

# Homework 5

**This homework is due on Friday, February 25, 2022, at 11:59PM. Self-grades and HW Resubmissions are due on the following Friday, March 4, 2022, at 11:59PM.**

## 1. Reading Lecture Notes

Staying up to date with lectures is an important part of the learning process in this course. Here are links to the notes that you need to read for this week: [Note 5](#) and [Note 6](#).

- Consider an RC circuit with a sinusoidal voltage input  $V(t) = A \cos(\omega t)$ . We are interested in finding the voltage on the capacitor in steady state (after a long time has passed). **Can we solve this using our standard differential equation techniques? Can we solve this with phasors? Which one is more concise and why?**
- There are two ways to make a low pass filter (discussed in the notes). **What are they?**
- Draw the voltage sources between terminals  $a$  and  $b$  in figure 1 as a single equivalent voltage source between terminals  $a$  and  $b$ , and label its voltage value. How does this equivalence relate to filtering and phasor analysis?**

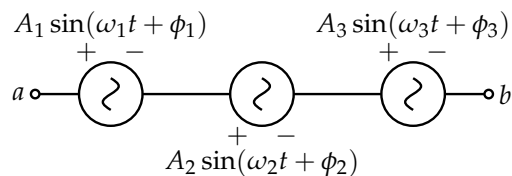


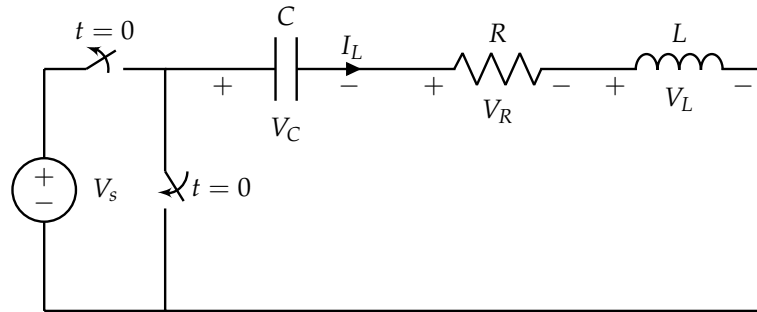
Figure 1: Three voltage sources in series.

- How we can address filter loading?**

## 2. Alternative “second order” perspective on solving the RLC circuit

In **Homework 4**, we solved an RLC circuit by setting state variables  $x_1(t) = V_C(t)$  and  $x_2(t) = I_L(t)$ , and using these to build a linear first-order vector differential equation. In this problem, we will see how to solve the same system by picking *different* state variables  $x_1(t) = V_C(t)$  and  $x_2(t) = \frac{d}{dt} V_C(t)$ , getting a linear *second order scalar* differential equation, and solving that differential equation.

Consider the following circuit like you saw in lecture, discussion, and the previous homework:



As before, assume that the system has reached steady-state for  $t < 0$ . At time  $t = 0$ , the switch changes state and disconnects the voltage source, replacing it with a short.

Suppose now we insisted on expressing everything in terms of one waveform  $V_C(t)$  instead of two of them (voltage across the capacitor and current through the inductor). This is called the “second-order” point of view, because we will end up using second derivatives.

For this problem, use  $R$  for the resistor,  $L$  for the inductor, and  $C$  for the capacitor in all the expressions.

- Write the current  $I_L(t)$  through the inductor in terms of the voltage  $V_C(t)$  across the capacitor.
- Now, notice that the voltage drop across the inductor involves  $\frac{d}{dt} I_L(t)$ . Write the voltage drop across the inductor,  $V_L(t)$ , in terms of the second derivative of  $V_C(t)$ .
- Show that a differential equation governing  $V_C(t)$  is

$$\frac{d^2}{dt^2} V_C(t) + \frac{R}{L} \frac{d}{dt} V_C(t) + \frac{1}{LC} V_C(t) = 0. \quad (1)$$

- Recall that in **Homework 3** Problem 5, we solved a second-order differential equation of the form

$$\frac{d^2 y(t)}{dt^2} + a \frac{dy(t)}{dt} + by(t) = 0 \quad (2)$$

by converting it into a matrix differential equation

$$\frac{d}{dt} \begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix} = \underbrace{\begin{bmatrix} 0 & 1 \\ -b & -a \end{bmatrix}}_A \begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix} \quad (3)$$

where  $x_1(t) := y(t)$  and  $x_2(t) := \frac{dy(t)}{dt}$ . It turned out that, if  $A$  has two distinct eigenvalues, the solution to this homogeneous differential equation have the form

$$\begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix} = \begin{bmatrix} c_1 e^{\lambda_1 t} + c_2 e^{\lambda_2 t} \\ c_3 e^{\lambda_1 t} + c_4 e^{\lambda_2 t} \end{bmatrix}. \quad (4)$$

where  $\lambda_1, \lambda_2$  are the eigenvalues of  $A$ , and  $c_1, c_2, c_3, c_4$  are constants determined by the initial conditions and the coefficients  $a, b$  in the differential equation. We would like to use this to construct a solution for  $V_C(t)$ .

Show that, if we identify  $y(t) := V_C(t)$ , then

$$x_1(t) = V_C(t) \quad x_2(t) = \frac{d}{dt} V_C(t), \quad (5)$$

and that the matrix  $A$  for our RLC circuit is

$$A = \begin{bmatrix} 0 & 1 \\ -\frac{1}{LC} & -\frac{R}{L} \end{bmatrix}. \quad (6)$$

Then, show that the two eigenvalues of  $A$  are

$$\lambda_1 = -\frac{R}{2L} + \frac{1}{2} \sqrt{\frac{R^2}{L^2} - \frac{4}{LC}}, \quad \lambda_2 = -\frac{R}{2L} - \frac{1}{2} \sqrt{\frac{R^2}{L^2} - \frac{4}{LC}}. \quad (7)$$

NOTE: From this part onwards, we will assume that the circuit parameters  $R, L, C$  are chosen so that the eigenvalues of  $A$  are distinct.

- (e) Now, we solve for  $x_1(t) = V_C(t)$  by determining  $c_1$  and  $c_2$  and plugging those, along with  $\lambda_1$  and  $\lambda_2$ , into eq. (4). Note that determining  $c_3$  and  $c_4$  isn't necessary to find  $x_1(t)$ , but we need them to set up a system of equations to solve for  $c_1$  and  $c_2$ . **Show that**

$$c_3 = \lambda_1 c_1, \quad c_4 = \lambda_2 c_2. \quad (8)$$

Then use the initial conditions of the RLC circuit to show that

$$c_1 = \frac{\lambda_2}{\lambda_2 - \lambda_1} V_s, \quad c_2 = -\frac{\lambda_1}{\lambda_2 - \lambda_1} V_s. \quad (9)$$

(HINT: This part is complicated, so here's some scaffolding to start you off. First, differentiate the expression we have  $x_1(t) = c_1 e^{\lambda_1 t} + c_2 e^{\lambda_2 t}$  to get a form for  $x_2(t)$ , and match coefficients of  $e^{\lambda_1 t}$  and  $e^{\lambda_2 t}$  to get the desired expressions for  $c_3$  and  $c_4$ . Next, use the initial conditions for RLC to see what  $V_C(0)$  are and  $\left. \frac{d}{dt} V_C(t) \right|_{t=0}$  are. This corresponds to  $x_1(0)$  and  $x_2(0)$ . Plug  $t = 0$  into the "sum of exponentials" form for  $x_1$  and  $x_2$ . This will get you two equations, one for each  $x_i$ , for  $c_1$  and  $c_2$ , which you can then solve.)

(HINT: The following matrix inverse formula may be useful:

$$\begin{bmatrix} 1 & 1 \\ \lambda_1 & \lambda_2 \end{bmatrix}^{-1} = \frac{1}{\lambda_2 - \lambda_1} \begin{bmatrix} \lambda_2 & -1 \\ -\lambda_1 & 1 \end{bmatrix} \quad (10)$$

)

We have found  $\lambda_1, \lambda_2, c_1, c_2$ , so by substituting into eq. (4) we have solved for  $x_1(t) = V_C(t)$ !

### 3. Designing Filters

In the lab, we will design various filter circuits using low-pass, high-pass, and band-pass filter elements. In this problem, we will walk through the use cases of these filter elements.

- (a) First, you remember that you saw in lecture that you can build simple filters using a resistor and a capacitor. **Design a simple first-order passive low-pass filter with the following specification using a  $1\ \mu\text{F}$  capacitor.** ("Passive" means that the filter does not require any power supply to operate on the input signal. Passive components include resistors, capacitors, inductors, diodes, etc., while an example of an active component would be an op-amp).

- Low-pass filter: cut-off frequency  $f_c = 2400\ \text{Hz}$ ,  $\omega_c = 2\pi \cdot 2400\ \frac{\text{rad}}{\text{s}}$ . Hz can be interpreted as "cycles/sec", and  $\frac{\text{rad}}{\text{s}}$  can be interpreted as " $2\pi$  radians/cycle".

Recall that the cutoff-frequency of such a filter is just where the magnitude of the filter is  $\frac{1}{\sqrt{2}}$  of its peak value.

**Show your work to find the resistor value that creates this low-pass filter. Draw the schematic-level representation of your design. Please mark  $V_{in}$ ,  $V_{out}$ , and the ground node(s) in your schematic.** Round your results to two significant figures.

- (b) **Now design a simple first-order passive high-pass filter with the following specification using a  $1\ \mu\text{F}$  capacitor.**

- High-pass filter: cut-off frequency  $f_c = 100\ \text{Hz}$ ,  $\omega_c = 2\pi \cdot 100\ \frac{\text{rad}}{\text{s}}$

**Show your work to find the resistor value that creates this high-pass filter. Draw the schematic-level representation of your design. Please mark  $V_{in}$ ,  $V_{out}$ , and the ground node(s) in your schematic.** Round your results to two significant figures.

- (c) You can try to build a bandpass filter by cascading the first-order low-pass and high-pass filters you designed in parts (a) and (b). To do this, you might be tempted to connect the  $V_{out}$  node of your low-pass filter directly to the  $V_{in}$  node of your high-pass filter. However, if you did this, just as you saw in 16A for voltage dividers, the purported high-pass filter would "load" the low-pass filter and you might get some potentially complicated mess instead of what you wanted.

**Show how you can use an ideal op-amp configured as a unity gain buffer to eliminate this loading effect to cascade the low-pass and high-pass filters, and write the resulting transfer function of the combined circuit. What kind of filter is this?** You can optionally use the included Jupyter notebook `plot_tf.ipynb`. (*HINT: Read Section 2.1 in Note 7.*)

(NOTE: In Python, use `1j` when your transfer function has a `j`.)

- (d) **Write down an expression for the time-domain output waveform  $V_{out}(t)$  of this filter if the input voltage is  $V_{in}(t) = 1 \sin(1000t)\ \text{V}$ .** Round your answer to 2 significant digits.

#### 4. Phasors and Eigenvalues

Suppose that we have the two-dimensional system of differential equations expressed in matrix/vector form:

$$\frac{d}{dt}\vec{x}(t) = A\vec{x}(t) + \vec{b}u(t) \quad (11)$$

where for this problem, the matrix  $A$  and the vector  $\vec{b}$  are both real.

- (a) Give a necessary condition on the eigenvalues  $\lambda_k$  of  $A$  such that any impact of an initial condition will eventually completely die out. (i.e. the system will reach steady-state.)

You don't have to prove this. The idea here is to make sure that you understand what kind of thing is required. (*HINT: Read Section 2 in Note 5.*)

- (b) Now assume that  $u(t)$  has a phasor representation  $\tilde{U}$ . In other words,  $u(t) = \tilde{U}e^{+j\omega t} + \overline{\tilde{U}}e^{-j\omega t}$ . Assume that the vector solution  $\vec{x}(t)$  to the system of differential equations (11) can also be written in phasor form as

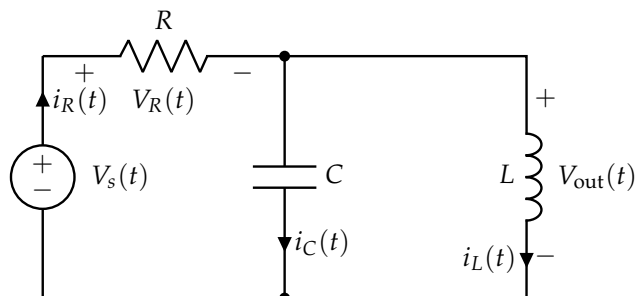
$$\vec{x}(t) = \vec{\tilde{X}}e^{+j\omega t} + \overline{\vec{\tilde{X}}}e^{-j\omega t}. \quad (12)$$

**Derive an expression for  $\vec{\tilde{X}}$  involving  $A, \vec{b}, j\omega, \tilde{U}$ , and the identity matrix  $I$ .** (Here, we assume that  $j\omega$ , and  $-j\omega$  are not eigenvalues of  $A$ , which indicates that  $\det(j\omega I - A)$  and  $\det(-j\omega I - A)$  are non-zero.)

### 5. Phasor-Domain Circuit Analysis

The analysis techniques you learned previously in 16A for resistive circuits are equally applicable for analyzing circuits driven by sinusoidal inputs in the phasor domain. In this problem, we will walk you through the steps with a concrete example.

Consider the following circuit where the input voltage is sinusoidal. The end goal of our analysis is to find an equation for  $V_{\text{out}}(t)$ .



The components in this circuit are given by:

$$V_s(t) = 10\sqrt{2} \cos\left(100t - \frac{\pi}{4}\right) \quad (13)$$

$$R = 5 \Omega \quad (14)$$

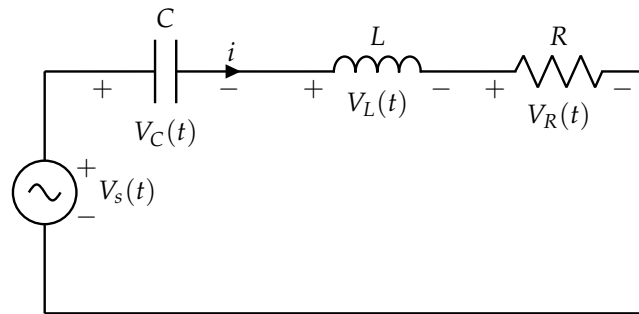
$$L = 50 \text{ mH} \quad (15)$$

$$C = 2 \text{ mF} \quad (16)$$

- Give the amplitude  $V_0$ , input frequency  $\omega$ , and phase  $\phi$  of the input voltage  $V_s$ .
- Transform the circuit into the phasor domain. **What are the impedances of the resistor, capacitor, and inductor? What is the phasor  $\tilde{V}_S$  of the input voltage  $V_s(t)$ ?**
- Use the circuit equations to **solve for  $\tilde{V}_{\text{out}}$** , the phasor representing the output voltage.
- Convert the phasor  $\tilde{V}_{\text{out}}$  back to get the time-domain signal  $V_{\text{out}}(t)$ .**

## 6. RLC filter

Consider the following RLC circuit:



- Write down the impedance of a series RLC circuit in the form  $Z_{RLC}(j\omega) = X(\omega) + jY(\omega)$ , where  $X(\omega)$  and  $Y(\omega)$  are real-valued functions of  $\omega$ . (HINT: Rationalize denominators before simplifying by collecting real and imaginary like terms.)
- Write the transfer function from  $V_S$  to  $V_R$  — the voltage drop across the resistor.
- Find the magnitude  $|H(j\omega)|$  and phase  $\angle H(j\omega)$  of the transfer function in terms of  $\omega, R, L, C$ .
- For the specific values for  $R, L, C$  given by overdamped and underdamped cases in the previous HW, use a computer to sketch plots of the magnitude and phase of the transfer function above. You can optionally use the included Jupyter notebook `plot_tf.ipynb`. Recall that the overdamped case had  $R = 1\text{ k}\Omega, C = 10\text{ nF}, L = 25\text{ }\mu\text{H}$ ; underdamped had  $R = 1\text{ }\Omega, C = 10\text{ nF}, L = 25\text{ }\mu\text{H}$ . (NOTE: In Python, use `1j` when your transfer function has a `j`.)
- To see how the values of  $R, L, C$  impact the impedance at different frequencies, run the included Jupyter notebook `hw5r1c_transfer.ipynb`. The script will generate two plots, the transfer function of the circuit as a function of frequency and the location of the eigenvalues in the imaginary, real plane. Explain qualitatively how the peak (resonant) frequency, the width of the peak, and the location of the eigenvalues depend on the resistance, inductance, and capacitance:

	$R$	$L$	$C$
I	1	2.5E-5	1E-8
II	10	2.5E-5	1E-8
III	10	2.5E-5	2E-9
IV	500	1.0E-4	2E-8

Table 1: Values for RLC Bandwidth problem, part e

## 7. Uniqueness justification for solutions to matrix/vector differential equations

In general, we have seen that we need to justify our methods of solving differential equations with a uniqueness proof. This is important as it allows us to trust our solution as being the only one for the problem at hand.

Consider matrix-vector differential equations of the form:

$$\frac{d}{dt}\vec{x}(t) = A\vec{x}(t) + \vec{b}u(t) \quad (17)$$

with some initial condition  $\vec{x}(0) = \vec{x}_0$ .

All the uniqueness proofs that you have done for yourself have been concerned with scalar differential equations, and scalar differential equations driven by inputs. So, why can we trust the solutions that we are getting for such matrix-vector differential equations?

This question takes us part of the way to the answer.

- (a) Suppose that the  $n \times n$  matrix  $A$  has  $n$  distinct eigenvalues and corresponding eigenvectors  $\vec{v}_1, \vec{v}_2, \dots, \vec{v}_n$ , so that the matrix  $V = [\vec{v}_1 \ \vec{v}_2 \ \dots \ \vec{v}_n]$  has linearly independent columns.

**Find the diagonalized system corresponding to (17). Show that if you are given any valid solution for the original system (17), you can change coordinates to the eigenbasis and also get a valid solution for the diagonalized system. List the new initial conditions that are satisfied in the diagonalized system.**

- (b) You have already proved the uniqueness of solutions for any scalar differential equation of the form  $\frac{d}{dt}x(t) = \lambda x(t) + u(t)$  with specified initial condition  $x(0) = x_0$ . How can you use this fact and the result of the previous part to argue that the solution must be unique for the matrix/vector differential equation?

*(HINT: (Start by assuming that you have two solutions to the original problem. Use the result of part a) to formulate solutions to the transformed, diagonal problem. Apply the uniqueness results for scalar ODEs to the solutions of the diagonalized problem. Finally conclude by considering what the invertibility of the transformation matrix  $V$  implies about the two solutions of the original problem?))*

We will see later in the course how the assumption we made on the eigenvectors of  $A$  is not actually needed for this proof to hold. But for now, it is important to understand this case first.



**8. (OPTIONAL) Make Your Own Problem.**

**Write your own problem about content covered in the course thus far, and provide a thorough solution to it.**

*NOTE:* This can be a totally new problem, a modification on an existing problem, or a Jupyter part for a problem that previously didn't have one. Please cite all sources for anything (including course material) that you used as inspiration.

*NOTE:* High-quality problems may be used as inspiration for the problems we choose to put on future homeworks or exams.

**9. Homework Process and Study Group**

Citing sources and collaborators are an important part of life, including being a student!

We also want to understand what resources you find helpful and how much time homework is taking, so we can change things in the future if possible.

- (a) **What sources (if any) did you use as you worked through the homework?**
- (b) **If you worked with someone on this homework, who did you work with?**  
List names and student ID's. (In case of homework party, you can also just describe the group.)
- (c) **Roughly how many total hours did you work on this homework? Write it down here where you'll need to remember it for the self-grade form.**

**Contributors:**

- Kuan-Yun Lee.
- Anant Sahai.
- Druv Pai.
- Kyoungtae Lee.
- Nikhil Shinde.
- Nathan Lambert.
- Ayan Biswas.
- Jaijeet Roychowdhury.
- Geoffrey Négier.
- Regina Eckert.
- Antroy Roy Chowdhury.
- Sanjit Batra.
- Kris Pister.
- Daniel Abraham.
- Wahid Rahman.
- Sidney Buchbinder.
- Ashwin Vangipuram.
- Tanmay Gautam.
- Sally Hui.