c = 2\pi \cdot 6000\,\frac{\mathrm{rad}}{\mathrm{s}} = 37\,699.1\,\frac{\mathrm{rad}}{\mathrm{s}} \tag{15}$$

Coincidentally, this is the same frequency as the geometric mean approach for the numbers given here. In practice, in the absence of anything else, one uses the second approach (put the cutoff 10x away from the desirable frequencies) when the desirable and undesirable frequencies are very far apart, and the first approach (geometric means) when they are closer together.

(d) **Draw the Bode plot (straight-line approximations to the transfer function) for the magnitude (using $20\log|H(\mathrm{j}\omega)|$) and phase of the lowpass filter.**





Solution: We note that the y-axis is $20\log|H(j\omega)|$. Therefore, when we hit a cut-off frequency, the role-off slope will be -20 dB instead of -1, which means that per each decade of movement on the ω-axis, our y-axis drops by 20 units.

The magnitude is given by $|H(j\omega)| = \frac{1}{\sqrt{1+\frac{\omega^2}{\omega_c^2}}}$ where $\omega_c = 37\,699.1\,\frac{\text{rad}}{\text{s}}$. Hence we have the following properties for the magnitude Bode plot approximation:

i. At $\omega < \omega_c$, the magnitude is $20\log(1) = 0$ dB.

ii. At $\omega > \omega_c$, the magnitude drops by 20 dB per decade of increase in ω.

The phase is given by

$$\measuredangle(H(j\omega)) = \measuredangle(1) - \measuredangle\left(1 + \frac{j\cdot\omega}{\omega_c}\right) \tag{16}$$

$$= 0^\circ - \operatorname{atan2}\left(\frac{\omega}{\omega_c}, 1\right) \tag{17}$$

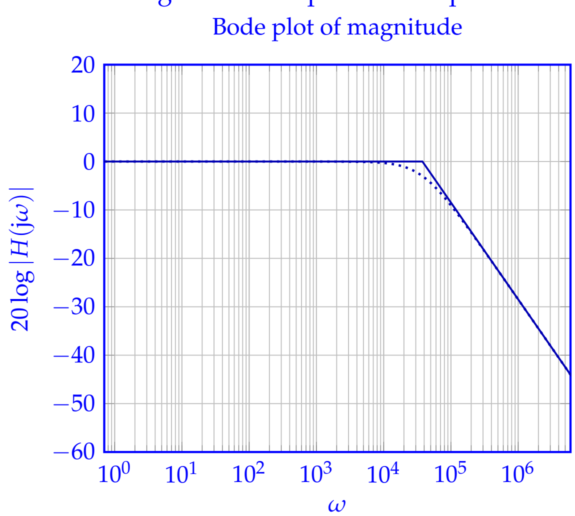
Hence we have the following properties for the phase Bode plot:

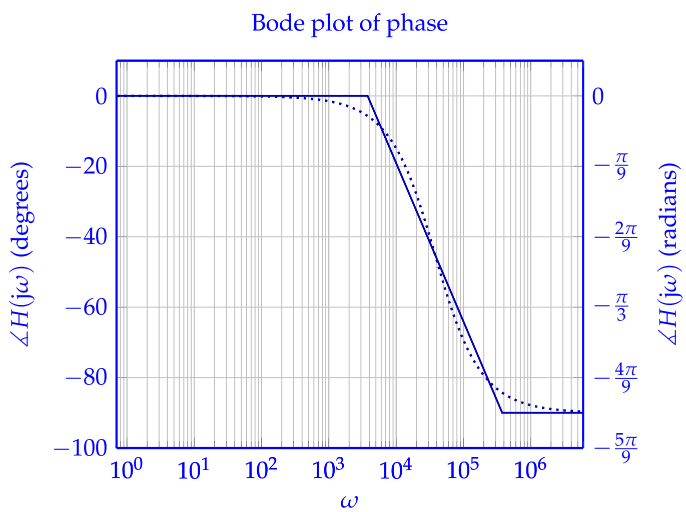
i. At $\omega < \frac{\omega_c}{10}$, the phase is $-\operatorname{atan2}(0,1) = 0^\circ$.

ii. At $\omega > 10\omega_c$, the phase is $-\operatorname{atan2}(\infty,1) = -90^\circ$.

iii. At $\omega = \omega_c$, the phase is $-\operatorname{atan2}(1,1) = -45^\circ$.

Hence the magnitude and phase Bode plots look as follows:





Note that the plots also have the accurate transfer function magnitude and phase in dotted lines. **This is not required by the problem** — it is there just to help you understand how good the Bode approximations are.

In practice, when hand-sketching such plots, one just draws the straight lines. When one has a computer, one can just plot the actual functions. Engineering is about doing whatever is easier but still lets you achieve your objectives.

(e) To achieve your goal of keeping the voice tone but rejecting the noise from the power-lines and fluorescent lights, **at what frequency do you want to have the cutoff frequency for the highpass filters**?

(HINT: To arrive at a unique solution consider computing the geometric mean (the analogous quantity to the arithmetic mean on a log scale) of the two frequencies of interest.)

Solution: There are a wide range of possible cutoff-frequencies you can choose for the highpass filter. The trade-offs are that if you choose a cutoff frequency close to the voice frequency, you will suppress the power line frequency well, but will also start to attenuate the voice frequency a little. If you choose a cutoff frequency close to the power line frequency, you will minimize attenuation of the voice frequency, but also have very little suppression of the power line frequency. Choosing this value in a principled way depends on more contextual information.

Since 60 Hz and 600 Hz are fairly close together, one reasonable way to select a cutoff frequency is to choose the geometric mean of the two frequencies (as was mentioned in the hint):

$$f_c = \sqrt{f_p f_v} = \sqrt{60 \cdot 600}\,\text{Hz} \approx 189.7\,\text{Hz} \qquad \omega_c = 2\pi f_c = 2\pi \cdot 189\,\frac{\text{rad}}{\text{s}} = 1187.5\,\frac{\text{rad}}{\text{s}} \tag{18}$$

(f) **Draw the Bode plot (straight-line approximations to the transfer function) for the magnitude (using $20\log|H(\mathrm{j}\omega)|$) and phase of the highpass filter.**

Solution: Similar to the previous Bode plot, the slopes will be 20 dB instead of 1.

The magnitude is given by $|H(\mathrm{j}\omega)| = \frac{1}{\sqrt{1+\frac{\omega_c^2}{\omega^2}}}$ where $\omega_c = 1187.5\,\frac{\text{rad}}{\text{s}}$. Hence we have the following properties for the magnitude Bode plot approximation:

i. At $\omega > \omega_c$, the magnitude is $20\log(1) = 0\,\text{dB}$.

ii. At $\omega < \omega_c$, the magnitude drops by 20 dB per decade of decrease in ω.

The phase is given by

$$\measuredangle(H(\mathrm{j}\omega)) = \measuredangle(1) - \measuredangle\left(1 - \frac{\mathrm{j}\cdot\omega_c}{\omega}\right) \tag{19}$$

$$= 0^\circ - \text{atan2}\left(-\frac{\omega_c}{\omega}, 1\right) \tag{20}$$

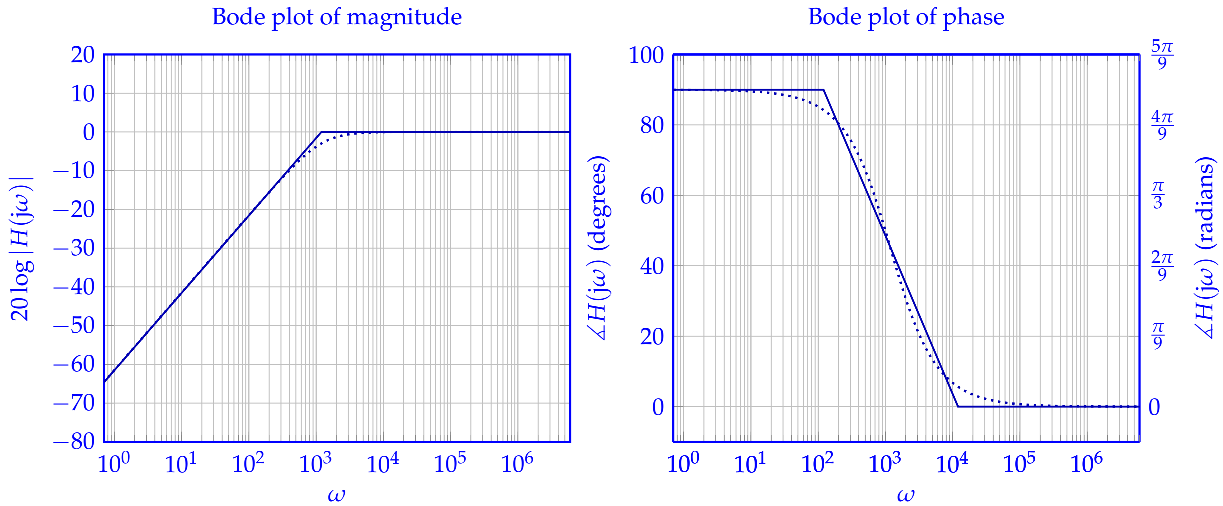
Hence we have the following properties for the phase Bode plot:

i. At $\omega < \frac{\omega_c}{10}$, the phase is $-\,\text{atan2}(-\infty, 1) = 90^\circ$.

ii. At $\omega > 10\omega_c$, the phase is $-\,\text{atan2}(0, 1) = 0^\circ$.

iii. At $\omega = \omega_c$, the phase is $-\,\text{atan2}(-1, 1) = 45^\circ$.

Hence the magnitude and phase Bode plots look as follows:



Note that the plots also have the accurate transfer function magnitude and phase in dotted lines. **This is not required by the problem —** it is there just to help you understand how good the Bode approximations are.

(g) For the following questions, assume your cut-off frequencies for lowpass and highpass are 6 kHz and 189 Hz respectively. Suppose that you only had 1 µF capacitors to use. **What resistance values would you choose for your highpass and lowpass filters so that they have the desired cutoff frequencies?**

Solution: For the highpass filter, the cutoff frequency is $f_c = \frac{1}{2\pi RC}$. Solving for R:

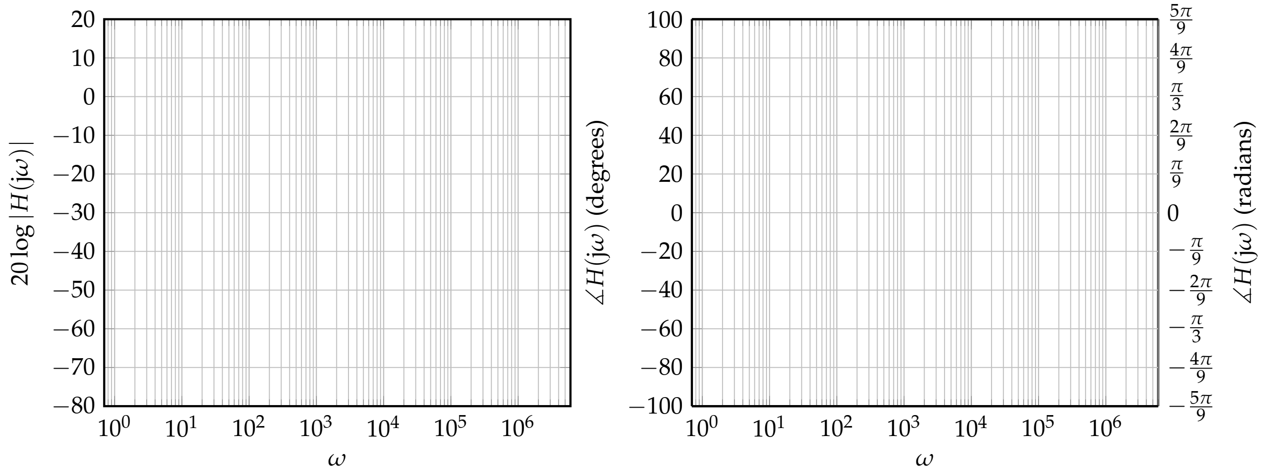
$$R = \frac{1}{2\pi f_c C} \tag{21}$$

In our case, we selected $f_c = 189\,\text{Hz}$, so $R_{\text{HP}} = \frac{1}{(1\times10^{-6})(2\pi\cdot189)} = 842\,\Omega$.

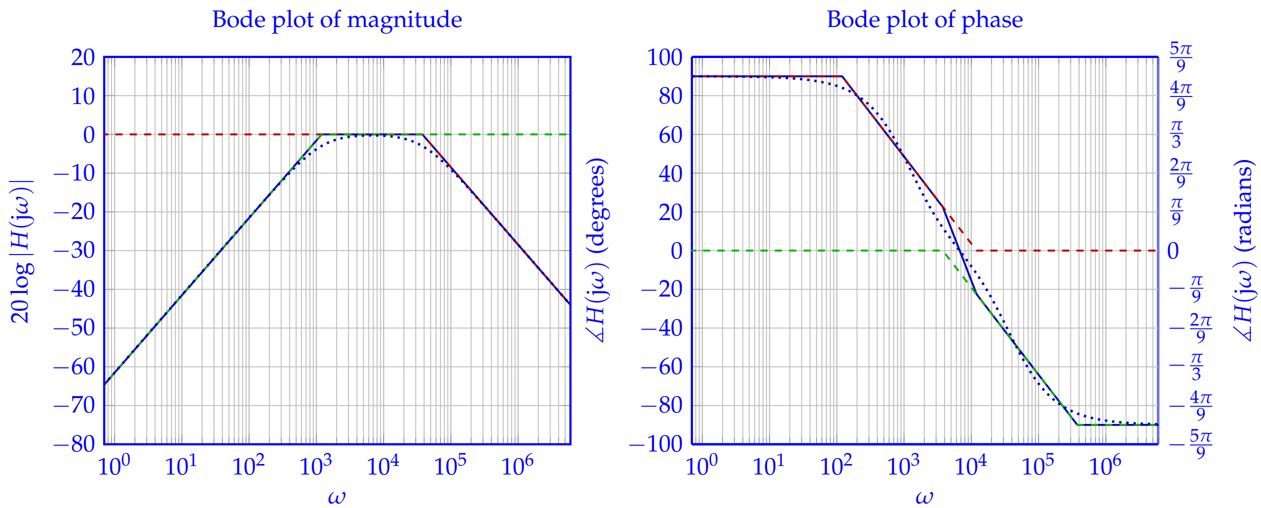
For the RC lowpass filter, the cutoff frequency expression is the same ($f_c = \frac{1}{2\pi RC}$), so $R_{\text{LP}} = \frac{1}{(1\times10^{-6})(2\pi\cdot6000)} = 26.5\,\Omega$.

(h) The overall bandpass filter that is created by cascading the lowpass and highpass with ideal buffers in between. **Draw the Bode plot (straight-line approximations to the transfer function) for the magnitude and phase of the overall bandpass transfer function.**

(HINT: You should think about how the Bode plot of a cascade of two filters can be derived based on the Bode plots of the lower-level filters.)



Solution: Since the bandpass filter is a cascade of the lowpass and highpass filters, for the bandpass Bode plot we can add the two previous plots. The previous Bode plots are drawn in the dashed format in the following figure, and their aggregate is drawn in solid blue.



Note that the plots also have the accurate transfer function magnitude and phase in dotted lines. **This is not required by the problem —** it is there just to help you understand how good the Bode approximations are.

(i) Suppose that the bandpass filter does not have enough suppression at 60 Hz and 60 kHz. You decide to use a cascade of three bandpass filters (with unity-gain buffers in between) (as shown in Figures 1 and 2). **What are the phasors for each of the frequency tones after all three bandpass filters?**

(HINT: Remember how you determined the transfer function of the bandpass filter from the transfer functions of the lowpass and highpass filters.)

Feel free to use a computer to help you evaluate both the magnitudes and the phases here.

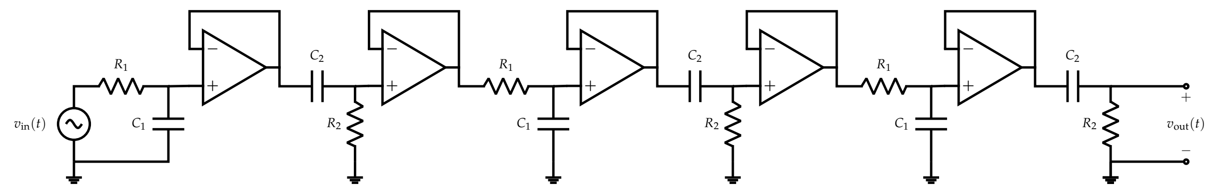


Figure 1: “Time-domain" circuit: Cascade of the three bandpass filters, using buffers to avoid loading.

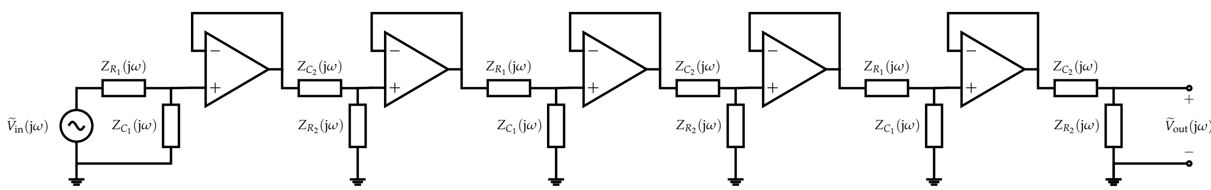


Figure 2: “Phasor-domain" circuit: Cascade of the three bandpass filters, using buffers to avoid loading.

Solution: Since each of the bandpass filters are isolated from each other with unity-gain buffers, the total transfer function is simply the single filter transfer function cubed:

$$H_{\text{total}}(j\omega) = H_{\text{BP}}(j\omega)^3 = \left(\frac{1}{1 + j\frac{\omega}{\omega_c^{\text{LP}}}} \times \frac{j\frac{\omega}{\omega_c^{\text{HP}}}}{1 + j\frac{\omega}{\omega_c^{\text{HP}}}}\right)^3 \tag{22}$$

Remember that $H(j\omega)$ can be expressed in polar form as well:

$$H_{\text{BP}}(j\omega) = |H(j\omega)|e^{j\measuredangle H(j\omega)} \tag{23}$$

$$H_{\text{total}}(j\omega) = |H(j\omega)|^3 e^{j3\measuredangle H(j\omega)} \tag{24}$$

Plugging in our chosen cutoff frequency for the bandpass filter and evaluating the magnitude and phase of our transfer function with a computer:

$$\text{Power line:} \quad H_{\text{BP}}(j\omega_p) = 0.30e^{j1.26}$$
$$\text{Voice:} \quad H_{\text{BP}}(j\omega_v) = 0.95e^{j0.21}$$
$$\text{Fluorescent light:} \quad H_{\text{BP}}(j\omega_f) = 0.099e^{-j1.48}$$

Then, the output phasors can be calculated from the total transfer function multiplied by the initial phasors for each frequency:

$$\widetilde{V}_{\text{out}} = H_{\text{total}}(j\omega) \cdot \widetilde{V}_{\text{in}} \tag{25}$$

Power line:

$$\widetilde{V}_{\text{out},p} = H_{\text{BP}}(j\omega_p)^3 \cdot \widetilde{V}_{\text{in},p} = 0.30^3 e^{j3\cdot1.26} \cdot 20e^{j\phi_p} \times 10^{-3} = 0.00054e^{j(3.76+\phi_p)}$$

Voice:

$$\widetilde{V}_{\text{out},v} = H_{\text{BP}}(j\omega_v)^3 \cdot \widetilde{V}_{\text{in},v} = 0.95^3 e^{j3\cdot0.21} \cdot e^{j\phi_v} \times 10^{-3} = 0.00086e^{j(0.62+\phi_v)}$$

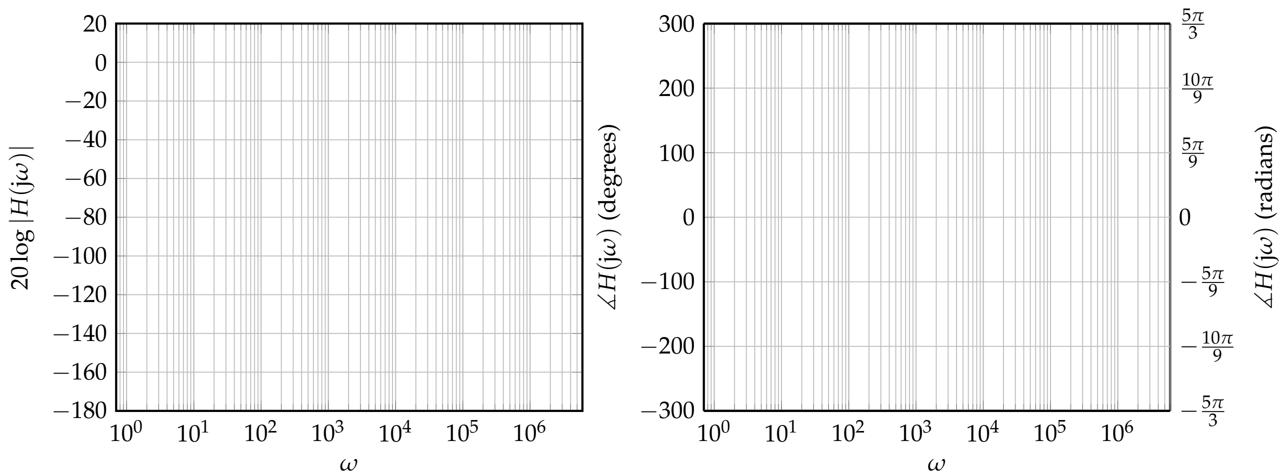
Fluorescent light:

$$\widetilde{V}_{\text{out},f} = H_{\text{BP}}(j\omega_f)^3 \cdot \widetilde{V}_{\text{in},f} = 0.099^3 e^{-j3\cdot1.48} \cdot e^{j\phi_f} \times 10^{-2} = 9.8 \times 10^{-6} e^{-j(4.40-\phi_f)}$$

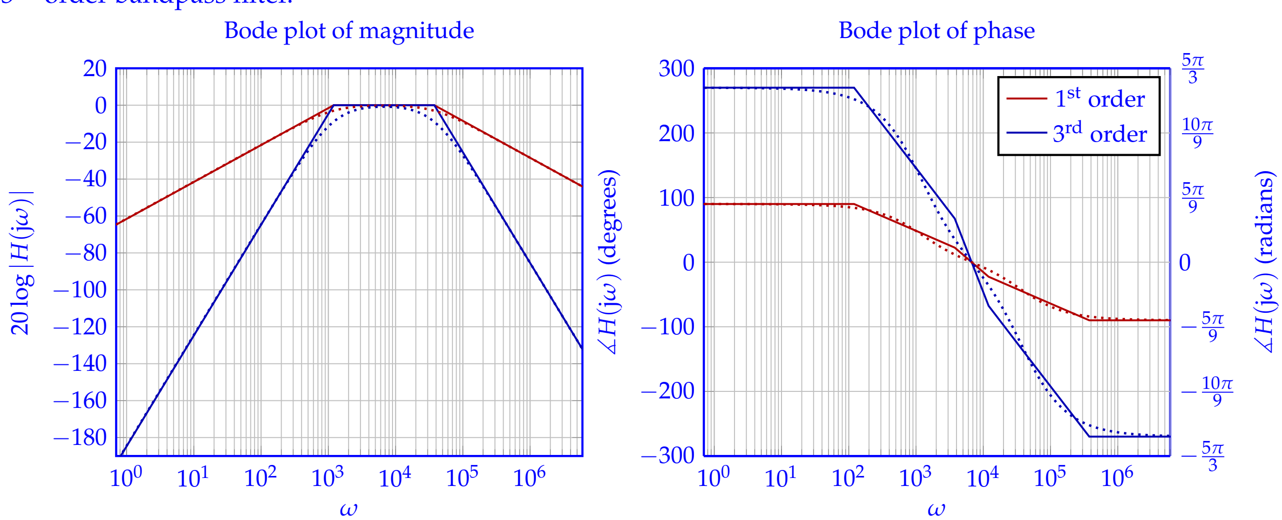
Compared to the original signals, we see that the voice signal has not been significantly attenuated, while the power line and fluorescent light noise have been significantly attenuated.

Sidenote: If you directly calculate the phase of $H_{\text{total}}(j\omega)$ at ω_p or ω_f, you may notice that you get a different answer (in fact, your answer and the answer here will be different by a multiple of 2π). This is due to *phase wrapping*, which constrains phase offsets to a range of $(-\pi, \pi]$ radians. Phase is often constrained to this range because of convention — there are infintely many phase offsets that correspond to the same signal (any multiple of 2π added to a given phase results in the same signal), so we typically choose the one within this range. However, in this solution, we leave the phase unwrapped so as to avoid confusion.

(j) **Draw the Bode plots (straight-line approximations to the transfer function) for the magnitude and phase of the 3^{rd} order bandpass filter.** To highlight the difference between the 3^{rd} and 1^{st} order filters, please draw both Bode plots on a single figure.



Solution: A bandpass with order $N = 3$ is just the cascade of 3 bandpass filters. Therefore, we can plot the original bandpass filter plot and multiply each point by 3 to get the Bode plot for the 3^{rd}-order bandpass filter.



Note that the plots also have the accurate transfer function magnitude and phase in dotted lines. **This is not required by the problem** — it is there just to help you understand how good the

Bode approximations are.

(k) **Write the final time domain voltage waveform that would be present after the filter.**

Solution: From the previous part, we have phasors for each tone at the output of the filter. We can simply take each of these phasors, and convert them back to the time domain.

$$\begin{aligned} V_{\text{out}}(t) &= 0.00055\cos\big((2\pi\cdot 60)t + 3.77 + \phi_p\big) && (26)\\ &+ 0.00085\cos((2\pi\cdot 600)t + 0.62 + \phi_v) && (27)\\ &+ 9.85\times 10^{-6}\cos\big((2\pi\cdot 60000)t + \phi_p - 4.40\big) && (28) \end{aligned}$$

Contributors:

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