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## Homework 2

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**This homework is due on Friday, February 4, 2022 at 11:59PM. Self-grades and HW Resubmissions are due the following Friday, February 11, 2022 at 11:59PM.**

### 1. Reading Lecture Notes

Staying up to date with lectures is an important part of the learning process in this course. Here are links to the notes that you need to read for this week: [Note 2](#) and [Note 3](#).

- (a) **How do we deal with piecewise constant inputs as introduced in the notes?**

**Solution:** For piecewise constant inputs, we can just treat them in the same way that we dealt with circuits with switches changing configuration. Make the state be instantaneously constant across the configuration change, and solve the differential equation with that initial condition.

- (b) **What conclusions do we get after approximating any function  $u(t)$  as being piecewise constant over fixed interval widths  $\Delta$ ?**

**Solution:** After taking the limit  $\Delta \rightarrow 0$ , we get the solution to the differential equation  $\frac{d}{dt}V(t) = \lambda V(t) - \lambda u(t)$  to be  $V(t) = v_0 e^{\lambda t} - \lambda \int_0^t u(\tau) e^{\lambda(t-\tau)} d\tau$ .

## 2. Uniqueness Counterexample

This problem explores an example of a differential equation that does not have a unique solution. The purpose is to show that uniqueness cannot always be assumed.

Along the way, this problem will also show you a heuristic way to guess the solutions to differential equations that is often called “separation-of-variables.” The advantage of the separation-of-variables technique is that it can often be helpful in systematically coming up with guesses for nonlinear differential equations. However, as with any technique for guessing, it is not a proof and the guess definitely needs to be checked and uniqueness verified before proceeding.

The idea of separation-of-variables is to pretend that  $\frac{d}{dt}x(t) = \frac{dx}{dt}$  is a ratio of quantities rather than what it is — a shorthand for taking the derivative of the function  $x(\cdot)$  with respect to its single argument, and then writing the result in terms of the free variable “ $t$ ” for that argument. This little bit of make-believe (sometimes called “an abuse of notation”) allows one the freedom to do calculations.

To demonstrate, let’s do this for a case where we know the correct solution:  $\frac{d}{dt}x(t) = \lambda x(t)$ . This is how a separation-of-variables approach would try to get a guess:

$$\frac{d}{dt}x(t) = \lambda x(t) \tag{1}$$

$$\frac{dx}{dt} = \lambda x \tag{2}$$

$$\frac{dx}{x} = \lambda dt \quad \text{separating variables to sides} \tag{3}$$

$$\int \frac{dx}{x} = \int \lambda dt \quad \text{integrating both sides} \tag{4}$$

$$\ln(x) + C_1 = \lambda t + C_2 \tag{5}$$

$$x(t) = Ke^{\lambda t} \quad \text{exponentiating both sides and folding constants} \tag{6}$$

With the above guess obtained,  $x(t) = Ke^{\lambda t}$  can be plugged in and seen to solve the original differential equation. Then of course, a uniqueness proof is required, but you did that in the previous homework.

To see why this technique can cause trouble, we will consider the following nonlinear differential equation involving a third root<sup>1</sup>.

$$\frac{d}{dt}x(t) = \alpha x^{\frac{1}{3}} \tag{7}$$

with the initial condition

$$x(0) = 0. \tag{8}$$

Let’s apply separation-of-variables and see what happens:

$$\frac{d}{dt}x(t) = \alpha x^{\frac{1}{3}} \tag{9}$$

$$\frac{dx}{dt} = \alpha x^{\frac{1}{3}} \tag{10}$$

$$x^{-\frac{1}{3}} dx = \alpha dt \tag{11}$$

$$\int x^{-\frac{1}{3}} dx = \int \alpha dt \tag{12}$$

$$\frac{3}{2}x^{\frac{2}{3}} + C_1 = \alpha t + C_2 \tag{13}$$

<sup>1</sup>This type of differential equation can arise from a physical setting of a inverted pyramidal container that had  $x(t)$  liters of water in it, where the rate of water being poured in is proportional to the height of the water  $x^{\frac{1}{3}}$ . This fractional power arises since volume is a cubic quantity while the water is being poured in at a rate governed by a one-dimensional quantity of length. Similar equations can arise in microfluidic dynamics.

$$x = \left( \frac{2}{3}\alpha t + C_3 \right)^{\frac{3}{2}} \quad (14)$$

(a) Given our separation-of-variables based calculation, let us guess a solution of the form

$$x(t) = \left( \frac{2}{3}\alpha t + c \right)^{\frac{3}{2}} \quad (15)$$

**Show that this is a solution to the differential equation (7), and find the  $c$  that satisfies the initial condition.** (HINT: You'll need to use the power rule and chain rule.)

**Solution:** We check that this is a solution by differentiating the guess for  $x(t)$ , and showing that it satisfies (7).

$$\begin{aligned} \frac{d}{dt}x(t) &= \frac{d}{dt} \left( \left( \frac{2}{3}\alpha t + c \right)^{\frac{3}{2}} \right) \\ &= \frac{3}{2} \cdot \left( \frac{2}{3}\alpha t + c \right)^{\frac{1}{2}} \cdot \frac{2}{3}\alpha \\ &= \alpha \left( \frac{2}{3}\alpha t + c \right)^{\frac{1}{2}} \\ &= \alpha \left( \left( \frac{2}{3}\alpha t + c \right)^{\frac{3}{2}} \right)^{\frac{1}{3}} \\ &= \alpha x^{\frac{1}{3}} \end{aligned}$$

We thus satisfy (7).

To solve for the initial condition, we evaluate

$$\begin{aligned} 0 &= x(0) = \left( \frac{2}{3}\alpha \cdot 0 + c \right)^{\frac{3}{2}} \\ &= \left( \frac{2}{3} \cdot c \right)^{\frac{3}{2}} \\ &\rightarrow c = 0 \end{aligned}$$

This is the only solution to the found form that satisfies the initial condition.

The equation  $x(t) = \left( \frac{2}{3}\alpha t \right)^{\frac{3}{2}}$  thus satisfies the differential equation and the initial value condition.

(b) Let us guess a second solution:

$$x(t) = 0 \quad (16)$$

**Show that this new guess also satisfies (7), and the initial condition ( $x(0) = 0$ ).**<sup>2</sup>

**Solution:**

$$\frac{d}{dt}x(t) = \frac{d}{dt}0 = 0 = \alpha \cdot 0^{\frac{1}{3}} = \alpha \cdot x(t) \quad (18)$$

Furthermore,  $x(0) = 0$ .

So this second solution also satisfies the differential equation and the initial value condition.

<sup>2</sup>Indeed, any solution of the form

$$x(t) = \begin{cases} 0, & \text{if } t < t_0 \\ \left( \frac{2}{3}\alpha (t - t_0) \right)^{\frac{3}{2}}, & \text{if } t \geq t_0 \end{cases} \quad (17)$$

also satisfies (7) and the initial condition  $x(0) = 0$ , for any  $t_0 > 0$ , which concludes that (7) has infinitely many solutions. We leave the verification of this solution for those who are interested.

- (c) A known (not by you yet, but by the mathematical community) sufficient condition for the uniqueness of solutions to differential equations of the form  $\frac{d}{dt}x(t) = f(x(t))$  is that the function  $f(x)$  be continuously differentiable (i.e.  $\frac{d}{dx}f(x)$  is a continuous function of  $x$ ) with a bounded derivative  $\frac{d}{dx}f(x)$  at the initial condition  $x(0)$  and everywhere that the solution  $x(t)$  purports to go. (You will understand the importance of this condition and where it comes from better when we are in Module 2 of 16B. We are not going to prove it.)

**Does this differential equation problem satisfy this condition that would let us trust guessing and checking?**

**Solution:** No. To see this, consider our differential equation (7) at  $t = 0$ . In this case,  $f(x) = \alpha x^{\frac{1}{3}}$ .

We see that  $\frac{d}{dx}f(x) = \frac{d}{dx}\left(\alpha x^{\frac{1}{3}}\right) = \frac{\alpha}{3}x^{-\frac{2}{3}} = \frac{\alpha}{3x^{\frac{2}{3}}}$ .

Note that at  $x(t = 0) = 0$ , our initial condition, the derivative becomes discontinuous in an unbounded fashion. It blows up. To be precise, if we take the limit

$$\lim_{x \rightarrow 0^+} \frac{d}{dx}f(x) = \lim_{x \rightarrow 0^+} \frac{\alpha}{3x^{\frac{2}{3}}} = +\infty.$$

Thus this differential equation does not satisfy the uniqueness condition that we have provided.

- (d) The separation-of-variables technique may involve steps that may not agree with the initial condition. **Explain why (11) might be a bit problematic.** (HINT: When is it not permissible to divide both sides of an equation by the same thing?)

**Solution:** The problem with (11) is that there is  $x^{-\frac{1}{3}}$  on the left hand side. If  $x = 0$ , this results in division by 0. As  $x(0) = 0$  is the initial condition, this unbounded point becomes a problem for the separation-of-variables technique.

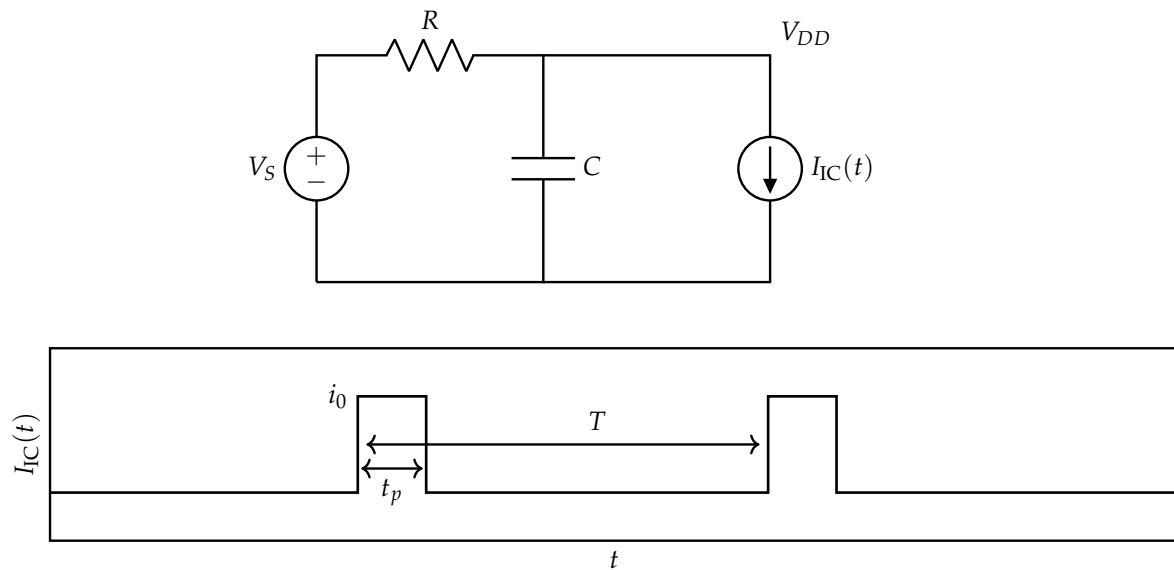
- (e) **Write an example of a differential equation that satisfies this uniqueness condition, and explain why.**

**Solution:** The simplest example would be  $\frac{d}{dt}x(t) = \lambda x(t)$  that we introduced earlier. Setting  $f(x) = \lambda x(t)$ , the derivative  $\frac{d}{dx}f(x) = \lambda$  is continuous and bounded.

### 3. IC Power Supply

Digital integrated circuits (ICs) often have very non-uniform current requirements which can cause voltage noise on the supply lines. If one IC is adding a lot of noise to the supply line, it can affect the performance of other ICs that use the same power supply, which can hinder performance of the entire device. For this reason, it is important to take measures to mitigate, or “smooth out”, the power supply noise that each IC creates. A common way of doing this is to add a “supply capacitor” between each IC and the power supply. (If you look at a circuit board, and the supply capacitor is the small capacitor next to each IC.)

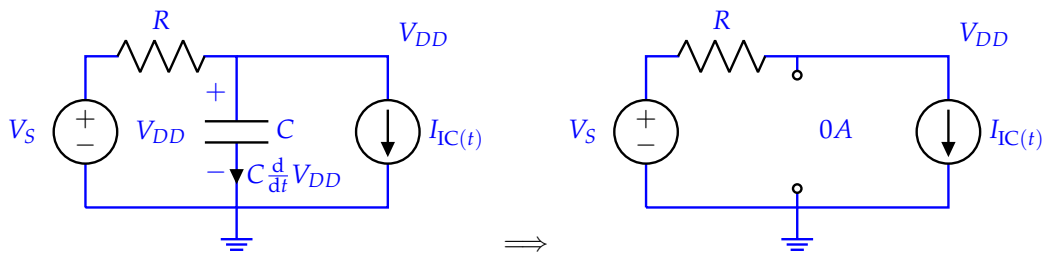
Here’s a simple model for a power supply and digital circuit:



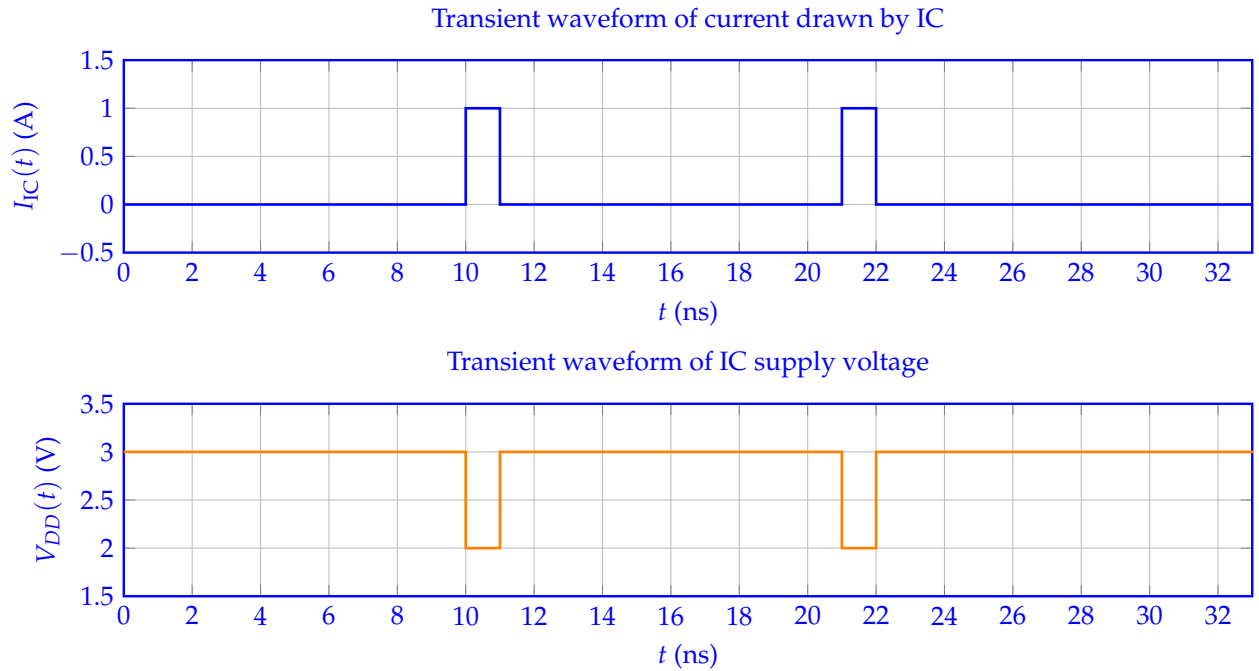
The current source is modeling the “spiky,” non-uniform nature of digital circuit current consumption. The resistor represents the sum of the source resistance of the supply and any wiring resistance between the supply and the load.

The capacitor is added to minimize the noise on  $V_{DD}$ . Assume that  $V_S = 3\text{ V}$ ,  $R = 1\ \Omega$ ,  $i_0 = 1\text{ A}$ ,  $T = 11\text{ ns}$ , and  $t_p = 1\text{ ns}$ .

(a) Sketch the voltage  $V_{DD}$  vs. time for two  $T$  periods assuming that  $C = 0$ . **Solution:**



If  $C = 0$ , the capacitor behaves as an open circuit with  $I_C = C \frac{d}{dt} V_{DD} = 0\text{ A}$ . This circuit will respond instantaneously to changes in the current; thus we may break this down into two different time segments, wherein the current source  $I_{IC}(t)$  equals 0 (and thus  $V_{DD} = V_S$  as no current flows through the resistor), and where the current source  $I_{IC}(t)$  equals  $i_0$  (and thus  $V_{DD} = V_S - i_0 R = 2\text{ V}$ ). These will follow the current source’s flips precisely. With that in mind, your sketch should look something like this:



In the above waveform, we have the first current spike at  $t = 10$  ns. Yours doesn't have to align with that: for example, if you had the first current spike at  $t = 0$ , that's okay. However, what *does* matter is that you have the timing between the current spikes drawn correctly.

- (b) **Give expressions for and sketch the voltage  $V_{DD}$  vs. time for two  $T$  periods for each of three different capacitor values for  $C$ : 1 pF, 1 nF, 1  $\mu$ F. (1 pF =  $1 \times 10^{-12}$  F, 1 nF =  $1 \times 10^{-9}$  F, 1  $\mu$ F =  $1 \times 10^{-6}$  F). For this part, to find the initial condition for  $V_{DD}$ , feel free to assume that for a very long time,  $I_{IC} = 0$ .**

**Solution:** Since the current through the source is a series of repeating pulses, where there is no current for a long time and then there is a constant current for a short time, we solve for  $V_{DD}(t)$  by writing a piecewise constant function for the current flowing through the circuit,  $I_{IC}$ . Starting with KVL:

$$V_S = V_R + V_{DD} \quad (19)$$

$$= \left( I_{IC} + C \frac{d}{dt} V_{DD} \right) R + V_{DD} \quad (20)$$

$$\frac{d}{dt} V_{DD} = \frac{1}{RC} (V_S - R I_{IC} - V_{DD}). \quad (21)$$

At this point we can use substitution with  $\tilde{V} = V_S - R I_{IC} - V_{DD}$ .

$$\frac{d}{dt} \tilde{V} = -\frac{d}{dt} V_{DD} \quad (22)$$

$$\frac{d}{dt} V_{DD} = \frac{\tilde{V}}{RC} \quad (23)$$

$$\frac{d}{dt} \tilde{V} = -\frac{\tilde{V}}{RC} \quad (24)$$

$$\tilde{V}(t) = A e^{-\frac{t}{RC}} \quad (25)$$

Substituting back to solve for  $V_{DD}(t)$  we get the following general expression for the voltage during any one piecewise constant time interval:

$$V_{DD}(t) = V_S - RI_{IC} - Ae^{-\frac{t}{RC}} \quad (26)$$

From here, based on the value of the piecewise current  $I_{IC}$  and the initial conditions imposed by previous time segments we can simplify the  $V_{DD}$  expression and solve for  $A$ . At the start, we will assume a convenient initial condition which corresponds with the behavior of  $V_{DD}$  if  $I_{IC} = 0$  for a long time before  $t = 0$ . In this case,  $V_{DD} = V_S$  and there is zero current flowing through the circuit. Things get exciting once the first current pulse starts such that  $I_{IC} = i_0$ . The initial condition for this piecewise section is the final voltage  $V_{DD}$  from the previous section,  $V_S$ .

$$V_S = V_S - Ri_0 - Ae^0 \quad (27)$$

$$A = -Ri_0 \quad (28)$$

$$V_{DD}(t) = V_S - Ri_0(1 - e^{-\frac{t}{RC}}) \quad (29)$$

You can compute the voltage at the end of the pulse by plugging in  $t = t_p$  and the  $R$  and  $C$  values for your scenario. This voltage will serve as the initial condition for the next piecewise constant section. The process of simplifying the general piecewise differential equation and solving for  $A$  can be performed repeatedly to determine the shape of the plot for further pulses. Below shows the steps to compute  $V_{DD}(t)$  over  $t$  for the piecewise function  $I_{IC}(t)$ .

$$I_{IC}(t) = \begin{cases} 0 \text{ A} & t < 10\text{ns} \\ 1 \text{ A} & 10\text{ns} \leq t < 11\text{ns} \\ 0 \text{ A} & 11\text{ns} \leq t < 21\text{ns} \\ 1 \text{ A} & 21\text{ns} \leq t < 22\text{ns} \\ 0 \text{ A} & 22\text{ns} \leq t \end{cases} \quad (30)$$

For  $t < 10\text{ns}$ :

$$V_{DD}(t) = V_S = 3\text{V} \quad (31)$$

For  $10\text{ns} \leq t < 11\text{ns}$ :

$$V_{DD}(t) = V_S - R \cdot I_{IC} - A_{10\text{ns}}e^{-(t-10\text{ns})/(RC)}, \quad I_{IC} = 1 \text{ A} \quad (32)$$

$$V_{DD}(10\text{ns}) = 3\text{V} - 1\text{V} - A_{10\text{ns}} = 3\text{V} \quad (33)$$

$$A_{10\text{ns}} = -1\text{V} \quad (34)$$

$$V_{DD}(t) = 3\text{V} - 1\text{V} \left(1 - e^{-(t-10\text{ns})/(RC)}\right) \quad (35)$$

For  $11\text{ns} \leq t < 21\text{ns}$ :

$$V_{DD}(t) = V_S - R \cdot I_{IC} - A_{11\text{ns}}e^{-(t-11\text{ns})/(RC)}, \quad I_{IC} = 0 \text{ A} \quad (36)$$

$$V_{DD}(11\text{ns}) = 3\text{V} - A_{11\text{ns}} = 3\text{V} - 1\text{V} \left(1 - e^{-(11\text{ns}-10\text{ns})/(RC)}\right) \quad (37)$$

$$A_{11\text{ns}} = 1\text{V} \left(1 - e^{-(1\text{ns})/(RC)}\right) \quad (38)$$

$$V_{DD}(t) = 3\text{V} - A_{11\text{ns}}e^{-(t-11\text{ns})/(RC)} \quad (39)$$

For  $21\text{ns} \leq t < 22\text{ns}$ :

$$V_{DD}(t) = V_S - R \cdot I_{IC} - A_{21\text{ns}}e^{-(t-21\text{ns})/(RC)}, \quad I_{IC} = 1 \text{ A} \quad (40)$$

$$V_{DD}(21\text{ns}) = 3\text{V} - 1\text{V} - A_{21\text{ns}} = 3\text{V} - A_{11\text{ns}}e^{-(21\text{ns}-11\text{ns})/(RC)} \quad (41)$$

$$A_{21ns} = -1V + A_{11ns}e^{-(10ns)/(RC)} \quad (42)$$

$$V_{DD}(t) = 2V - A_{21ns}e^{-(t-21ns)/(RC)} \quad (43)$$

For  $22ns \leq t$ :

$$V_{DD}(t) = V_S - R \cdot I_{IC} - A_{22ns}e^{-(t-22ns)/(RC)}, \quad I_{IC} = 0 \text{ A} \quad (44)$$

$$V_{DD}(22ns) = 3V - A_{22ns} = 2V - A_{21ns}e^{-(22ns-21ns)/(RC)} \quad (45)$$

$$A_{22ns} = 1V + A_{21ns}e^{-(1ns)/(RC)} \quad (46)$$

$$V_{DD}(t) = 3V - A_{22ns}e^{-(t-22ns)/(RC)} \quad (47)$$

By substituting the different values of  $RC$ , we have expressions for each of the different capacitances.

For  $RC = 1\Omega \cdot 1 \text{ pF} = 1 \text{ ps}$ , the  $A$  values can be found as:

$$A_{11ns} = 1V(1 - e^{-1ns/1ps}) = 1V(1 - e^{-1000}) \approx 1V \quad (48)$$

$$A_{21ns} = -1V + A_{11ns}e^{-10ns/1ps} \approx -1V \quad (49)$$

$$A_{22ns} = 1V + A_{21ns}e^{-1ns/1ps} \approx 1V \quad (50)$$

We can plug in these  $A$  values into the  $V_{DD}(t)$  expressions we already found which will give the following result

$$V_{DD}(t) = \begin{cases} 3V & t < 10ns \\ 3V - 1V \left(1 - e^{-(t-10ns)/1ps}\right) & 10ns \leq t < 11ns \\ V_{DD}(t) = 3V - 1Ve^{-(t-11ns)/1ps} & 11ns \leq t < 21ns \\ V_{DD}(t) = 2V + 1Ve^{-(t-21ns)/1ps} & 21ns \leq t < 22ns \\ V_{DD}(t) = 3V - 1Ve^{-(t-22ns)/1ps} & 22ns \leq t \end{cases} \quad (51)$$

Similarly for  $RC = 1\Omega \cdot 1 \text{ nF} = 1 \text{ ns}$ :

$$A_{11ns} = 1V(1 - e^{-1ns/1ns}) = 1V(1 - e^{-1}) \approx 0.632V \quad (52)$$

$$A_{21ns} = -1V + 0.632e^{-10ns/1ns} \approx -0.999V \quad (53)$$

$$A_{22ns} = 1V + A_{21ns}e^{-1ns/1ns} \approx 0.632V \quad (54)$$

$$V_{DD}(t) = \begin{cases} 3V & t < 10ns \\ 3V - 1V \left(1 - e^{-(t-10ns)/1ns}\right) & 10ns \leq t < 11ns \\ V_{DD}(t) = 3V - 0.632Ve^{-(t-11ns)/1ns} & 11ns \leq t < 21ns \\ V_{DD}(t) = 2V + 0.999Ve^{-(t-21ns)/1ns} & 21ns \leq t < 22ns \\ V_{DD}(t) = 3V - 0.632Ve^{-(t-22ns)/1ns} & 22ns \leq t \end{cases} \quad (55)$$

For  $RC = 1\Omega \cdot 1 \mu\text{F} = 1 \mu\text{s}$ :

$$A_{11ns} = 1V(1 - e^{-1ns/1\mu s}) = 1V(1 - e^{-0.001}) \approx 0.001V \quad (56)$$

$$A_{21ns} = -1V + 0.001e^{-10ns/1\mu s} \approx -0.999V \quad (57)$$

$$A_{22ns} = 1V + A_{21ns}e^{-1ns/1\mu s} \approx 0.002V \quad (58)$$

$$V_{DD}(t) = \begin{cases} 3V & t < 10ns \\ 3V - 1V \left(1 - e^{-(t-10ns)/1\mu s}\right) & 10ns \leq t < 11ns \\ V_{DD}(t) = 3V - 0.001Ve^{-(t-11ns)/1\mu s} & 11ns \leq t < 21ns \\ V_{DD}(t) = 2V + 0.999Ve^{-(t-21ns)/1\mu s} & 21ns \leq t < 22ns \\ V_{DD}(t) = 3V - 0.002Ve^{-(t-22ns)/1\mu s} & 22ns \leq t \end{cases} \quad (59)$$

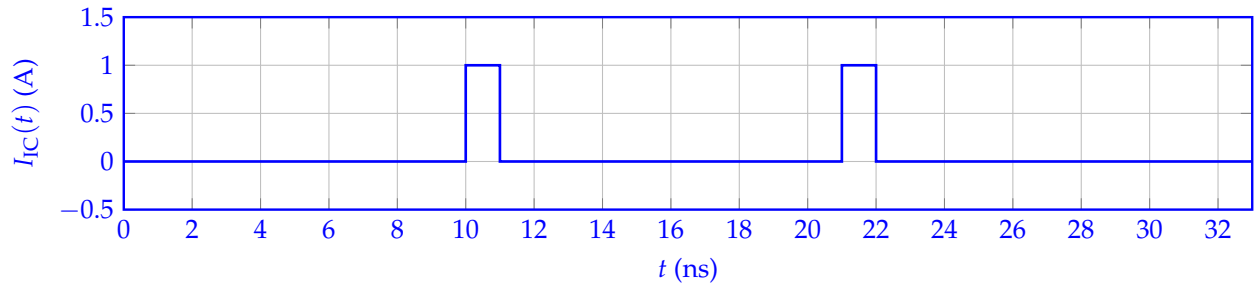
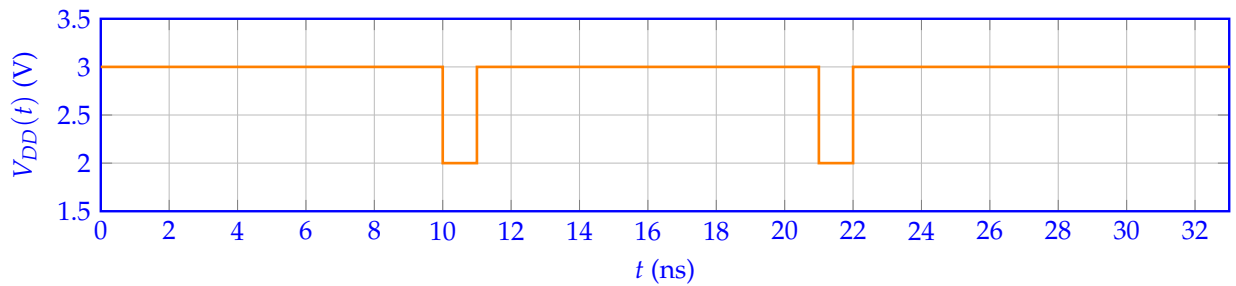
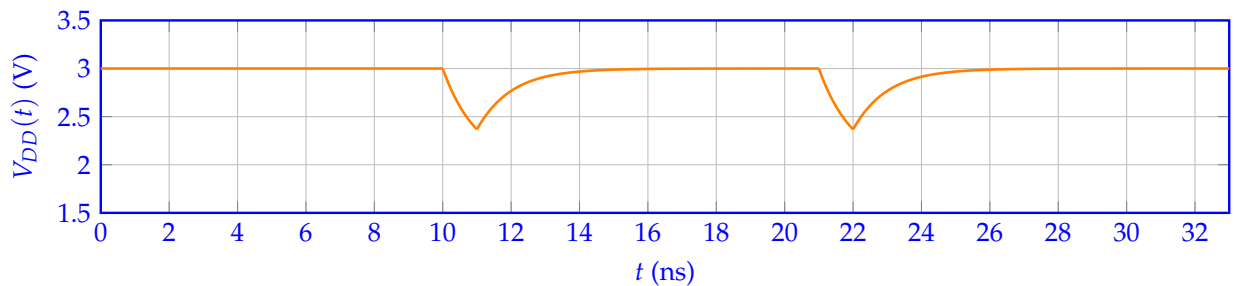
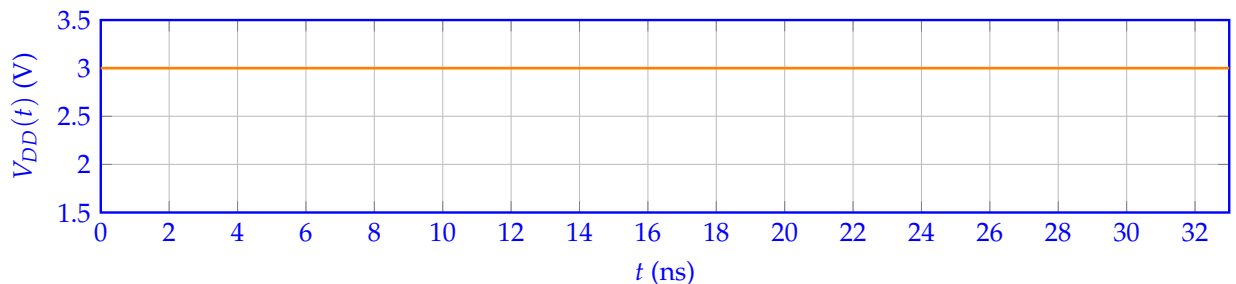


In general: each of the three curves will tend towards a final value  $V_{DD} = V_S$ , growing exponentially slower towards this goal as time progresses. However, on each time interval  $t_p$ , the current source will start drawing charge from both  $V_S$ —whose current decreases as time proceeds—and  $C$ —whose charge, and therefore whose potential to contribute voltage, tends to increase with time. With each  $t_p$ ,  $V_{DD}$  decreases nonlinearly, as there are both exponential and linear factors contributing to the rise and fall of the voltage.

With a lower capacitance, we will see the capacitor charge and discharge faster with time—this means that  $V_{DD}$  will fluctuate more/change more drastically on  $t_p$ ; as you increase capacitance, this fluctuation is less evident, as  $C$  has more charge to pull from, and will thus be less affected by the change of charge incurred by the current source.

The final sketches should look something like this:

Sketch of transient current drawn by IC

 $V_{DD}(t)$  with 1 pF capacitor $V_{DD}(t)$  with 1 nF capacitor $V_{DD}(t)$  with 1  $\mu$ F capacitor

If you have the plots and the expressions and work for  $V_{DD}(t)$ , but not the verbose explanation regarding the plots, you still deserve full credit - this is here for your understanding.

The idea here is to see the effect that the capacitors have on  $V_{DD}(t)$  when viewed at the time scale of the current spikes.

- the 1 pF capacitor causes the RC circuit to have a time constant of  $\tau = RC = 1$  ps, that is 1 picosecond =  $10^{-12}$  seconds, and the effect that this has on  $V_{DD}(t)$  is invisible at the nanosecond time scale. For this reason, we can conclude that the 1 pF capacitor would not be adequate to mitigate the noise that the IC will put on the power supply.
- the 1 nF capacitor causes the RC circuit to have a time constant of  $\tau = RC = 1$  ns. This is a

long enough time scale that the effect on  $V_{DD}(t)$  will be visible. At the end of the 1 ns current spike,  $V_{DD}(t)$  will have dropped from 3 V to  $2 + \exp(-1) \approx 2.37$  V. This means that the 1 nF capacitor is actually reducing the power supply noise a little bit, but not much.

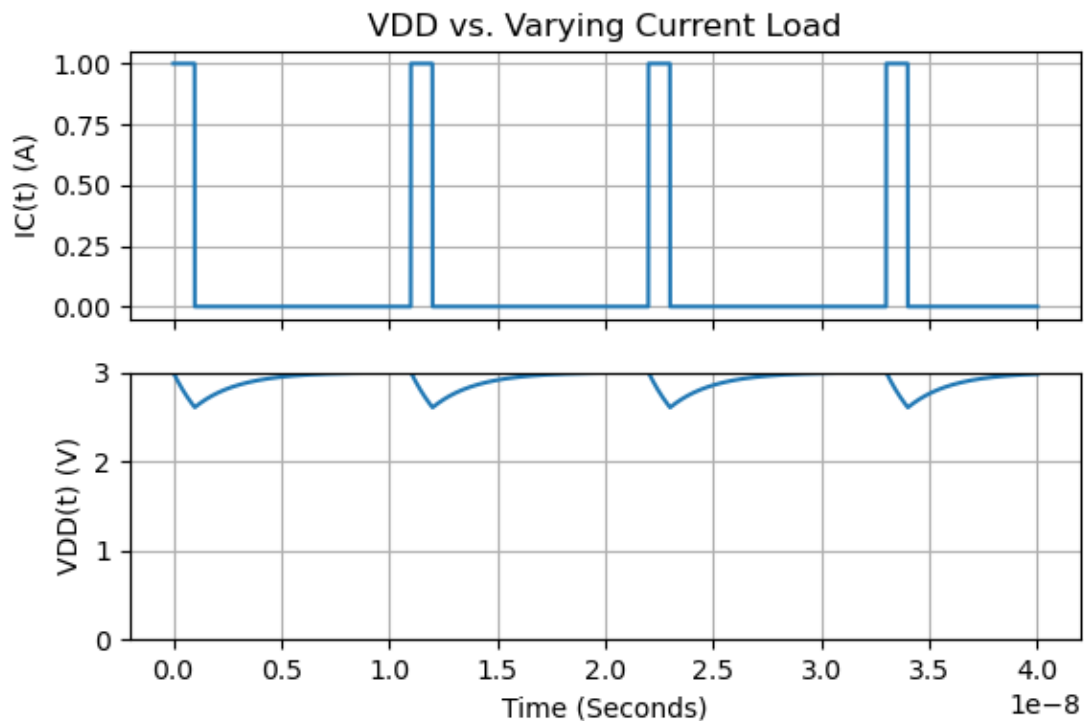
- the 1  $\mu$ F capacitor causes the RC circuit to have a time constant of  $\tau = RC = 1 \mu$ s. This time constant is 1000 times longer than the duration of the current spike. At the end of the current spike,  $V_{DD}(t)$  will have dropped by only 1 mV, so at the scale at which these sketches are drawn, there is no visible change. The 1  $\mu$ F capacitor has almost totally removed the power supply noise.

- (c) Launch the attached Jupyter notebook to interact with a simulated version of this IC power supply. Try to simulate the scenarios outlined in the previous parts. For one of these scenarios, keep the RC time constant fixed, but vary the relative value of  $R$  vs.  $C$  (e.g. compare  $R = 1, C = 2e-9$  to the case where  $R = 2, C = 1e-9$ ). **Is it better to have a lower  $R$  or lower  $C$  value for a fixed RC time constant when attempting to minimize supply noise? Give an intuitive explanation for why this might be the case.**

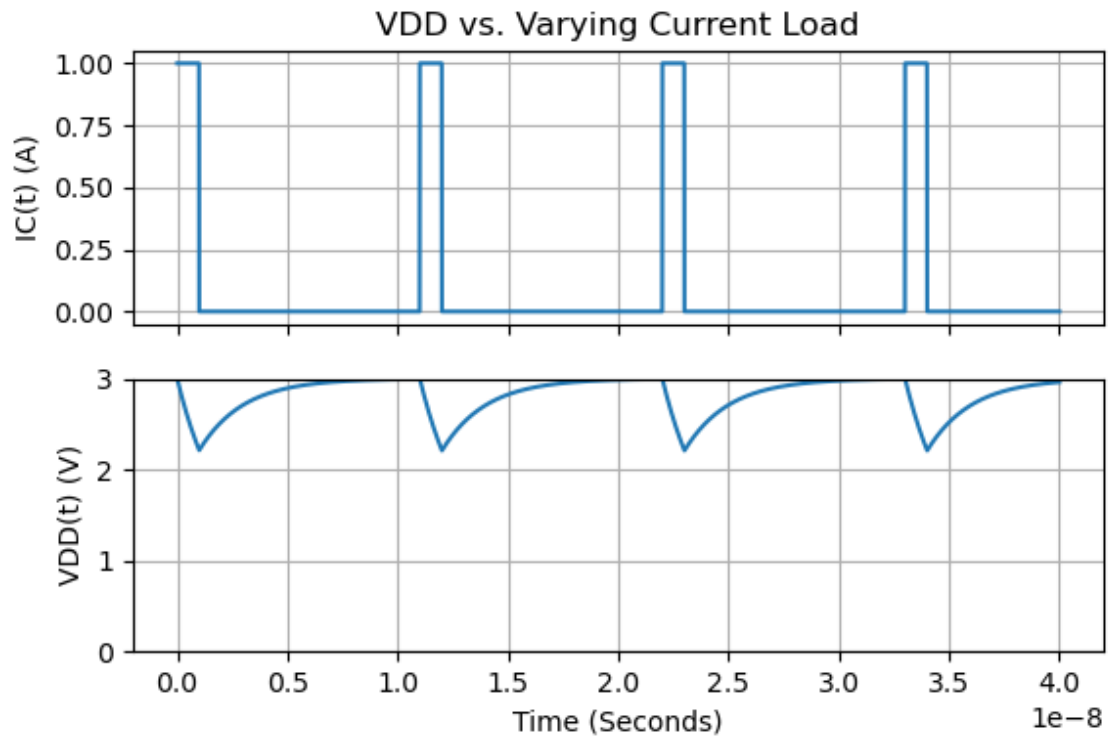
Be sure to play with the y limits on the graph as well as how long the simulation runs to best understand what is going on here.

**Solution:** A lower resistance and higher capacitance leads to smaller variation in the supply voltage with each current spike. One intuitive way to see this is to think about where the charge comes from whenever the current source turns on. The charge comes from the capacitor and from the voltage source through the resistor. By  $Q = CV$  for a constant amount of charge drawn, a larger capacitor results in lower voltage change. By  $V = IR$  for a constant amount of current drawn through the resistor, a larger resistor leads to a larger voltage drop.

In the case where  $R = 1$  and  $C = 2e-9$ , we get the following plot:



In the case where  $R = 2$  and  $C = 1e-9$ , we get the following plot:



Notice that the shape of the  $V_{DD}$  curves is the same because the RC constant is the same. However they drop to different voltages by the end of each pulse.

#### 4. Simple Scalar Differential Equations Driven by an Input

In this question, we will show the existence and uniqueness of solutions to differential equations with inputs. In particular, we consider the scalar differential equation

$$\frac{d}{dt}x(t) = \lambda x(t) + bu(t) \quad (60)$$

$$x(0) = x_0 \quad (61)$$

where  $u: \mathbb{R} \rightarrow \mathbb{R}$  is a known function of time. Feel free to assume  $u$  is "nice" in the sense that it is integrable, continuous, and differentiable with bounded derivative – basically, let  $u$  be nice enough that all the usual calculus theorems work.

- (a) We will first demonstrate the existence of a solution to eqs. (60) and (61).

Define  $x_d: \mathbb{R} \rightarrow \mathbb{R}$  by

$$x_d(t) := e^{\lambda t}x_0 + \int_0^t e^{\lambda(t-\tau)}bu(\tau) d\tau. \quad (62)$$

**Show that  $x_d$  satisfies eqs. (60) and (61).**

(HINT: When showing that  $x_d$  satisfies eq. (60), one possible approach to calculate the derivative of the integral term is to use the fundamental theorem of calculus and the product rule.)

**Solution:** We first show that  $x_d$  satisfies eq. (60). Using the fundamental theorem of calculus and the product rule, we can calculate

$$\frac{d}{dt}x_d(t) = \frac{d}{dt}\left(e^{\lambda t}x_0 + \int_0^t e^{\lambda(t-\tau)}bu(\tau) d\tau\right) \quad (63)$$

$$= \frac{d}{dt}\left(e^{\lambda t}x_0\right) + \frac{d}{dt}\int_0^t e^{\lambda(t-\tau)}bu(\tau) d\tau \quad (64)$$

$$= x_0\left(\frac{d}{dt}e^{\lambda t}\right) + \frac{d}{dt}\left(e^{\lambda t}\int_0^t e^{-\lambda\tau}bu(\tau) d\tau\right) \quad (65)$$

$$= x_0\left(\frac{d}{dt}e^{\lambda t}\right) + \frac{d}{dt}\left(e^{\lambda t}\int_0^t e^{-\lambda\tau}bu(\tau) d\tau\right) \quad (66)$$

$$= x_0\left(\lambda e^{\lambda t}\right) + \left\{\left(\frac{d}{dt}e^{\lambda t}\right)\left(\int_0^t e^{-\lambda\tau}bu(\tau) d\tau\right) + \left(e^{\lambda t}\right)\left(\frac{d}{dt}\int_0^t e^{-\lambda\tau}bu(\tau) d\tau\right)\right\} \quad (67)$$

$$= \lambda x_0 e^{\lambda t} + \lambda e^{\lambda t}\int_0^t e^{-\lambda\tau}bu(\tau) d\tau + e^{\lambda t}\left(e^{-\lambda\tau}bu(\tau)\Big|_{\tau=t}\right) \quad (68)$$

$$= \lambda\left(x_0 e^{\lambda t} + e^{\lambda t}\int_0^t e^{-\lambda\tau}bu(\tau) d\tau\right) + e^{\lambda t}\left(e^{-\lambda t}bu(t)\right) \quad (69)$$

$$= \lambda x_d(t) + bu(t) \quad (70)$$

so  $x_d$  satisfies eq. (60).

Alternatively, we could have used the **Leibniz rule** (not in scope) to get the terms in the square brackets:

$$\frac{d}{dt}x_d(t) = \lambda e^{\lambda t}x_0 + \left[e^{\lambda(t-t)}bu(t) \cdot 1 - e^{\lambda(t-0)} \cdot 0 + \lambda \int_0^t e^{\lambda(t-\tau)}bu(\tau) d\tau\right] \quad (71)$$

$$= \lambda\left[e^{\lambda t}x_0 + \int_0^t e^{\lambda(t-\tau)}bu(\tau) d\tau\right] + bu(t) \quad (72)$$

$$= \lambda x_d(t) + bu(t) \quad (73)$$

which again shows that  $x_d$  satisfies eq. (60).

Now we show that  $x_d$  satisfies eq. (61). Indeed,

$$x_d(0) = \left(e^{\lambda t}x_0 + \int_0^t e^{\lambda(t-\tau)}bu(\tau) d\tau\right)\Big|_{t=0} = \underbrace{e^{\lambda \cdot 0}}_{=1}x_0 + \underbrace{\int_0^0 e^{\lambda(0-\tau)}bu(\tau) d\tau}_{=0} = x_0. \quad (74)$$

Thus  $x_d$  satisfies eq. (61).

- (b) Now, we will show that  $x_d$  is the unique solution to eqs. (60) and (61).

Suppose that  $y: \mathbb{R} \rightarrow \mathbb{R}$  also satisfies eqs. (60) and (61). **Show that  $y(t) = x_d(t)$  for all  $t$ .**

(HINT: This time, show that  $z(t) := y(t) - x_d(t) = 0$  for all  $t$ . Do this by showing that  $z(0) = 0$  and  $\frac{d}{dt}z(t) = \lambda z(t)$ , then use the uniqueness theorem for homogeneous first-order linear differential equations from the last homework. Note that the specific form of  $x_d(t)$  in eq. (62) is irrelevant for the solution and should not be used.)

**Solution:** Again, the solution is in some parts.

Step 1. We show that  $z(0) = 0$ . Indeed,

$$z(0) = y(0) - x_d(0) = x_0 - x_0 = 0. \quad (75)$$

Step 2. We show that  $\frac{d}{dt}z(t) = \lambda z(t)$ . Indeed,

$$\frac{d}{dt}z(t) = \frac{d}{dt}(y(t) - x_d(t)) \quad (76)$$

$$= \frac{d}{dt}y(t) - \frac{d}{dt}x_d(t) \quad (77)$$

$$= (\lambda y(t) + bu(t)) - (\lambda x_d(t) + bu(t)) \quad (78)$$

$$= \lambda y(t) - \lambda x_d(t) \quad (79)$$

$$= \lambda z(t). \quad (80)$$

Step 3. We show that  $z(t) = 0$  for all  $t$ . Indeed, we know that  $z(t)$  satisfies the differential equation

$$\frac{d}{dt}z(t) = \lambda z(t) \quad (81)$$

$$z(0) = 0. \quad (82)$$

This is a first-order linear differential equation, so we know from the previous homework that its unique solution is

$$z(t) = z(0) \cdot e^{\lambda t} = 0 \cdot e^{\lambda t} = 0. \quad (83)$$

This is what was claimed, so we are done.

- (c) In this part, we will calculate some values of  $x_d$  for common values of  $u$ .

- i. If  $u(t) := u$  is a constant function, **what is  $x_d(t)$ ?**
- ii. If  $u(t) := e^{\alpha t}$  for some real number  $\alpha \neq \lambda$ , **what is  $x_d(t)$ ?**
- iii. If  $u(t) := e^{\lambda t}$ , **what is  $x_d(t)$ ?**

NOTE: Assume for simplicity that  $\lambda \neq 0$ .

**Solution:**

- i. We calculate

$$x_d(t) = e^{\lambda t} x_0 + \int_0^t e^{\lambda(t-\tau)} bu(\tau) d\tau \quad (84)$$

$$= e^{\lambda t} x_0 + bu \int_0^t e^{\lambda(t-\tau)} d\tau \quad (85)$$

$$= e^{\lambda t} x_0 + \frac{e^{\lambda t} - 1}{\lambda} bu. \quad (86)$$

ii. We calculate

$$x_d(t) = e^{\lambda t} x_0 + \int_0^t e^{\lambda(t-\tau)} b u(\tau) d\tau \quad (87)$$

$$= e^{\lambda t} x_0 + b \int_0^t e^{\lambda(t-\tau)} e^{\alpha \tau} d\tau \quad (88)$$

$$= e^{\lambda t} x_0 + b e^{\lambda t} \int_0^t e^{-\lambda \tau} e^{\alpha \tau} d\tau \quad (89)$$

$$= e^{\lambda t} x_0 + b e^{\lambda t} \int_0^t e^{(\alpha-\lambda)\tau} d\tau \quad (90)$$

$$= e^{\lambda t} x_0 + b \frac{e^{\alpha t} - e^{\lambda t}}{\alpha - \lambda} \quad (91)$$

$$= \left( x_0 - \frac{b}{\alpha - \lambda} \right) e^{\lambda t} + b \frac{e^{\alpha t}}{\alpha - \lambda}. \quad (92)$$

iii. We calculate

$$x_d(t) = e^{\lambda t} x_0 + \int_0^t e^{\lambda(t-\tau)} b u(\tau) d\tau \quad (93)$$

$$= e^{\lambda t} x_0 + b \int_0^t e^{\lambda(t-\tau)} e^{\lambda \tau} d\tau \quad (94)$$

$$= e^{\lambda t} x_0 + b \int_0^t e^{\lambda t} e^{-\lambda \tau} e^{\lambda \tau} d\tau \quad (95)$$

$$= e^{\lambda t} x_0 + b \int_0^t e^{\lambda t} d\tau \quad (96)$$

$$= e^{\lambda t} x_0 + b e^{\lambda t} \int_0^t d\tau \quad (97)$$

$$= e^{\lambda t} x_0 + b t e^{\lambda t}. \quad (98)$$

## 5. Op-Amp Stability

In this question we will revisit the basic op-amp model that was introduced in EECS 16A and we will add a capacitance  $C_{out}$  to make the model more realistic (refer to figure 1). Now that we have the tools to do so, we will study the behavior of the op-amp in positive and negative feedback (refer to figure 2). Furthermore, we will begin looking at the integrator circuit (refer to figure 4) to see how a capacitor in the negative feedback can behave. In the next homework, you will see why it ends up being close to an integrator.

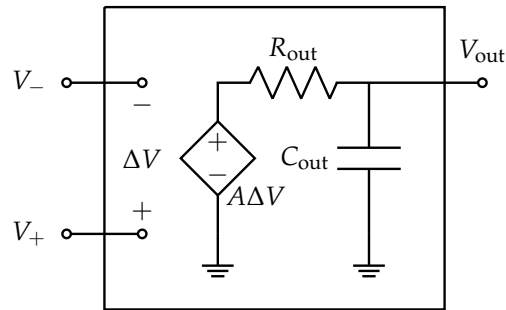
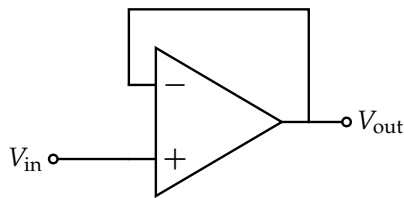
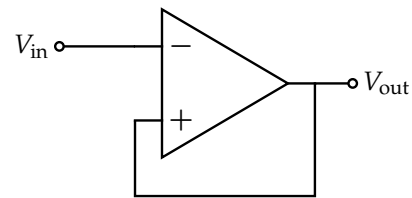


Figure 1: Op-amp model:  $\Delta V = V_+ - V_-$



(a) Buffer in negative feedback



(b) "Buffer" in positive feedback that doesn't actually work as a buffer.

Figure 2: Op-amp in buffer configuration

- (a) Using the op-amp model in figure 1 and the buffer in negative-feedback configuration in figure 2a, **draw a combined circuit**. Remember that  $\Delta V = V_+ - V_-$ , the voltage difference between the positive and negative labeled input terminals of the op-amp. (*HINT: Look at figure 5 to see how this was done for the integrator. That might help.*) **NOTE:** Note: here, we have used the equivalent model for the op-amp gain. In more advanced analog circuits courses, it is traditional to use a controlled current source with a resistor in parallel instead.

**Solution:** Please refer to Figure 3 for the completed circuit.

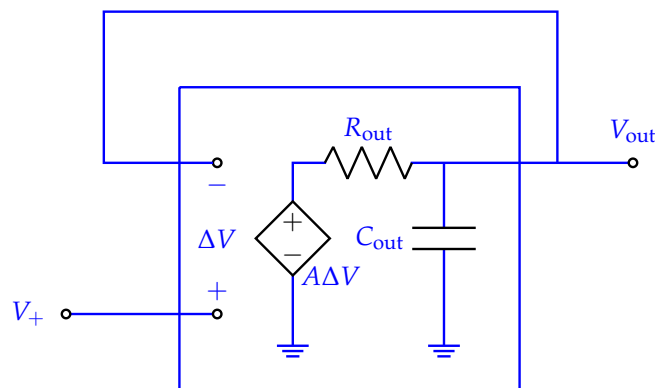


Figure 3: Negative-feedback buffer configuration using given op-amp model



- (b) Let's look at the op-amp in negative feedback. From our discussions in EECS 16A, we know that the buffer in figure 2a should work with  $V_{\text{out}} \approx V_{\text{in}}$  by the golden rules. **Write a differential-equation for  $V_{\text{out}}$  by replacing the op-amp with the given model and show what the solution will be as a function of time for a static  $V_{\text{in}}$ . What does it converge to as  $t \rightarrow \infty$ ?** Note: We assume the gain  $A > 1$  for all parts of the question. For this part, you can assume the initial condition  $V_{\text{out}}(0) = 0$ .

**Solution:** We have  $\Delta V = V_{\text{in}} - V_{\text{out}}$ . Next, we can write the following branch equations:

$$i = C_{\text{out}} \frac{d}{dt} V_{\text{out}} \quad (99)$$

$$A\Delta V = V_R + V_{\text{out}} \quad (100)$$

$$A(V_{\text{in}} - V_{\text{out}}) = RC \frac{d}{dt} V_{\text{out}} + V_{\text{out}} \quad (101)$$

Simplifying the last line, we get:

$$AV_{\text{in}} = RC \frac{d}{dt} V_{\text{out}} + (1 + A)V_{\text{out}} \quad (102)$$

$$\frac{d}{dt} V_{\text{out}} + \frac{1 + A}{RC} V_{\text{out}} - \frac{A}{RC} V_{\text{in}} = 0 \quad (103)$$

$$(104)$$

Solving the above differential equation, with the substitution  $\tilde{V}_{\text{out}} = V_{\text{out}} - \frac{A}{A+1} V_{\text{in}}$ , we get

$$\tilde{V}_{\text{out}}(t) = ke^{-\frac{A+1}{RC}t}. \quad (105)$$

Hence, the equation for  $V_{\text{out}}$  becomes

$$V_{\text{out}}(t) = \frac{A}{A+1} V_{\text{in}} + ke^{-\frac{A+1}{RC}t} \quad (106)$$

Substituting for the initial condition  $V_{\text{out}}(0) = 0$ , we get:

$$V_{\text{out}} = \frac{AV_{\text{in}}}{A+1} \left(1 - e^{-\frac{A+1}{RC}t}\right). \quad (107)$$

Since  $A > 1$ , the exponent is negative, hence as  $t \rightarrow \infty$ , the solution will converge to  $V_{\text{out}} \rightarrow \frac{A}{A+1} V_{\text{in}}$ .

- (c) Next, let's look at the op-amp in positive feedback. We know that the configuration given in figure 2b is unstable and  $V_{\text{out}}$  will just rail. **Again, using the op-amp model in figure 1, show that  $V_{\text{out}}$  does not converge and hence the output will rail. For positive DC input  $V_{\text{in}} > 0$ , will  $V_{\text{out}}$  rail to the positive or negative side? Explain.** For this part, you can assume the initial condition  $V_{\text{out}}(0) = 0$ .

**Solution:** This time, we have  $\Delta V = V_{\text{out}} - V_{\text{in}}$ . Next, we can write the following branch equations:

$$i = C_{\text{out}} \frac{d}{dt} V_{\text{out}} \quad (108)$$

$$A\Delta V = V_R + V_{\text{out}} \quad (109)$$

$$A(V_{\text{out}} - V_{\text{in}}) = RC \frac{d}{dt} V_{\text{out}} + V_{\text{out}} \quad (110)$$

Simplifying the last line, we get:

$$-AV_{\text{in}} = RC \frac{d}{dt} V_{\text{out}} + (1 - A)V_{\text{out}} \quad (111)$$

$$\frac{d}{dt}V_{\text{out}} + \frac{1-A}{RC}V_{\text{out}} + \frac{A}{RC}V_{\text{in}} = 0 \quad (112)$$

$$(113)$$

Solving the above differential equation, with the substitution  $\tilde{V}_{\text{out}} = V_{\text{out}} - \frac{A}{A-1}V_{\text{in}}$ , we get:

$$\tilde{V}_{\text{out}} = ke^{-\frac{1-A}{RC}t} \quad (114)$$

$$= ke^{\frac{A-1}{RC}t} \quad (115)$$

Substituting for the hypothetical initial condition  $V_{\text{out}}(0) = 0$ , we get:

$$V_{\text{out}} = -\frac{AV_{\text{in}}}{A-1} \left( e^{\frac{A-1}{RC}t} - 1 \right) \quad (116)$$

Since  $A > 1$ , the exponent is positive, hence as  $t \rightarrow \infty$ , the solution will be unbounded, and  $V_{\text{out}} \rightarrow -\infty$ . Of course, it can't grow to negative infinity, and so we can conclude that  $V_{\text{out}}$  will rail to the negative side if  $V_{\text{out}}$  had started at zero. The same exact story would hold if  $V_{\text{out}}$  started anywhere below  $V_{\text{in}}$ .

The case of  $V_{\text{out}}$  starting out significantly greater than  $V_{\text{in}}$  deserves some mention, although not necessary for full credit on this question. In this case, the initial condition for  $\tilde{V}_{\text{out}}$  would be positive, and would proceed to unstably attempt to run away to positive infinity. It would be stopped at the positive rail, where it would stay.

Essentially, all that matters is the initial condition of  $\tilde{V}_{\text{out}}$  — start out positive, then we rail to a positive rail for  $V_{\text{out}}$ . Start out negative, then we rail to the negative rail for  $V_{\text{out}}$ . An op-amp in positive feedback retains its comparator-like character.

- (d) For an ideal op-amp, we can assume that it has an infinite gain, i.e.,  $A \rightarrow \infty$ . **Under these assumptions, show that the op-amp in negative feedback behaves as an ideal buffer, i.e.,  $V_{\text{out}} = V_{\text{in}}$ .**

**Solution:** Taking the limit of our solution in part (a),

$$V_{\text{out}} = \lim_{A \rightarrow \infty} \frac{AV_{\text{in}}}{A+1} \left( 1 - e^{-\frac{A+1}{RC}t} \right) \quad (117)$$

$$= V_{\text{in}} \quad (118)$$

The coefficient of the exponent goes to 1 whereas the exponent itself goes to  $-\infty$ , and hence  $1 - e^{-\infty} \rightarrow 1$ .

- (e) Let's extend our analysis to the integrator circuit shown below (figure 4). Simplifying all the equations, we get a system of differential equations in two variables  $V_C$  and  $V_{\text{out}}$ , where  $V_C$  and  $V_{\text{out}}$  are the voltage drops across the capacitors  $C$  and  $C_{\text{out}}$ . **Fill in the missing term in the following matrix differential equation.**

$$\frac{d}{dt} \begin{bmatrix} V_{\text{out}}(t) \\ V_C(t) \end{bmatrix} = \begin{bmatrix} -\left(\frac{A+1}{R_{\text{out}}C_{\text{out}}} + \frac{1}{RC_{\text{out}}}\right) & -\left(\frac{1}{RC_{\text{out}}} + \frac{A}{R_{\text{out}}C_{\text{out}}}\right) \\ -\frac{1}{RC} & ? \end{bmatrix} \begin{bmatrix} V_{\text{out}}(t) \\ V_C(t) \end{bmatrix} + \begin{bmatrix} \frac{1}{RC_{\text{out}}} \\ \frac{1}{RC} \end{bmatrix} V_{\text{in}}(t) \quad (119)$$

(HINT: We picked an easier term to hide. You don't have to write out all the equations and do a lot of algebra to figure out what the missing term is.)

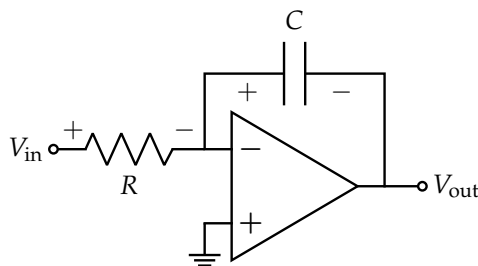


Figure 4: Integrator circuit

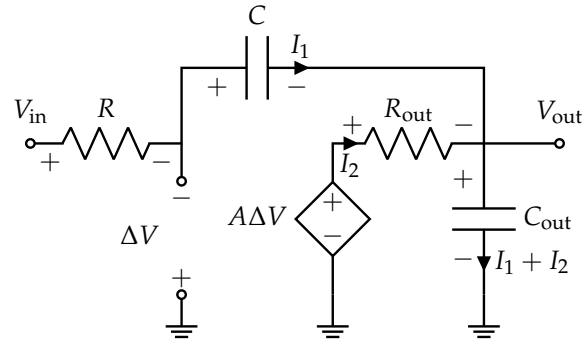


Figure 5: Integrator circuit with op-amp model.

**Solution:** We can define a convenient variable as follows:

$$\Delta V = 0 - (V_{\text{in}} - i_1 R) \quad (120)$$

$$i_1 = C \frac{d}{dt} V_C = \frac{1}{R} (V_{\text{in}} - V_C - V_{\text{out}}) \quad (121)$$

$$i_2 = \frac{1}{R_{\text{out}}} (A\Delta V - V_{\text{out}}) \quad (122)$$

$$= \frac{AR}{R_{\text{out}}} i_1 - \frac{A}{R_{\text{out}}} V_{\text{in}} - \frac{1}{R_{\text{out}}} V_{\text{out}} \quad (123)$$

Next, we can write the branch equations as follows:

$$V_{\text{out}} = V_{\text{in}} - i_1 R - V_C \quad (124)$$

$$C_{\text{out}} \frac{d}{dt} V_{\text{out}} = i_1 + i_2 \quad (125)$$

Substituting from equations (120), (121) and (123),

$$\frac{d}{dt} V_C = -\frac{1}{RC} V_{\text{out}} - \frac{1}{RC} V_C + \frac{1}{RC} V_{\text{in}} \quad (126)$$

$$C_{\text{out}} \frac{d}{dt} V_{\text{out}} = \frac{1}{R} (V_{\text{in}} - V_C - V_{\text{out}}) + \frac{A}{R_{\text{out}}} (V_{\text{in}} - V_C - V_{\text{out}}) - \frac{A}{R_{\text{out}}} V_{\text{in}} - \frac{1}{R_{\text{out}}} V_{\text{out}} \quad (127)$$

$$= \frac{1}{R} V_{\text{in}} - \left( \frac{1}{R} + \frac{A}{R_{\text{out}}} \right) V_C - \left( \frac{1}{R_{\text{out}}} + \frac{A}{R_{\text{out}}} + \frac{1}{R} \right) V_{\text{out}} \quad (128)$$

Hence, we can write the matrix differential equation as follows:

$$\frac{d}{dt} \begin{bmatrix} V_{\text{out}} \\ V_C \end{bmatrix} = \begin{bmatrix} -\left( \frac{A+1}{R_{\text{out}} C_{\text{out}}} + \frac{1}{RC_{\text{out}}} \right) & -\left( \frac{1}{RC_{\text{out}}} + \frac{A}{R_{\text{out}} C_{\text{out}}} \right) \\ -\frac{1}{RC} & -\frac{1}{RC} \end{bmatrix} \begin{bmatrix} V_{\text{out}} \\ V_C \end{bmatrix} + \begin{bmatrix} \frac{1}{RC_{\text{out}}} \\ \frac{1}{RC} \end{bmatrix} V_{\text{in}} \quad (129)$$

Note that for the given problem, it's not required to do this entire derivation. The missing term can be found out simply from eq. (121)

## 6. Successive Approximation Register Analog-to-Digital Converter (SAR ADC)

An analog-to-digital converter (ADC) is a circuit for converting an analog voltage into an approximate digital representation of that voltage. One commonly used circuit architecture for analog-to-digital converters is the Successive Approximation Register ADC (SAR ADC), which you will see in Lab 3. An  $N$ -bit SAR ADC converts an input analog voltage to an  $N$ -bit binary string between 0 and  $2^N - 1$ . This binary string represents an integer, which in turn approximates the value of our analog input voltage.

The SAR ADC does this by following one of the key themes in 16B: reducing a problem into sub-problems that we already know how to solve. In this case, the two ingredients are the DAC (digital to analog converter) that we saw in HW 1, and a binary search tree that you saw in 61A. As you remember from 61A, the key operation required for a binary search is dividing a group into two halves, solving the problem at the current step with a less-than/greater-than comparison, and descending into one of the halves. We continue this until we can no longer subdivide the problem. The comparison operation is therefore key to a binary search tree. Fortunately, we have a circuit element from 16A that lets us do that: a comparator.

Explicitly, the SAR ADC implements the binary search algorithm by feeding trial digital codes into a DAC, like the one we analyzed in Homework 1. The circuit then takes the resulting analog voltage from the DAC and compares it with the analog input voltage using a **comparator**. It then uses feedback (**SAR logic**) to adjust the DAC output voltage to get as close as possible to the input analog voltage, step by step. The algorithm starts by determining the most significant bit (MSB), which is the bit with the largest binary weight (i.e. furthest to the left in a traditional binary number), and then moving on to determine the next bit.

If this is not clear to you, think about how you would play 20 questions to guess an integer between 0 and  $2^{20} - 1$ , which is approximately 1 million. This is a game where you have to guess what number your friend is thinking of, and they can only tell you if your guess is too high or too low. You have 20 guesses to get your number as close as possible to your friend's. Would you start by guessing 0, then 1, then 2, etc.? Or would you start in the middle, let's say 500,000, see if you're too high or low, then move into the next half, e.g. 250,000 or 750,000 depending on your result? The latter approach of divide-and-conquer is a faster way of solving the problem! The SAR ADC is basically a circuit implementation of this game. The input voltage is the "number" your friend has in mind, the DAC code is your guess, the comparator is your friend's response (too high vs. too low), and the SAR logic used to adjust the next code is you deciding what to guess next. For a 20-bit SAR DAC, you have 20 guesses; for an  $N$ -bit SAR DAC, you have  $N$  guesses.

Figure 6 illustrates a high-level circuit diagram of a SAR ADC. The analog input voltage is one of the inputs to the comparator. The other comparator input,  $V_{DAC}$ , is the DAC voltage output, i.e. the analog representation of your code/guess. The comparator compares these two values and outputs a logical high (1) if  $V_{IN} \geq V_{DAC}$  or a logical low (0) if  $V_{IN} < V_{DAC}$ . This comparator decision then feeds into the SAR logic, implemented in a microcontroller in this case. This logic is basically the brain that implements the binary search pattern, deciding which bits in the code need to be changed. The updated code then propagates back to the DAC, which updates  $V_{DAC}$ , and the cycle continues  $N$  times. Once this is done, the final  $N$ -bit output code comes out of the register storage units for use by other circuits. If you're wondering what  $V_{REF}$  does for the DAC, recall from Homework 1 that it tells the DAC what the maximum possible input voltage is.

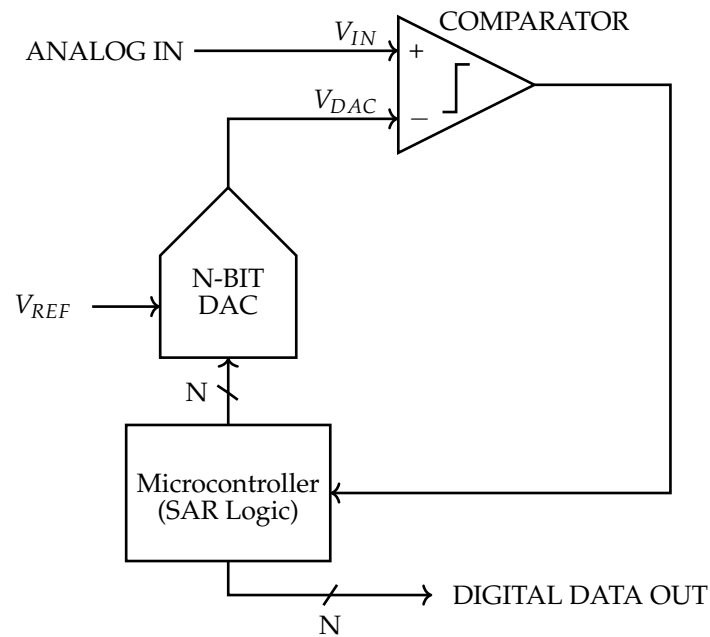


Figure 6: SAR ADC circuit diagram

Here is an illustration of the algorithm, which the SAR logic takes care of.

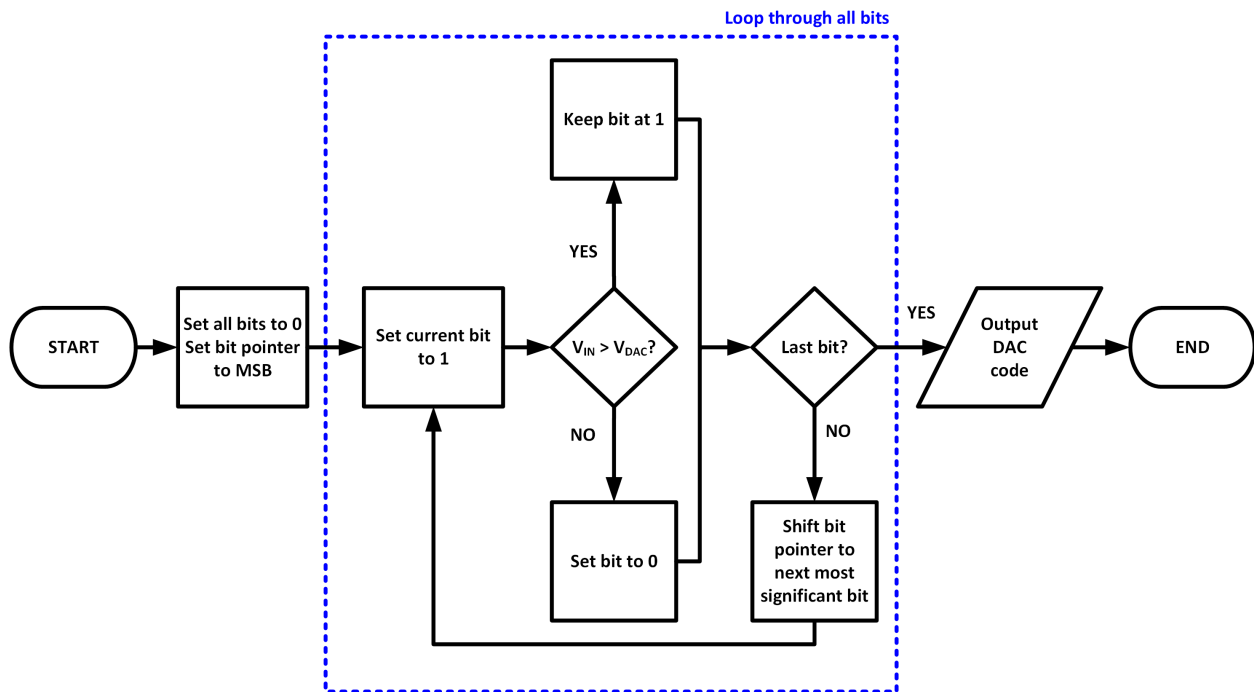


Figure 7: Flow chart of SAR ADC algorithm.

Let's look at a 3-bit SAR ADC as an example. The voltage transfer curve of the 3-bit SAR ADC is shown in Figure 8. For  $V_{IN} = 0.3V_{REF}$ , the operation of SAR ADC is shown in Figure 9.

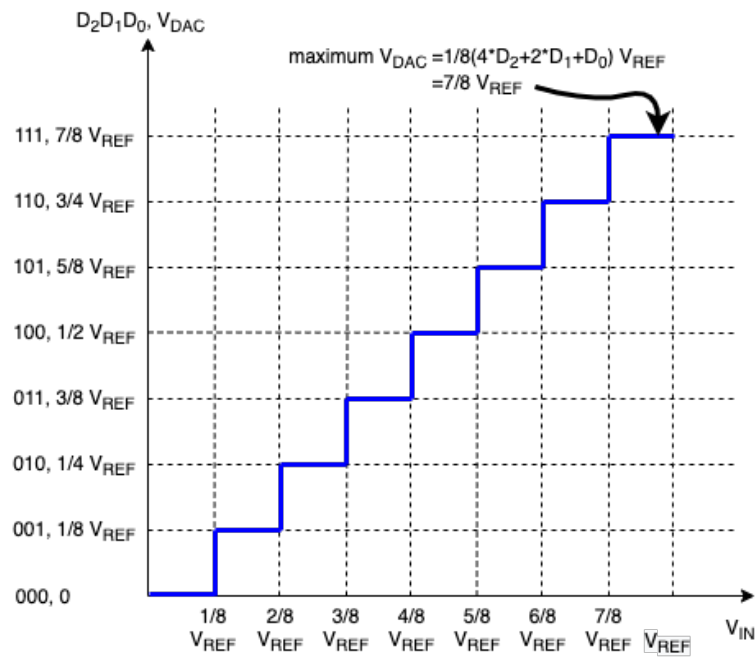


Figure 8: Voltage transfer curve of 3-bit SAR ADC:  $D_2D_1D_0$  is the output digital code of the ADC. Notice that the maximum voltage of the  $V_{DAC}$  is  $\frac{2^N-1}{2^N} V_{REF}$  where  $N$  is the number of bits ( $N = 3$  in this case)

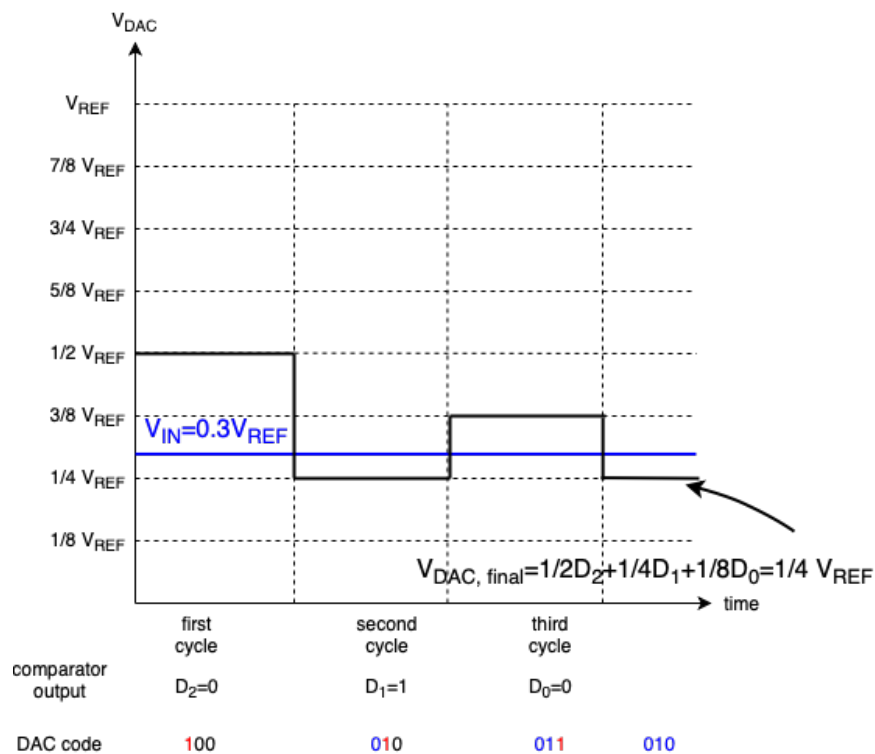


Figure 9: Timing diagram of SAR output code. Red digit in DAC code is the bit being decided in the cycle; blue digits are the bits determined by the previous conversion cycles.

From the timing diagram in Figure 9, it can be seen that the final digital output code is 010 which

represents the closest DAC output voltage  $\left(\frac{1}{4}V_{\text{REF}}\right)$  to the input analog voltage  $V_{\text{IN}} = 0.3V_{\text{REF}}$  with a 3 bit binary representation.

We will now analyze the operation of a 4-bit SAR ADC with the input voltage range between 0 V and 5 V and output code range between 0000 (0) and 1111 (15). In other words,  $V_{\text{REF}}$  is 5 V and  $0\text{ V} < V_{\text{in}} < 5\text{ V}$  in Figure 6.

- (a) **If the ADC output code is 0000, what is the corresponding DAC voltage? What about code 1111? Code 1001?** (HINT: Try to draw the transfer curve for 4-bit SAR ADC, similar to that for the 3-bit SAR ADC in Figure 8.)

**Solution:** Output code 0000 corresponds to  $V_{\text{DAC}} = 0\text{ V}$ .

Output code 1111 corresponds to  $V_{\text{DAC}} = \frac{1 \times 2^3 + 1 \times 2^2 + 1 \times 2^1 + 1 \times 2^0}{2^4} \times V_{\text{REF}} = \frac{15}{16} \times 5\text{ V} = 4.6875\text{ V}$ .

Output code 1001 corresponds to  $V_{\text{DAC}} = \frac{1 \times 2^3 + 0 \times 2^2 + 0 \times 2^1 + 1 \times 2^0}{2^4} \times V_{\text{REF}} = \frac{9}{16} \times 5\text{ V} = 2.8125\text{ V}$ .

- (b) **If the input analog voltage is  $V_{\text{IN}} = 3\text{ V}$ , what is the ratio between the input voltage  $V_{\text{IN}}$  and the maximum input voltage  $V_{\text{REF}} = 5\text{ V}$ ? What should be the output code of the SAR ADC?**

**Solution:** The ratio is  $\frac{V_{\text{IN}}}{V_{\text{REF}}} = \frac{3}{5} = 0.6$ .

The output code can be computed in 4 cycles using the SAR ADC algorithm.

In the first cycle, the DAC code is 1000, resulting in comparator output  $D_3 = 1$  since  $0.6 > \frac{8}{16}$ .

In the second cycle, the DAC code is 1100, resulting in comparator output  $D_2 = 0$  since  $0.6 < \frac{12}{16}$ .

In the third cycle, the DAC code is 1010, resulting in comparator output  $D_1 = 0$  since  $0.6 < \frac{10}{16}$ .

In the fourth cycle, the DAC code is 1001, resulting in comparator output  $D_0 = 1$  since  $0.6 > \frac{9}{16}$ .

Hence the final DAC output code is  $D_3D_2D_1D_0 = 1001$ .

- (c) Again, if the input analog voltage is  $V_{\text{IN}} = 3\text{ V}$ , draw the operation of a 4-bit SAR ADC resolving its output code (see Figure 9 as reference). Specifically:
- Plot the output voltage of the DAC in the timing diagram in Figure 10.
  - Fill out the output of the comparator in each conversion cycle.
  - Fill out the ADC output code ( $D_3D_2D_1D_0$ ) in each conversion cycle.

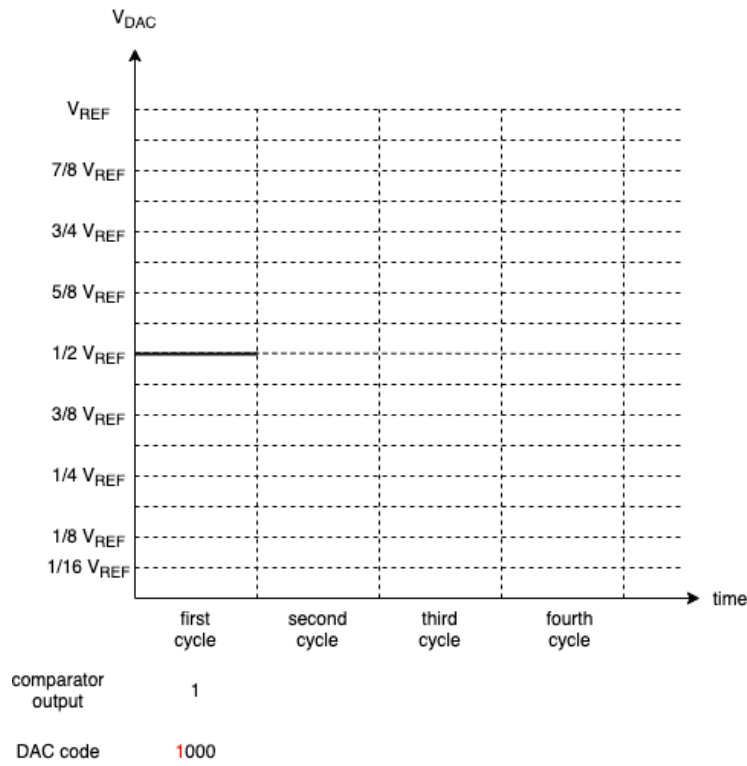


Figure 10: SAR ADC  $V_{DAC}$  timing diagram

Solution: See Figure 11.

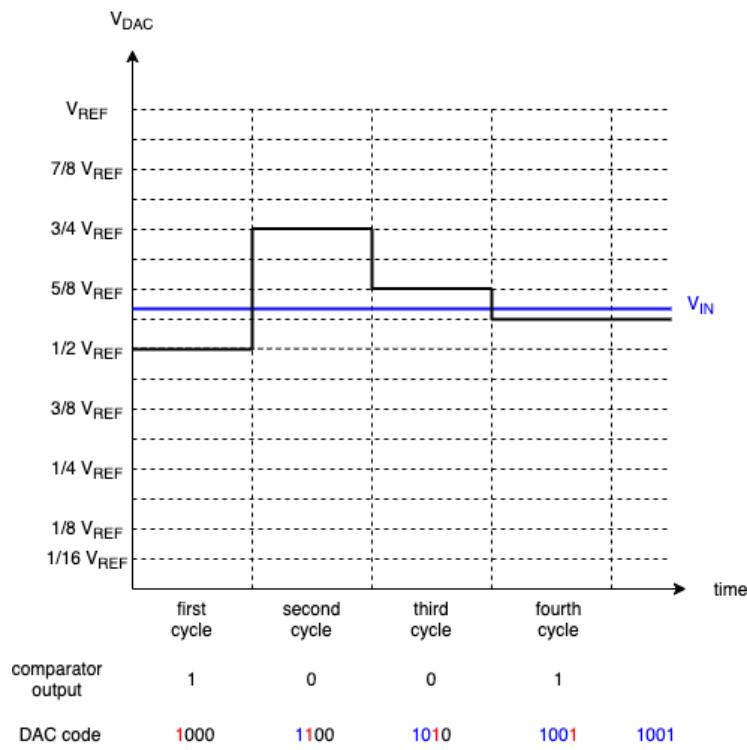


Figure 11: SAR ADC  $V_{DAC}$  timing diagram answer



**7. (OPTIONAL) Make Your Own Problem.**

**Write your own problem about content covered in the course thus far, and provide a thorough solution to it.**

*NOTE:* This can be a totally new problem, a modification on an existing problem, or a Jupyter part for a problem that previously didn't have one. Please cite all sources for anything (including course material) that you used as inspiration.

*NOTE:* High-quality problems may be used as inspiration for the problems we choose to put on future homeworks or exams.

**8. Homework Process and Study Group**

Citing sources and collaborators are an important part of life, including being a student!

We also want to understand what resources you find helpful and how much time homework is taking, so we can change things in the future if possible.

- (a) **What sources (if any) did you use as you worked through the homework?**
- (b) **If you worked with someone on this homework, who did you work with?**  
List names and student ID's. (In case of homework party, you can also just describe the group.)
- (c) **Roughly how many total hours did you work on this homework? Write it down here where you'll need to remember it for the self-grade form.**

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