Homework 4

This homework is due on Saturday, September 23, 2023 at 11:59PM. Self-grades and HW Resubmissions are due the following Saturday, September 30, 2023 at 11:59PM.

1. **Adapted from Hambley P6.27**

Suppose you have a filter with transfer function $H(\mathrm{j}\omega) = \frac{1}{1+\mathrm{j}\frac{\omega}{\omega_c}}$, with $\omega_c = 400\ \frac{\mathrm{rad}}{\mathrm{s}}$.

The input signal of the filter with this transfer function is

$$v_{\mathrm{in}}(t) = 1 + 2\cos(400t + 30^\circ) + 3\cos\left(10^{10}t\right) \tag{1}$$

Find an expression for the output voltage (you may approximate).

Solution: The given input signal is

$$v_{\mathrm{in}}(t) = 1 + 2\cos(400t + 30^\circ) + 3\cos\left(10^{10}t\right) \tag{2}$$

which has components with frequencies of 0, 400, and $10^{10}\ \frac{\mathrm{rad}}{\mathrm{s}}$. Evaluating the transfer function for these frequencies yields

$$H(\mathrm{j}0) = \frac{1}{1+\mathrm{j}\frac{0}{400}} = 1 \tag{3}$$

$$H(\mathrm{j}400) = \frac{1}{\sqrt{2}}\mathrm{e}^{-\mathrm{j}45^\circ} \tag{4}$$

$$H(10^{10}) \approx 0 \tag{5}$$

Applying the appropriate value of the transfer function to each component of the input signal yields the output:

$$v_{\mathrm{out}}(t) = 1 + \sqrt{2}\cos(400t - 15^\circ) \tag{6}$$

2. Hambley P6.33

Consider the circuit shown in Figure 1. This circuit consists of a source having an internal resistance of R_s, an RC lowpass filter, and a load resistance of R_l.

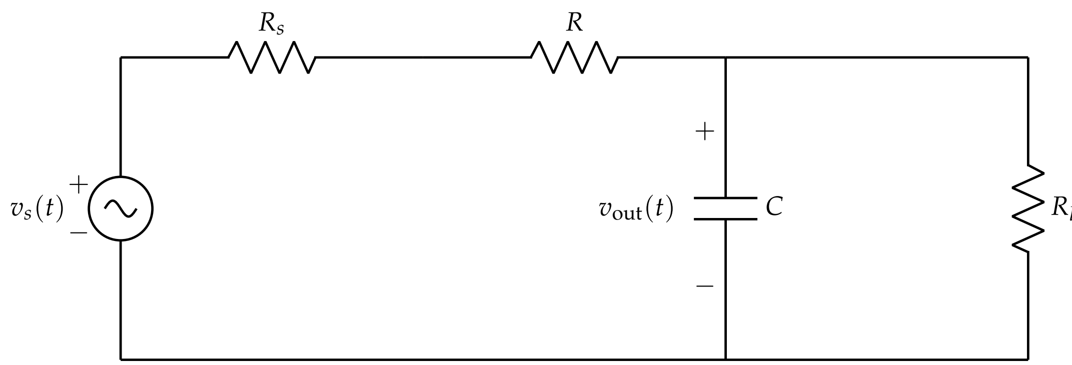


Figure 1: P6.33(a)

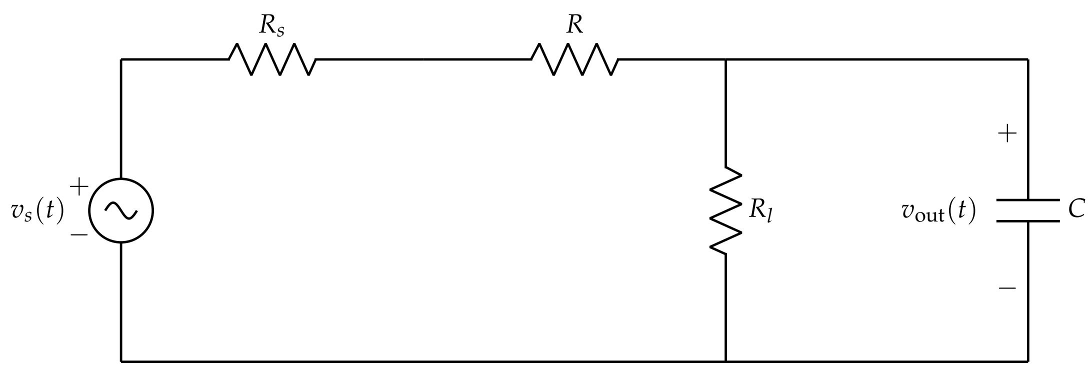


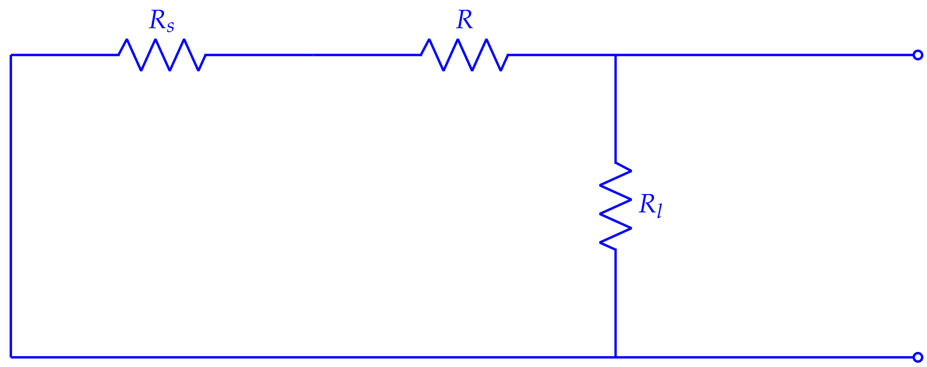
Figure 2: P6.33(b)

Show that the transfer function of this circuit is given by

$$H(\mathrm{j}\omega) = \frac{\widetilde{V}_{\text{out}}}{\widetilde{V}_s} = \frac{R_l}{R_s + R + R_l} \times \frac{1}{1 + \mathrm{j}\frac{\omega}{\omega_c}} \tag{7}$$

in which the cutoff frequency ω_c is given by $\omega_c = \frac{1}{R_t C}$ where $R_t = \frac{R_L(R_s+R)}{R_l+R_s+R}$. Notice that R_t is the parallel combination of R_l and $(R_s + R)$. *(HINT: One way to make this problem easier is to rearrange the circuit as shown in Figure 2 and then to find the Thevenin equivalent for the source and resistances.)*

Solution: First, we find the Thevenin equivalent for the source and resistances.



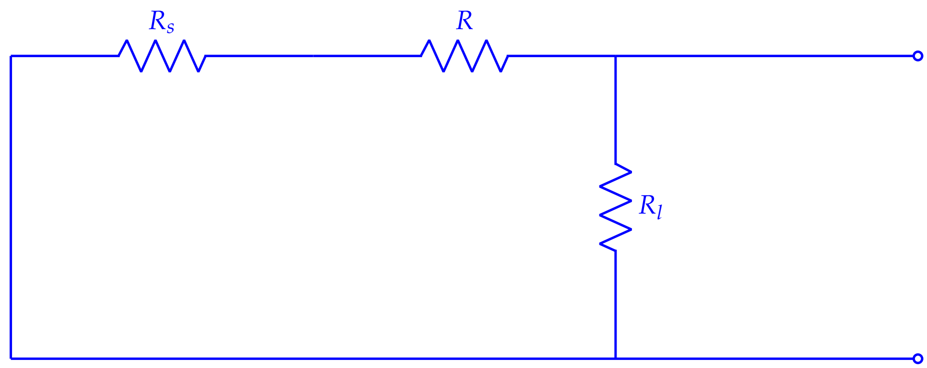
The open-circuit voltage is given by

$$v_t(t) = v_s(t)\frac{R_l}{R_s + R + R_l} \tag{8}$$

In terms of phasors, this becomes

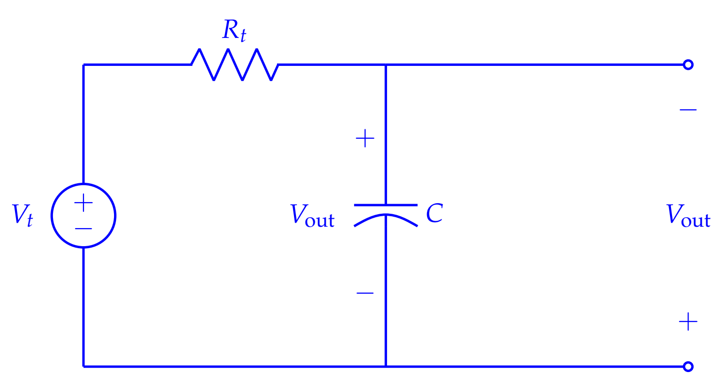
$$\widetilde{V}_t = \widetilde{V}_s\frac{R_l}{R_s + R + R_l} \tag{9}$$

Zeroing the source, we find the Thevenin resistance:



$$R_t = \frac{1}{\frac{1}{R_l} + \frac{1}{R+R_s}} \tag{10}$$

Thus, the original circuit has the equivalent:



The transfer function for this circuit is

$$\frac{\widetilde{V}_{\text{out}}}{\widetilde{V}_t} = \frac{1}{1 + \mathrm{j}\frac{\omega}{\omega_c}} \tag{11}$$

where $\omega_c = \frac{1}{R_t C}$. Using eq. (9) to substitute for $\widetilde{V}_t$ in eq. (11) and rearranging, we have

$$H(\mathrm{j}\omega_c) = \frac{\widetilde{V}_{\text{out}}}{\widetilde{V}_s} = \frac{R_l}{R_s + R + R_l} \times \frac{1}{1 + \mathrm{j}\frac{\omega}{\omega_c}} \tag{12}$$

3. Hambley P6.74

Derive an expression for the resonant frequency of the circuit shown in Figure 3. We define the resonant frequency to be the frequency at which the impedance is purely real (no imaginary component).

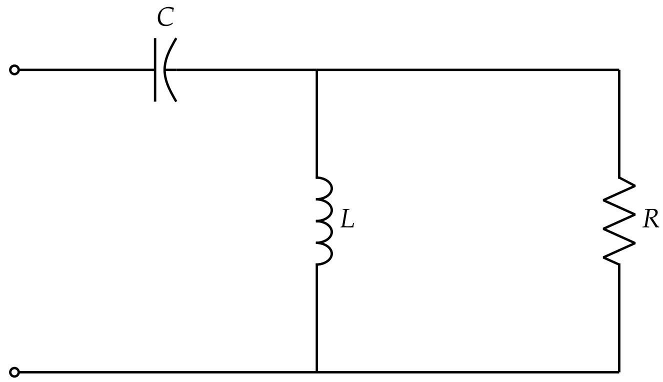


Figure 3: P6.74

Solution: The combined impedance is

$$Z = \frac{1}{j\omega C} + \frac{1}{\frac{1}{R} + \frac{1}{j\omega L}} \tag{13}$$

$$= -\frac{j}{\omega C} + \frac{\frac{1}{R} + j\frac{1}{\omega L}}{\left(\frac{1}{R}\right)^2 + \left(\frac{1}{\omega L}\right)^2} \tag{14}$$

At the resonance frequency, the imaginary part must go to zero, so

$$\frac{\frac{1}{\omega L}}{\left(\frac{1}{R}\right)^2 + \left(\frac{1}{\omega L}\right)^2} - \frac{1}{\omega C} = 0 \tag{15}$$

Solving for ω yields $\omega = \frac{1}{\sqrt{LC - \left(\frac{L}{R}\right)^2}}$.

4. Circuit Design

In this problem, you will find a circuit where several components have been left *blank* for you to fill in.

Assume that the op-amp is *ideal*. A special note on op amps in frequency domain analysis: The op-amps you learned about in 16A can be used in exactly the same way for setting up differential equations and even Phasor analysis in 16B. Treat them as ideal op-amps and invoke the Golden Rules.

You have at your disposal *only one of each* of the following components (not including R_1):

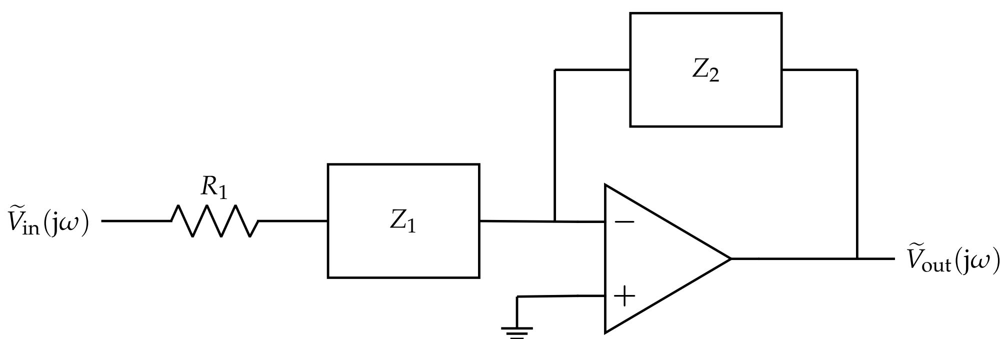
(a) an open circuit

(b) a short circuit

(c) a resistor (you choose from the values $R = 1\,\mathrm{k\Omega}, 10\,\mathrm{k\Omega}, 20\,\mathrm{k\Omega}$)

(d) a capacitor (you choose from the values $C = 0.5\,\mu\mathrm{F}, 1\,\mu\mathrm{F}, 2\,\mu\mathrm{F}$)

Consider the circuit below. The labeled voltages $\widetilde{V}_{\text{in}}(\mathrm{j}\omega)$ and $\widetilde{V}_{\text{out}}(\mathrm{j}\omega)$ are the phasor representations of $v_{\text{in}}(t)$ and $v_{\text{out}}(t)$ respectively, where $v_{\text{in}}(t)$ has the form $v_{\text{in}}(t) = v_0 \cos(\omega t + \phi)$. The transfer function $H(\mathrm{j}\omega)$ is defined as $H(\mathrm{j}\omega) = \frac{\widetilde{V}_{\text{out}}(\mathrm{j}\omega)}{\widetilde{V}_{\text{in}}(\mathrm{j}\omega)}$.



(a) Let $Z_1(\mathrm{j}\omega)$ and $Z_2(\mathrm{j}\omega)$ are the impedances of the boxes shown in the circuit diagram. **Write the expression of the transfer function** $H(\mathrm{j}\omega)$**.**

Solution: The circuit is in the inverting amplifier configuration. Hence the transfer function is given by

$$H(\mathrm{j}\omega) = -\frac{Z_2}{R_1 + Z_1} \tag{16}$$

(b) Let R_1 be $1\,\mathrm{k\Omega}$. We have to find Z_1 and Z_2, such that the circuit's transfer function $H(\mathrm{j}\omega)$ has the following properties:

- $|H(\mathrm{j}0)| = 0$.
- $|H(\mathrm{j}\infty)| = 10$.
- $|H(\mathrm{j}10^3)| = \sqrt{50}$.

Using the fact that the circuit is a high pass filter, infer the components (we will find values later) of Z_1 **and** Z_2**. Write the transfer function** $H(\mathrm{j}\omega)$ **using these components.**

(HINT: Try method of elimination: figure out what Z_2 cannot be. Once you find what Z_2 is, what does Z_1 have to be for the circuit to be a filter?)

Solution: The circuit should be a high-pass filter, so Z_2 cannot be a short circuit or a capacitor, otherwise $\widetilde{V}_{\text{out}}(\mathrm{j}\infty) = 0$. Also Z_2 cannot be open circuit as that will break the negative feedback. So Z_2 is a resistor, i.e. $Z_2 = R$.

Since Z_2 cannot be a capacitor, Z_1 must be a capacitor. Otherwise, we would not have any frequency dependent impedance in the circuit, which means it wouldn't be a filter. So $Z_1 = \frac{1}{\mathrm{j}\omega C}$. Hence the transfer function is

$$H(\mathrm{j}\omega) = -\frac{R}{R_1 + \frac{1}{\mathrm{j}\omega C}} \tag{17}$$

$$= -\frac{R}{R_1} \cdot \frac{1}{1 - \frac{\mathrm{j}}{\omega C R_1}} \tag{18}$$

Observe that $|H(\mathrm{j}0)| = 0$ and $|H(\mathrm{j}\infty)| = \frac{R}{R_1}$, so it is a high-pass filter.

(c) **Now use the facts that $|H(\mathrm{j}\infty)| = 10$ and $R_1 = 1\,\mathrm{k}\Omega$ to find the component value of Z_2.**

Solution: From $|H(\mathrm{j}\infty)| = 10 = \frac{R}{R_1}$, we know that $R = 10R_1 = 10\,\mathrm{k}\Omega$, which is one of the options for resistor values.

(d) **Finally use the fact that $|H(\mathrm{j}10^3)| = \sqrt{50}$ and the values of R_1 and Z_2 to find the component value of Z_1.**

Solution: We can now write the transfer function as

$$H(\mathrm{j}\omega) = -10 \frac{1}{1 - \frac{\mathrm{j}}{1000\omega C}} \tag{19}$$

We get

$$|H(\mathrm{j}10^3)| = 10 \frac{1}{\sqrt{1 + \frac{1}{(10^6 C)^2}}} = \sqrt{50} \tag{20}$$

$$\implies 100 \frac{1}{1 + \frac{1}{(10^6 C)^2}} = 50 \tag{21}$$

$$\implies \frac{1}{1 + \frac{1}{(10^6 C)^2}} = \frac{1}{2} \tag{22}$$

$$\implies \frac{1}{(10^6 C)^2} = 1 \tag{23}$$

$$\implies C = 1\,\mu\mathrm{F} \tag{24}$$

5. Designing Filters

In the lab, we will design various filter circuits using low-pass, high-pass, and band-pass filter elements. In this problem, we will walk through the use cases of these filter elements.

(a) First, you remember that you saw in lecture that you can build simple filters using a resistor and a capacitor. **Design a simple first-order *passive* low-pass filter with the following specification using a 1 µF capacitor.** ("Passive" means that the filter does not require any power supply to operate on the input signal. Passive components include resistors, capacitors, inductors, diodes, etc., while an example of an active component would be an op-amp).

- Low-pass filter: cut-off frequency $f_c = 2400\,\text{Hz}$, $\omega_c = 2\pi \cdot 2400\frac{\text{rad}}{\text{s}}$. Hz can be interpreted as "cycles/sec", and $\frac{\text{rad}}{\text{s}}$ can be interpreted as "2π radians/cycle".

Recall that the cutoff-frequency of such a filter is just where the magnitude of the filter is $\frac{1}{\sqrt{2}}$ of its peak value.

Show your work to find the resistor value that creates this low-pass filter. Draw the schematic-level representation of your design. Please mark V_{in}, V_{out}, and the ground node(s) in your schematic. Round your results to two significant figures.

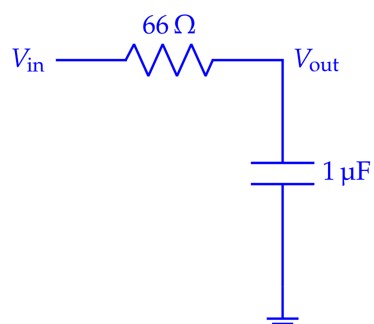
Solution: Low-pass filter

$$\omega_c = 2\pi f_c = \frac{1}{RC} \tag{25}$$

$$f_c = \frac{1}{2\pi RC} = 2400\,\text{Hz} \tag{26}$$

$$R = \frac{1}{2\pi \cdot 1\mu\text{F} \cdot 2400\,\text{Hz}} = 66\,\Omega \tag{27}$$

Therefore, we need a 66 Ω resistor.



(b) **Now design a simple first-order *passive* high-pass filter with the following specification using a 1 µF capacitor.**

- High-pass filter: cut-off frequency $f_c = 100\,\text{Hz}$, $\omega_c = 2\pi \cdot 100\frac{\text{rad}}{\text{s}}$

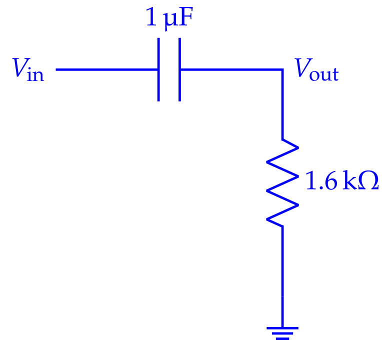
Show your work to find the resistor value that creates this high-pass filter. Draw the schematic-level representation of your design. Please mark V_{in}, V_{out}, and the ground node(s) in your schematic. Round your results to two significant figures.

Solution: High-pass filter

$$f_c = \frac{1}{2\pi RC} = 100\,\text{Hz} \tag{28}$$

$$R = \frac{1}{2\pi \cdot 1\mu\text{F} \cdot 100\,\text{Hz}} = 1.6\,\text{k}\Omega \tag{29}$$

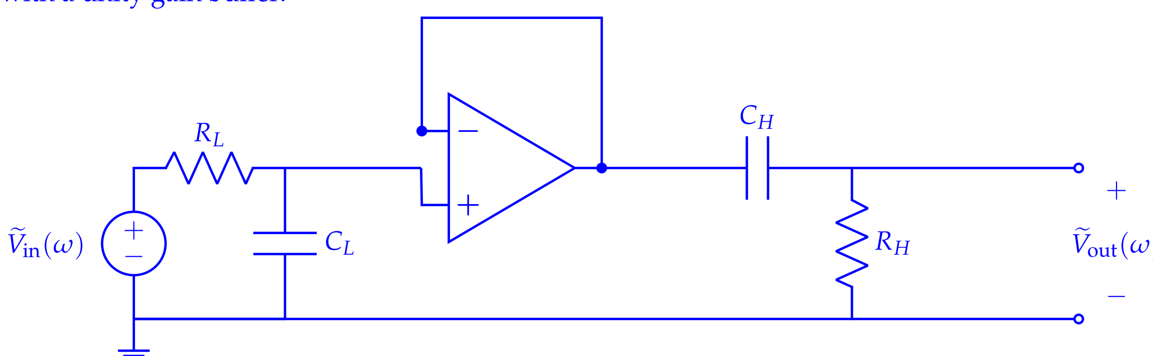
Therefore, we need a 1.6 kΩ resistor. Note that we want a 24 times lower frequency, which means a 24 times higher time constant, which means a 24 times higher resistor.



(c) You can try to build a bandpass filter by cascading the first-order low-pass and high-pass filters you designed in parts (a) and (b). To do this, you might be tempted to connect the V_{out} node of your low-pass filter directly to the V_{in} node of your high-pass filter. However, if you did this, just as you saw in 16A for voltage dividers, the purported high-pass filter would "load" the low-pass filter and you might get some potentially complicated mess instead of what you wanted.

Show how you can use an ideal op-amp configured as a unity gain buffer to eliminate this loading effect to cascade the low-pass and high-pass filters, and write the resulting transfer function of the combined circuit. Draw the magnitude and phase transfer functions of the combined circuit (you can use Bode Plot approximations). What kind of filter is this?

Solution: Consider the circuit given below, which is the low pass and the high pass, connected with a unity gain buffer:



We know that when we cascade circuits, the combined transfer function is the multiplication of the individual elements. For the Low Pass Filter $H_L(j\omega)$, Unity Gain Buffer $H_{\text{unity}}(j\omega)$, and High Pass Filter $H_H(j\omega)$.

$$H(j\omega) = H_L(j\omega) \cdot H_{\text{unity}}(j\omega) \cdot H_H(j\omega) \tag{30}$$

And we know that:

$$H_L(j\omega) = \frac{1}{1 + j\omega R_L C_L}, \quad H_{\text{unity}}(j\omega) = 1, \quad H_H(j\omega) = \frac{j\omega R_H C_H}{1 + j\omega R_H C_H} \tag{31}$$

Combining the transfer functions, we get:

$$H(\mathrm{j}\omega) = \frac{1}{(1 + \mathrm{j}\omega R_L C_L)} \cdot \frac{\mathrm{j}\omega R_H C_H}{(1 + \mathrm{j}\omega R_H C_H)} \tag{32}$$

The magnitude and phase transfer functions are shown below. We can see that this is a band pass filter.

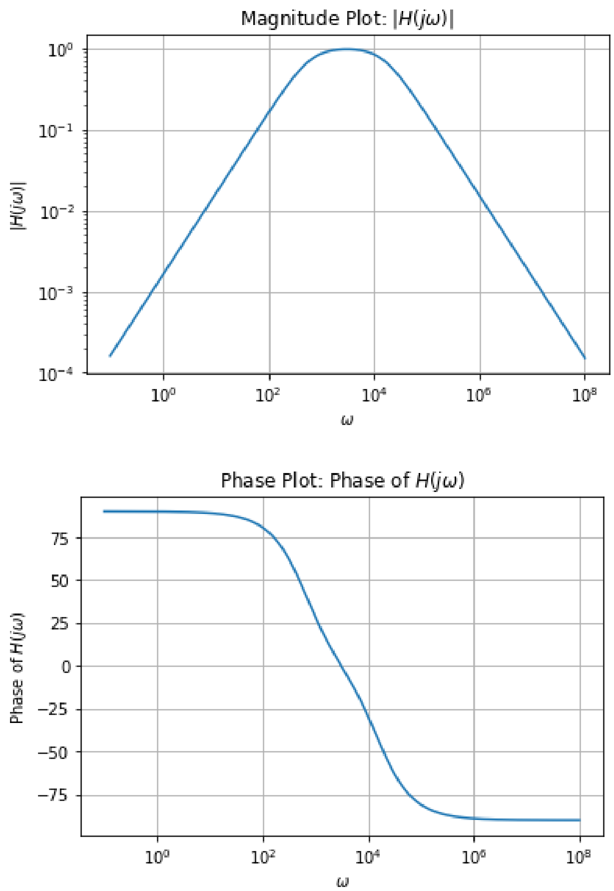


Figure 5: Magnitude and Phase transfer functions

(d) **Write down an expression for the time-domain output waveform $V_{\text{out}}(t)$ of this filter if the input voltage is $V_{\text{in}}(t) = 1\sin(1000t)$V.** Round your answer to 2 significant digits.

Solution: We can find the transfer function at this point:

$$\left| H(\mathrm{j}\omega = \mathrm{j}10^3) \right| = 0.85 \tag{33}$$

$$\angle H(\mathrm{j}\omega = \mathrm{j}10^3) = 0.49 \text{ rad} = 28.23^\circ \tag{34}$$

Therefore the output will be:

$$V_{\text{out}}(t) = 0.85\sin(1000t + 0.49)\text{V}. \tag{35}$$

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