Homework 3

This homework is due on Friday, February 10, 2023 at 11:59PM. Self-grades and HW Resubmissions are due the following Friday, February 17, 2023 at 11:59PM.

1. NAND Circuit

Let us consider a NAND logic gate. This circuit implements the boolean function $\overline{(A \cdot B)}$. The $\cdot$ stands for the AND operation, and the $\overline{}$ stands for NOT; combining them, we get NAND!

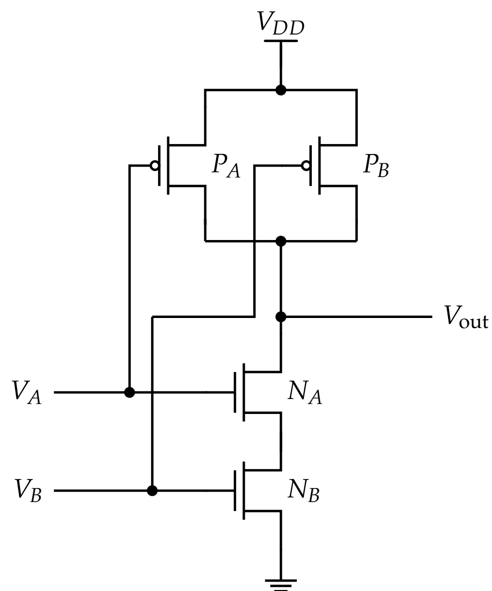


Figure 1: NAND gate transistor-level implementation.

V_{tn} and V_{tp} are the threshold voltages for the NMOS and PMOS transistors, respectively. Assume that $V_{DD} > V_{tn}, |V_{tp}| > 0$.

(a) **Label the gate, source, and drain nodes for the NMOS and PMOS transistors (please redraw the circuit).**

Solution: As a convention throughout the course, we will draw NMOS transistors with their source at the bottom (and drain at the top). On the other hand, PMOS transistors will have their source at the top. Therefore, the drains are at the top of N_A (connected to V_{out}) and the top of N_B (connected to N_A). The sources are at the bottom of N_A (connected to N_B) and the bottom of N_B (connected to ground). The gate terminal of N_A is connected to V_A; the gate of N_B is connected to V_B.

For the PMOS transistors, the source is at the top of P_A and P_B (connected to V_{DD}). The drain is at the bottom of P_A and P_B (connected to V_{out}). The gate terminal of P_A is connected to V_A; the gate of P_B is connected to V_B.

(b) If $V_A = V_{DD}$ and $V_B = V_{DD}$, **which transistors act like open switches? Which transistors act like closed switches? What is V_{out}?**

Solution: P_A and P_B are off (open switches). N_B and N_A are on (closed switches). $V_{\text{out}} = 0\text{V}$ because it is connected to ground through a closed circuit consisting of N_A and N_B (and detached from V_{DD}).

(c) **If $V_A = 0V$ and $V_B = V_{DD}$, what is V_{out}?**

Solution: P_B and N_A are off (open switches). P_A and N_B are on (closed switches). $V_{\text{out}} = V_{DD}$ because it is connected to V_{DD} through a closed circuit consisting of P_A (and detached from ground, since *both* N_A and N_B must be closed for V_{out} to be connected to ground).

(d) **If $V_A = V_{DD}$ and $V_B = 0V$, what is V_{out}?**

Solution: P_A and N_B are off (open switches), P_B is on (closed switch). So, $V_{\text{out}} = V_{DD}$ because it is connected to V_{DD} through a closed switch.

(e) **If $V_A = 0V$ and $V_B = 0V$, what is V_{out}?**

Solution: N_B is off, creating an open circuit. P_A and P_B are on, creating a closed circuit. $V_{\text{out}} = V_{DD}$ because it is connected by closed circuit to V_{DD}.

(f) **Write out the truth table for this circuit.**

V_A	V_B	V_{out}
0	0	
0	V_{DD}	
V_{DD}	0	
V_{DD}	V_{DD}	

Solution:

V_A	V_B	V_{out}
0	0	V_{DD}
0	V_{DD}	V_{DD}
V_{DD}	0	V_{DD}
V_{DD}	V_{DD}	0

2. Hambley P4.61

A DC source is connected to a series RLC circuit by a switch that closes at $t = 0$, as shown in Figure 2. The initial conditions are $i(0+) = 0$ and $v_C(0+) = 25$. Write the differential equation for $v_C(t)$. Solve for $v_C(t)$ given that $R = 80\,\Omega$.

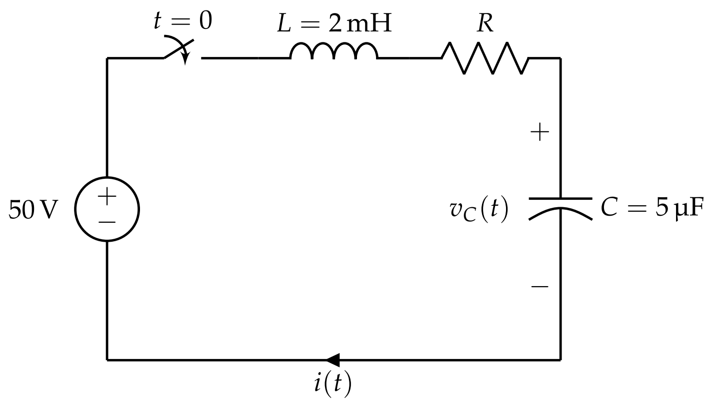


Figure 2: P4.61

Solution: We can apply KVL to the circuit to obtain:

$$
\begin{aligned}
50 &= v_L(t) + v_R(t) + v_C(t) && (1)\\
&= L\frac{\mathrm{d}i(t)}{\mathrm{d}t} + i(t)R + v_C(t) && (2)\\
&= L\frac{\mathrm{d}i(t)}{\mathrm{d}t} + RC\frac{\mathrm{d}v_C(t)}{\mathrm{d}t} + v_C(t) && (3)
\end{aligned}
$$

Now, we have that $L\frac{\mathrm{d}i(t)}{\mathrm{d}t} = L\frac{\mathrm{d}}{\mathrm{d}t}\left(C\frac{\mathrm{d}v_C(t)}{\mathrm{d}t}\right) = LC\frac{\mathrm{d}^2v_C(t)}{\mathrm{d}t^2}$. Plugging this in, we get

$$
\begin{aligned}
LC\frac{\mathrm{d}^2v_C(t)}{\mathrm{d}t^2} + RC\frac{\mathrm{d}v_C(t)}{\mathrm{d}t} + v_C(t) &= 50 && (4)\\
\frac{\mathrm{d}^2v_C(t)}{\mathrm{d}t^2} + \frac{R}{L}\frac{\mathrm{d}v_C(t)}{\mathrm{d}t} + \frac{1}{LC}v_C(t) &= \frac{50}{LC} && (5)
\end{aligned}
$$

We have a second order differential equation, so our solution will be of the form $v_C(t) = v_{C_P}(t) + v_{C_H}(t)$, where $v_{C_P}(t)$ is the particular solution and $v_{C_H}(t)$ is the homogeneous solution. Here, we have a DC forcing function (i.e., $f(t) = \frac{50}{LC}$). Hence, the particular solution would be the solution if we replaced inductors with short circuits and capacitances with open circuits. This yields $v_{C_P}(t) = 50$. To find α and ω_0, we can pattern match $\omega_0 = \sqrt{\frac{1}{LC}} = 10^4$ and $2\alpha = \frac{R}{2L} \implies \alpha = \frac{R}{2L} = 2 \times 10^4$ from Note 5. Since $\alpha > \omega_0$, the homogeneous solution will be of the form

$$v_{C_H}(t) = K_1 e^{s_1 t} + K_2 e^{s_2 t} \tag{6}$$

where $s_1 = -\alpha + \sqrt{\alpha^2 - \omega_0^2} = -2679.49$ and $s_2 = -\alpha - \sqrt{\alpha^2 - \omega_0^2} = -37320.5$. Hence, the final solution is of the form

$$v_C(t) = v_{C_P}(t) + v_{C_H}(t) = 50 + K_1 e^{s_1 t} + K_2 e^{s_2 t} \tag{7}$$

To find K_1 and K_2, we can utilize the fact that $v_C(0) = 25$ and $\frac{\mathrm{d}v_C(t)}{\mathrm{d}t}|_{t=0} = \frac{i(0)}{C} = 0$. Plugging these in, we get the following system:

$$v_C(0) = 25 = 50 + K_1 + K_2 \tag{8}$$

$$\frac{\mathrm{d}v_C(t)}{\mathrm{d}t}\Big|_{t=0} = 0 = s_1 K_1 + s_2 K_2 \tag{9}$$

Solving this system of equation yields $K_1 = -26.93$ and $K_2 = 1.93$. Hence, the final answer is

$$v_C(t) = 50 + (-26.93)\mathrm{e}^{-2679.49t} + (1.93)\mathrm{e}^{-37320.5t} \tag{10}$$

3. Hambley P4.64

Consider the circuit shown in Figure 3, with $R = 25\,\Omega$.

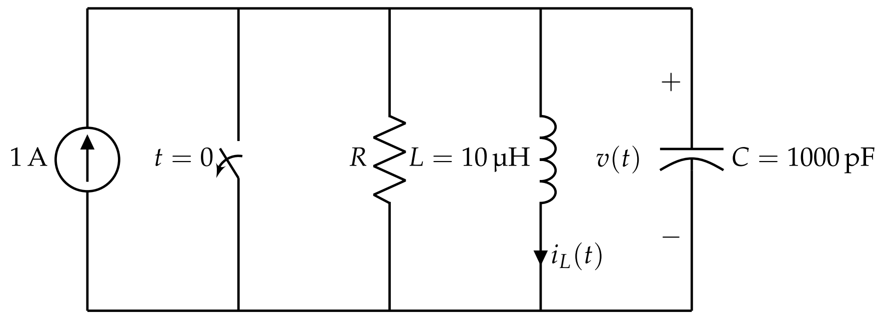


Figure 3: P4.64

(a) Compute the undamped resonant frequency, ω_0, and α.

Solution: From KCL, we have

$$1 = i_R(t) + i_L(t) + i_C(t) \tag{11}$$
$$= \frac{v(t)}{R} + i_L(t) + C\frac{\mathrm{d}v(t)}{\mathrm{d}t} \tag{12}$$

Taking derivatives on both sides, we get

$$C\frac{\mathrm{d}^2v(t)}{\mathrm{d}t^2} + \frac{1}{R}\frac{\mathrm{d}v(t)}{\mathrm{d}t} + \frac{\mathrm{d}i_L(t)}{\mathrm{d}t} = 0 \tag{13}$$

We also know that $v(t) = L\frac{\mathrm{d}i_L(t)}{\mathrm{d}t}$. Plugging this in, we get

$$C\frac{\mathrm{d}^2v(t)}{\mathrm{d}t^2} + \frac{1}{R}\frac{\mathrm{d}v(t)}{\mathrm{d}t} + \frac{1}{L}v(t) = 0 \tag{14}$$
$$\frac{\mathrm{d}^2v(t)}{\mathrm{d}t^2} + \frac{1}{RC}\frac{\mathrm{d}v(t)}{\mathrm{d}t} + \frac{1}{LC}v(t) = 0 \tag{15}$$

Pattern matching to Note 5, we get $\omega_0 = \sqrt{\frac{1}{LC}} = 1 \times 10^7$ and $2\alpha = \frac{1}{RC} = 4 \times 10^7 \implies \alpha = 2 \times 10^7$. Hence, it is an overdamped circuit ($\alpha > \omega_0$).

(b) The initial conditions are $v(0+) = 0$ and $i_L(0+) = 0$. Show that this requires $v'(0+) = 10^9\,\frac{\mathrm{V}}{\mathrm{s}}$.

Solution: We still must satisfy KCL at $t = 0+$, so we have

$$1 = i_R(0+) + i_L(0+) + i_C(0+) \tag{16}$$
$$= \frac{v(0+)}{R} + i_L(0+) + Cv'(0+) \tag{17}$$
$$= Cv'(0+) \tag{18}$$

This leaves us with $v'(0+) = \frac{1}{C} = 10^9\,\frac{\mathrm{V}}{\mathrm{s}}$.

(c) Find the particular solution for $v(t)$.

Solution: To find the particular solution, we first notice that the forcing function is $f(t) = 0$ which is a constant. Hence, we can replace capacitors with open circuits and inductors with short circuits. If we were to do this, all of the current flows through the branch with the inductor and the particular solution is $v_P(t) = 0$.

(d) Find the general solution for $v(t)$, including the numerical values of all parameters.

Solution: Since $\alpha > \omega_0$, the homogeneous solution will be of the form

$$v_C(t) = K_1 e^{s_1 t} + K_2 e^{s_2 t} \tag{19}$$

where $s_1 = -\alpha + \sqrt{\alpha^2 - \omega_0^2} = -2.68 \times 10^6$ and $s_2 = -\alpha - \sqrt{\alpha^2 - \omega_0^2} = -3.73 \times 10^7$. Since the particular solution $v_P(t) = 0$, we have that

$$v(t) = K_1 e^{s_1 t} + K_2 e^{s_2 t} \tag{20}$$

Now, we will use our initial conditions of $v(0) = 0$ and $v'(0) = 10^9$. Plugging these in, we get the following system of equations:

$$v(0) = 0 = K_1 + K_2 \tag{21}$$

$$v'(0) = 10^9 = s_1 K_1 + s_2 K_2 \tag{22}$$

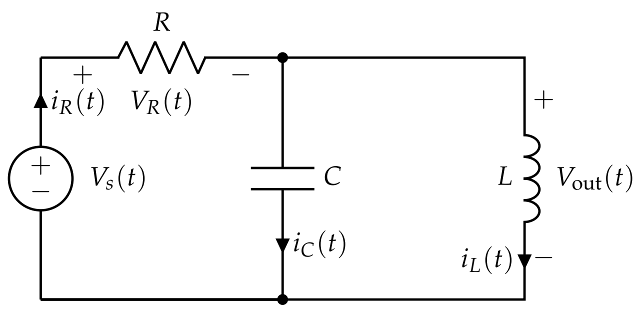
Solving the system of equations yields $K_1 = 28.89$ and $K_2 = -28.89$. Thus, the final answer is

$$v(t) = 28.89 e^{(-2.68 \times 10^6) t} - 28.89 e^{(-3.73 \times 10^7) t} \tag{23}$$

4. Phasor-Domain Circuit Analysis

The analysis techniques you learned previously in 16A for resistive circuits are equally applicable for analyzing circuits driven by sinusoidal inputs in the phasor domain. In this problem, we will walk you through the steps with a concrete example.

Consider the following circuit where the input voltage is sinusoidal. The end goal of our analysis is to find an equation for $V_{\text{out}}(t)$.



The components in this circuit are given by:

$$V_s(t) = 10\sqrt{2}\cos\left(100t - \frac{\pi}{4}\right) \tag{24}$$

$$R = 5\,\Omega \tag{25}$$

$$L = 50\,\text{mH} \tag{26}$$

$$C = 2\,\text{mF} \tag{27}$$

(a) **Give the amplitude V_0, input frequency ω, and phase ϕ of the input voltage V_s.**

Solution: A sinusoid takes the form $v(t) = V_0\cos(\omega t + \phi)$. Given $V_s(t)$, we find:

$$V_0 = 10\sqrt{2}\text{V} \tag{28}$$

$$\omega = 100\frac{\text{rad}}{\text{s}} \tag{29}$$

$$\phi = -\frac{\pi}{4}\text{rad} \tag{30}$$

(b) Transform the circuit into the phasor domain. **What are the impedances of the resistor, capacitor, and inductor? What is the phasor $\widetilde{V_S}$ of the input voltage $V_s(t)$?**

Solution:

$$Z_L = \text{j}\omega L = \text{j}5\Omega \tag{31}$$

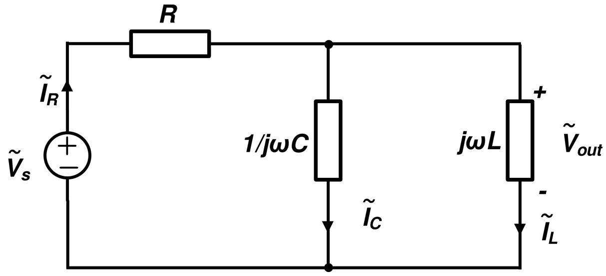
$$Z_C = \frac{1}{\text{j}\omega C} = -\text{j}5\Omega \tag{32}$$

$$Z_R = R = 5\Omega \tag{33}$$

$$\widetilde{V_s} = |V_s|\text{e}^{\text{j}\angle V_s} = 10\sqrt{2}\text{e}^{-\text{j}\frac{\pi}{4}} \tag{34}$$

(c) Use the circuit equations to **solve for** $\widetilde{V}_{\text{out}}$, the phasor representing the output voltage.

Solution: The phasor representation of the circuit is shown below:



Where

$$\widetilde{I}_R = \frac{\widetilde{V}_S - \widetilde{V}_{\text{out}}}{R} \tag{35}$$

$$\widetilde{I}_L = \frac{\widetilde{V}_{\text{out}}}{\mathrm{j}\omega L} \tag{36}$$

$$\widetilde{I}_C = \widetilde{V}_{\text{out}} \cdot \mathrm{j}\omega C \tag{37}$$

Rewriting the current relation in terms of voltage phasors gives:

$$\frac{\widetilde{V}_S - \widetilde{V}_{\text{out}}}{R} = \frac{\widetilde{V}_{\text{out}}}{\mathrm{j}\omega L} + \widetilde{V}_{\text{out}} \cdot \mathrm{j}\omega C \tag{38}$$

Substituting the component values in the above equation we get

$$\frac{\widetilde{V}_S - \widetilde{V}_{\text{out}}}{5} = \frac{\widetilde{V}_{\text{out}}}{5\mathrm{j}} + \frac{\widetilde{V}_{\text{out}} \cdot \mathrm{j}}{5} \tag{39}$$

$$= \frac{\widetilde{V}_{\text{out}}}{5\mathrm{j}} - \frac{\widetilde{V}_{\text{out}}}{5\mathrm{j}} \tag{40}$$

$$= 0 \tag{41}$$

Which gives:

$$\widetilde{V}_{\text{out}} = \widetilde{V}_S \tag{42}$$

We found that $\widetilde{V}_{\text{out}} = \widetilde{V}_S$ because this circuit is in resonance; i.e., the capacitor and inductor have the exact values that cause current and voltage to endlessly oscillate between them at this frequency. If we chose a different value for ω with these same component values, the circuit would not be in resonance and $\widetilde{V}_{\text{out}}$ and $\widetilde{V}_S$ would no longer be equal.

One may think that this answer seems weird. For $\widetilde{V}_{\text{out}}$ to equal $\widetilde{V}_S$ means that no current is flowing through the resistor. This means that somehow, the impedance of the parallel L and C combination would have to be infinity. Let's check what that is:

$$Z_L \parallel Z_C = \frac{(\mathrm{j}5) \cdot (-\mathrm{j}5)}{\mathrm{j}5 + (-\mathrm{j}5)} = +\infty \tag{43}$$

Wow! Indeed it is infinity. This shows something counterintuitive that can occur with phasors and impedances. For resistors, one may think that parallel connections always lower the resistance. However, since imaginary impedances can be positive imaginary and negative imaginary, a parallel connection can make the impedance bigger or smaller. The same kind of counterintuitive behavior is also possible for series combinations. Resistors in series always increase the resistance. But the same L and C in series can combine to have a zero impedance at the natural frequency.

If one wants to know why **something divided by 0** is ∞ in the complex plane, read this Wiki article: Riemann Sphere. This is another facet of complex analysis, and why engineers were drawn to it when modeling physical systems for design purposes.

5. Hambley P6.55

Consider the circuit shown in Figure 4. The input signal is given by

$$v_{\text{in}}(t) = 5 + 5\cos(2000\pi t) \tag{44}$$

Find an expression for the output $v_{\text{out}}(t)$ in steady-state conditions.

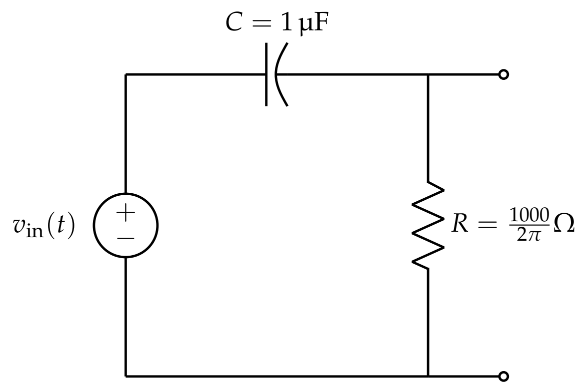


Figure 4: P6.55

(HINT: Use superposition. That is, find $v_{\text{out},1}(t)$ which is the output voltage if the input is $v_{\text{in},1}(t) = 5$, and then find $v_{\text{out},2}(t)$ which is the output voltage if the input is $v_{\text{in},2}(t) = 5\cos(2000\pi t)$. What is $v_{\text{out}}(t)$ in terms of $v_{\text{out},1}(t)$ and $v_{\text{out},2}(t)$?)

Solution: The input signal is given by

$$v_{\text{in}}(t) = 5 + 5\cos(2000\pi t) \tag{45}$$

We can find $\widetilde{V}_{\text{out}}$ as a function of ω and $\widetilde{V}_{\text{in}}$, then apply superposition. Applying the fact that $Z_C = \frac{1}{\text{j}\omega C}$ and $Z_R = R$, we can use the voltage divider formula to write

$$\begin{aligned} \widetilde{V}_{\text{out}} &= \frac{Z_R}{Z_R + Z_C}\widetilde{V}_{\text{in}} && (46)\\ &= \frac{R}{R + \frac{1}{\text{j}\omega C}}\widetilde{V}_{\text{in}} && (47)\\ &= \frac{\text{j}\omega RC}{\text{j}\omega RC + 1}\widetilde{V}_{\text{in}} && (48) \end{aligned}$$

Now, let's first consider $v_{\text{in},1}(t) = 5$ as per the hint. We have $\widetilde{V}_{\text{in},1} = 5e^{\text{j}0}$, and the corresponding frequency is $\omega = 0$ (we can write $v_{\text{in},1}(t) = 5 = 5\cos(0 \cdot t + 0)$). Thus, plugging into eq. (48), we get $\widetilde{V}_{\text{out},1} = \frac{\text{j}0}{\text{j}0+1} = 0$. Hence, $v_{\text{out},1}(t) = 0$. Next, we consider $v_{\text{in},2}(t) = 5\cos(2000\pi t)$. We have $\widetilde{V}_{\text{in},2} = 5e^{\text{j}0}$, and the corresponding frequency is $\omega = 2000\pi$. Thus, plugging into eq. (48), we get $\widetilde{V}_{\text{out},2} = \frac{\text{j}(2000\pi)RC}{\text{j}(2000\pi)RC+1} \cdot 5$. Plugging in for the provided values of R and C, we obtain $\widetilde{V}_{\text{out},2} = 5\frac{\text{j}}{1+\text{j}}$. To convert this to polar form, we compute the magnitude and phase as follows:

$$\left|\widetilde{V}_{\text{out},2}\right| = 5\left|\frac{\text{j}}{1+\text{j}}\right| = 5\frac{|\text{j}|}{|1+\text{j}|} = \frac{5}{\sqrt{2}} \tag{49}$$

$$\angle\widetilde{V}_{\text{out},2} = \angle\left(\frac{5\text{j}}{1+\text{j}}\right) = \angle(5\text{j}) - \angle(1+\text{j}) = \frac{\pi}{2} - \frac{\pi}{4} = \frac{\pi}{4} \tag{50}$$

Thus, we can equivalently write $\widetilde{V}_{\text{out},2} = \frac{5}{\sqrt{2}} e^{j\frac{\pi}{4}}$. Converting back into time domain, we get $v_{\text{out},2}(t) = \frac{5}{\sqrt{2}} \cos\left(2000\pi t + \frac{\pi}{4}\right)$. Now, we can apply superposition to find $v_{\text{out}}(t)$. Since we decomposed the input voltage as a sum of its components, we have

$$v_{\text{out}}(t) = \underbrace{v_{\text{out},1}(t)}_{0} + \underbrace{v_{\text{out},2}(t)}_{\frac{5}{\sqrt{2}}\cos\left(2000\pi t + \frac{\pi}{4}\right)} = \frac{5}{\sqrt{2}} \cos\left(2000\pi t + \frac{\pi}{4}\right) \tag{51}$$

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