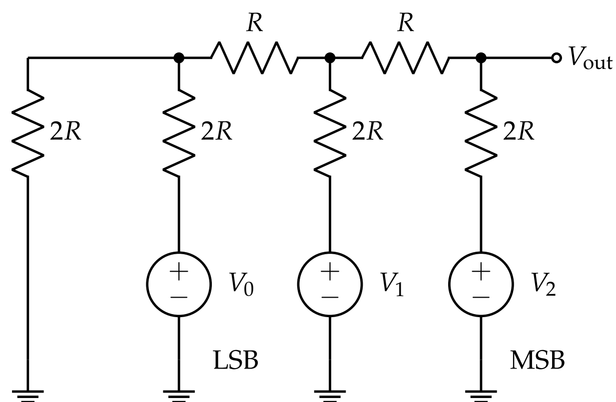
Homework 1

This homework is due on Saturday, September 2, 2023, at 11:59PM. Self-grades and HW Resubmissions are due on the following Saturday, Feburary 9, 2023, at 11:59PM.

1. **Digital-Analog Converter**

   A digital-analog converter (DAC) is one of the key interface components between the digital and the analog world. It is a circuit for converting a digital representation of a number (binary) into a corresponding analog voltage. In this problem, we will consider a DAC made out of resistors only (resistive DAC) called the R-$2R$ ladder. This DAC will help us generate the analog voltages from the digital representation, and later will also help us digitize the analog voltages when we will be building analog to digital interfaces in Lab 3, in part based on this ladder-DAC.

   Here is the circuit for a 3-bit resistive DAC.

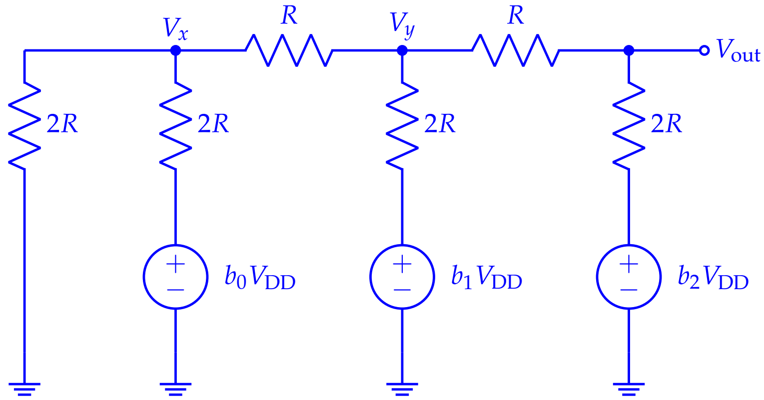


   Let $b_0, b_1, b_2 = \{0, 1\}$ (that is, either 1 or 0), and let the voltage sources $V_0 = b_0 V_{\text{DD}}$, $V_1 = b_1 V_{\text{DD}}$, $V_2 = b_2 V_{\text{DD}}$, where V_{DD} is the supply voltage.

   As you may have noticed, (b_2, b_1, b_0) represents a 3-bit binary (unsigned) number where each of b_i is a binary bit. b_0 is the least significant bit (LSB) and b_2 is the most significant bit (MSB). We will now analyze how this converter functions.

   (a) **Solve for V_{out} in terms of V_{DD} and the binary bits b_2, b_1, b_0.**

   Solution: There are several ways to solve this problem. The first way is to use KCL and create a system of equations which we solve for using Gaussian elimination.



Applying KCL at nodes V_x, V_y, and V_{out} and substituting in for the currents through the resistors, we get

$$\frac{V_x}{2R} + \frac{V_x - b_0 V_{\text{DD}}}{2R} + \frac{V_x - V_y}{R} = 0 \tag{1}$$
$$\frac{V_y - b_1 V_{\text{DD}}}{2R} + \frac{V_y - V_x}{R} + \frac{V_y - V_{\text{out}}}{R} = 0 \tag{2}$$
$$\frac{V_{\text{out}} - b_2 V_{\text{DD}}}{2R} + \frac{V_{\text{out}} - V_y}{R} = 0 \tag{3}$$

This system of equations can be solved using substitution or Gaussian elimination. One approach is shown below:

Multiplying (1), (2), (3) by R, we get

$$2V_x - \frac{b_0 V_{\text{DD}}}{2} - V_y = 0 \tag{4}$$
$$\frac{5V_y}{2} - \frac{b_1 V_{\text{DD}}}{2} - V_x - V_{\text{out}} = 0 \tag{5}$$
$$\frac{3V_{\text{out}}}{2} - \frac{b_2 V_{\text{DD}}}{2} - V_y = 0 \tag{6}$$

Adding $\frac{1}{4} \times$ (4), $\frac{1}{2} \times$ (5), and (6), we get

$$V_{\text{out}} - \frac{b_2 V_{\text{DD}}}{2} - \frac{b_1 V_{\text{DD}}}{4} - \frac{b_0 V_{\text{DD}}}{8} = 0 \tag{7}$$
$$\implies \frac{b_2 V_{\text{DD}}}{2} + \frac{b_1 V_{\text{DD}}}{4} + \frac{b_0 V_{\text{DD}}}{8} = V_{\text{out}} \tag{8}$$

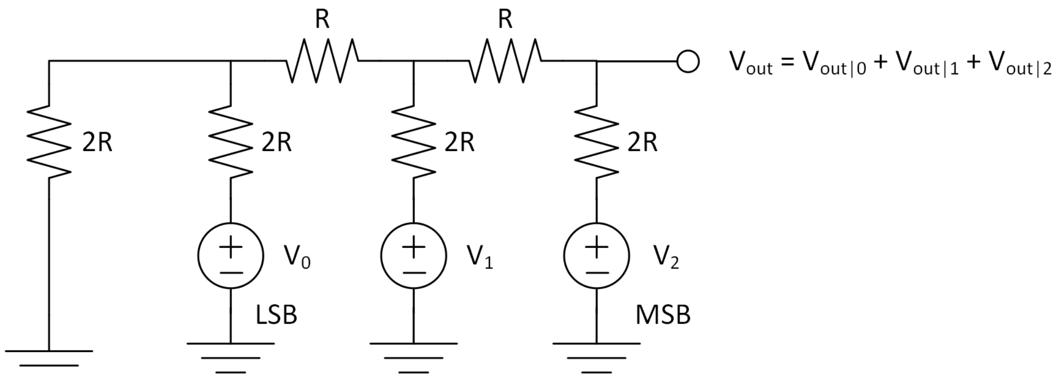
While using KCL and Gaussian elimination is a correct way of solving this problem, there is another way by using superposition and Thevenin equivalent circuits which you learned in EE16A. This approach tends to be more "intuitive": it helps you understand why the circuit was designed the way it was, how to quickly solve for the circuit (in your head, with enough practice), and transfer key design principles to invent new circuits for new problems. These skills are key to a successful career in engineering.

Recall that the output voltage is a superposition of all the independent voltage and current sources in the circuit as seen at the output. In other words, we can turn on each independent source separately, solve for the output voltage for that given source, repeat for each independent source, and sum the output voltages for each case. This will be the final output voltage.

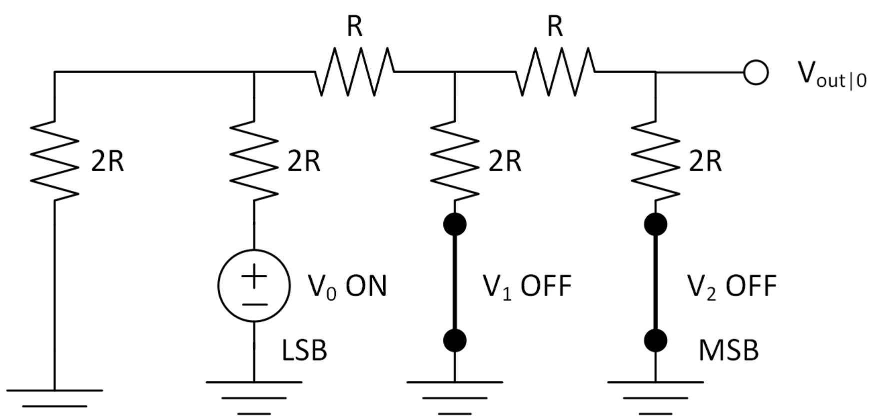
Mathematically, we can write this as:

$$V_{\text{out}} = V_{\text{out}\,|\,0} + V_{\text{out}\,|\,1} + V_{\text{out}\,|\,2} \tag{9}$$

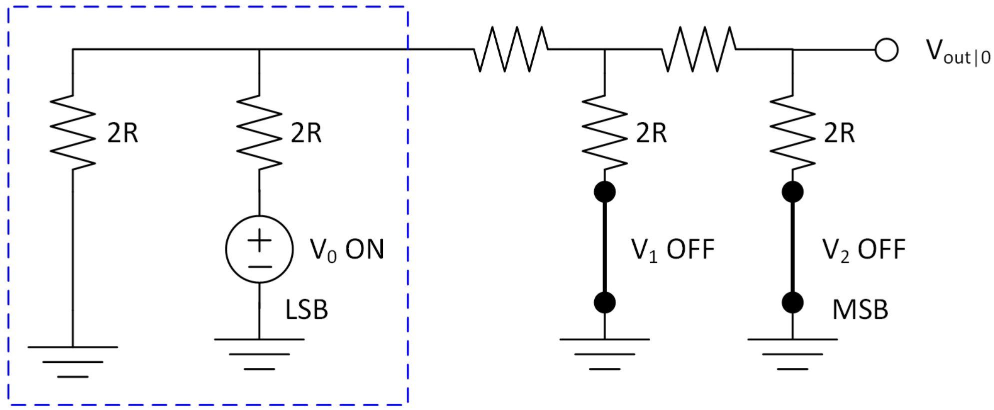
where $V_{\text{out}\,|\,0}$ refers to V_{out} due to independent source 0 on and all other sources off.

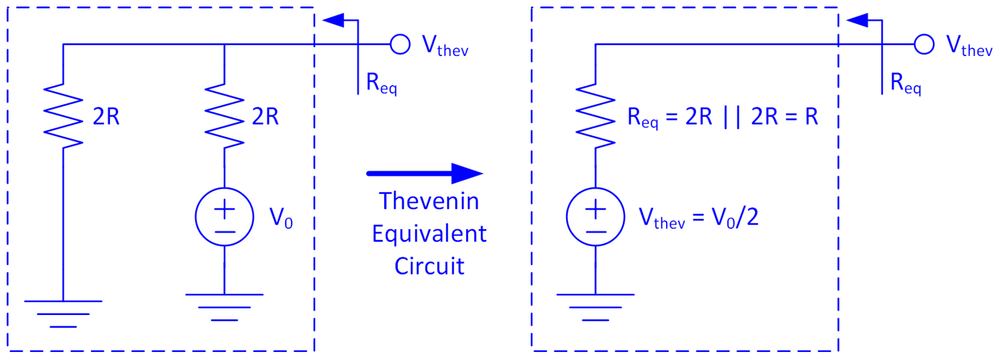


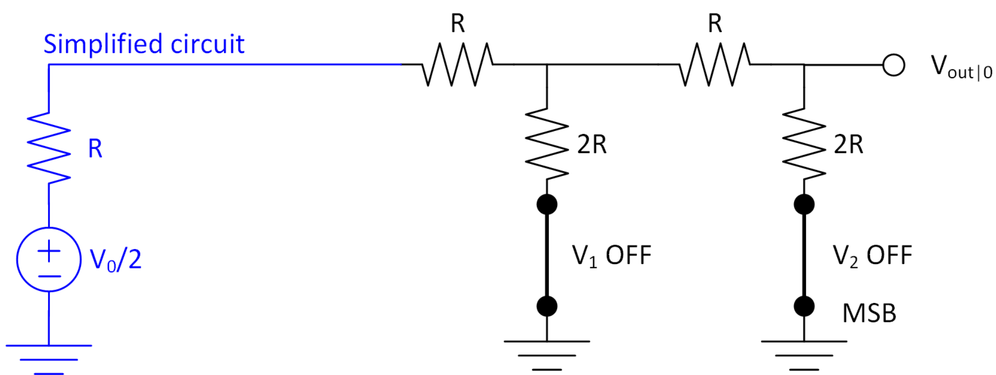
Let's first solve for $V_{\text{out}\,|\,0}$. To do so, re-draw the circuit with the independent voltage source V_0 on and all the other independent sources, voltage sources V_1 and V_2, off. Recall that when we turn a voltage source off, we treat it as a short circuit, i.e. 0 Volts (as opposed to turning a current source off, which we treat as an open circuit, i.e. 0 Amps). We show this circuit below.



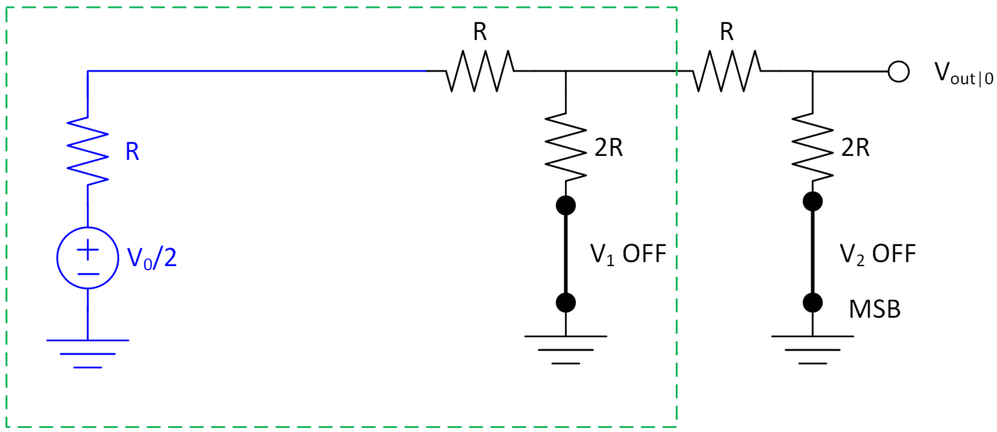
We then want to solve for $V_{\text{out}\,|\,0}$ by using Thevenin equivalent circuits to simplify the problem. We will conduct Thevenin simplification twice. First, draw a bounding box around the components we want to simplify. How you draw this box depends on how much you want to simplify at once (you get better with practice). We choose to draw the blue dotted box, find the corresponding Thevenin circuit, and replace the original components with their equivalent circuit.

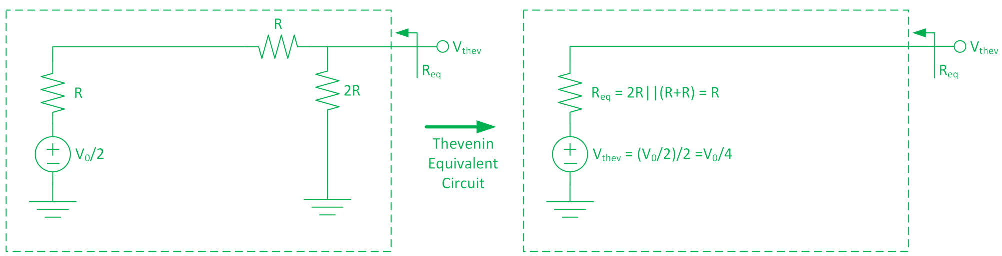


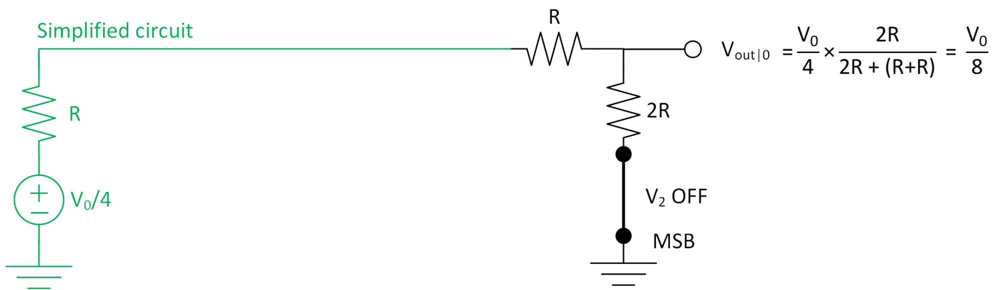




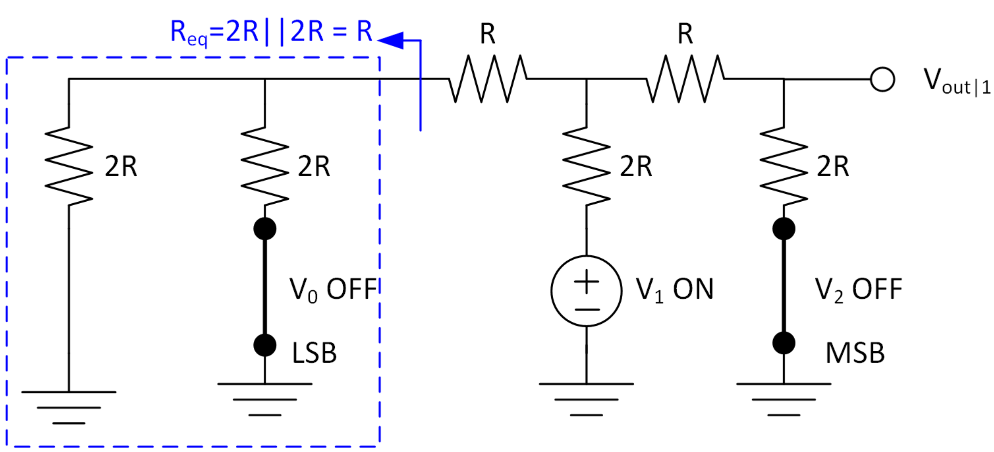
We then conduct a second round of Thevenin using the green dotted box below. From the resulting simplified circuit, we see what's left is a straightforward voltage divider, and solve for $V_{\text{out}|0}$ directly.

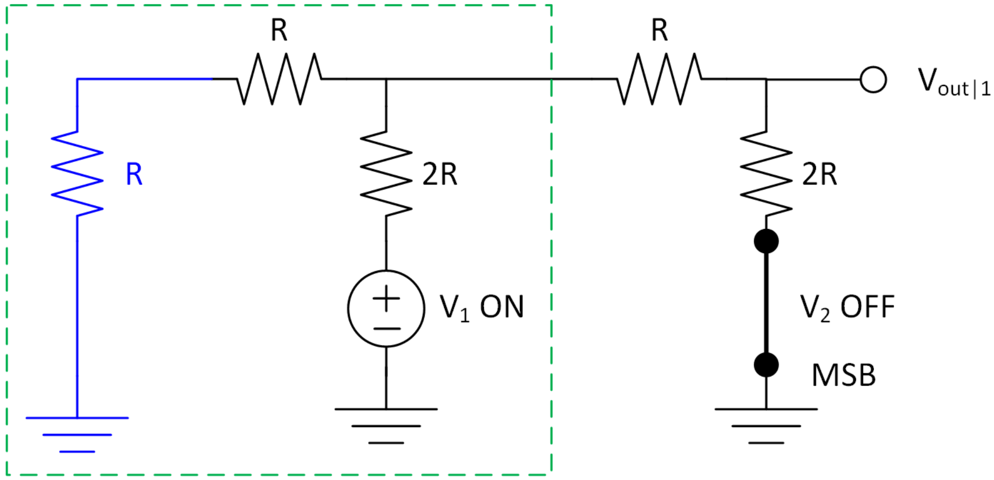


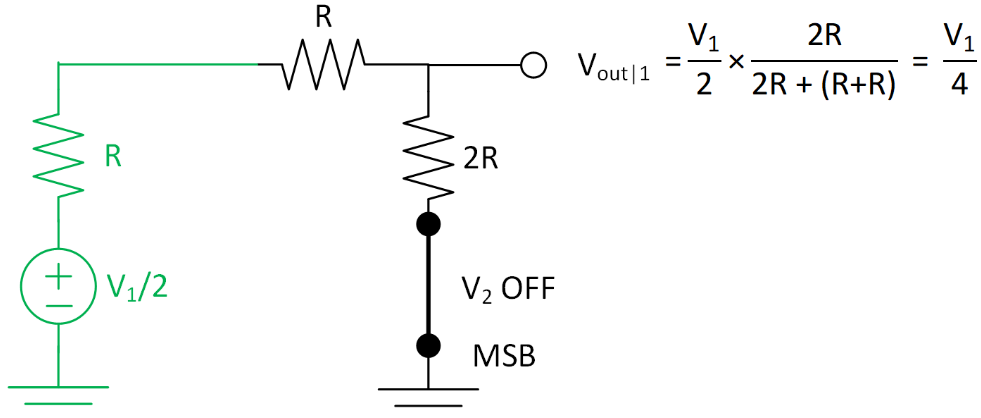




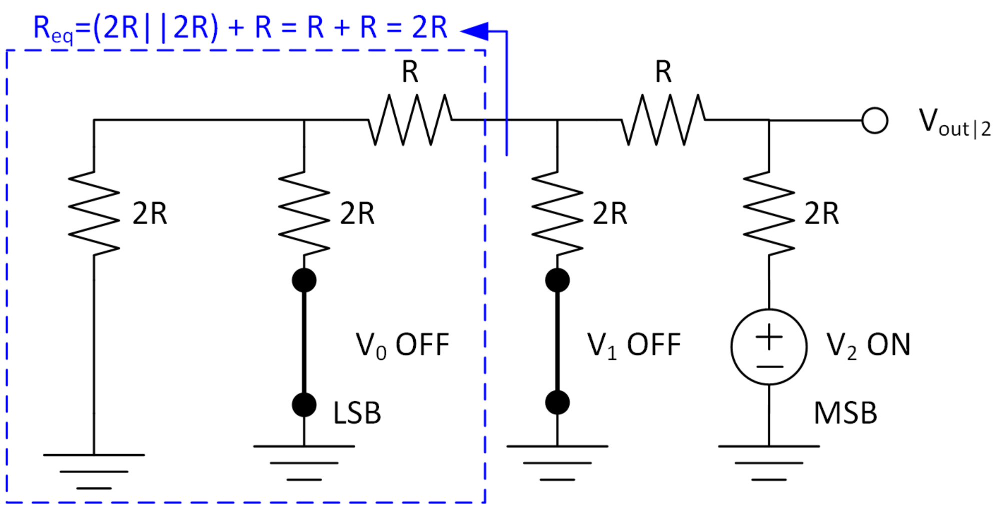
Now that we are done with the independent source V_0, we can repeat this for the other two independent sources. To find V_{out} due to independent source V_1 only, i.e. $V_{\text{out}|1}$:

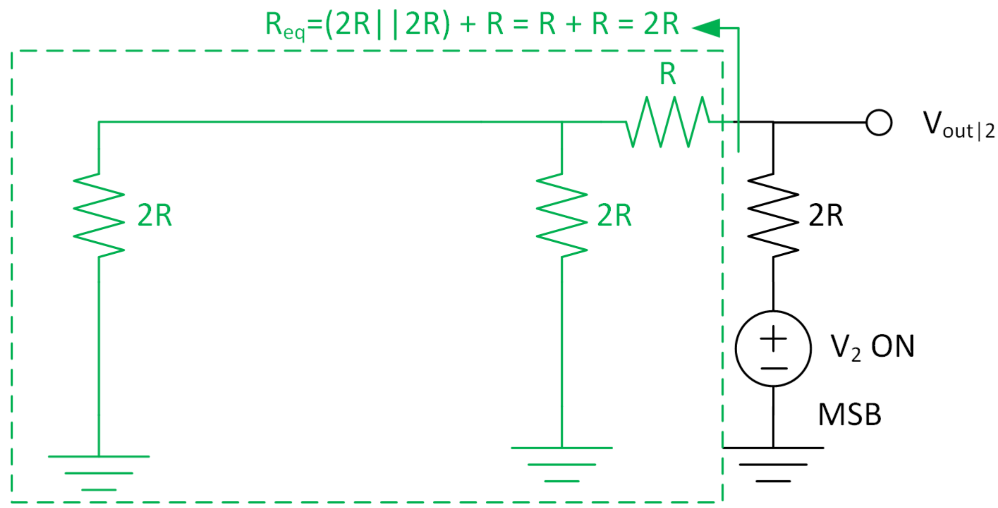


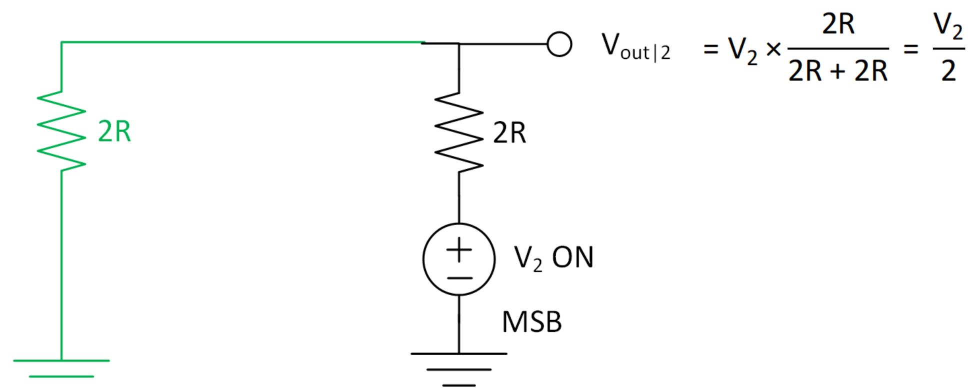




Similarly, we can solve for $V_{out|2}$:







Putting it together, we have:

$$V_{\text{out}} = V_{\text{out}\,|\,0} + V_{\text{out}\,|\,1} + V_{\text{out}\,|\,2} = \frac{V_0}{8} + \frac{V_1}{4} + \frac{V_2}{2} \tag{10}$$

If we substitute for $V_0 = b_0 V_{\text{DD}}$, $V_1 = b_1 V_{\text{DD}}$, $V_2 = b_2 V_{\text{DD}}$, we get:

$$V_{\text{out}} = \frac{b_0 V_{\text{DD}}}{8} + \frac{b_1 V_{\text{DD}}}{4} + \frac{b_2 V_{\text{DD}}}{2} \tag{11}$$

which is the same result we found using KCL and Gaussian elimination in (8).

If you've made it this far, well done! Here's a challenge question for you. If instead of binary, imagine we are dealing with ternary numbers, i.e. $V_0 = t_0 V_{\text{DD}}$, $V_1 = t_1 V_{\text{DD}}$, $V_2 = t_2 V_{\text{DD}}$ where t_0, t_1, and t_2 can be -1, 0, or 1 ("ternary" means three possible values per digit). Assume we also want the output voltage levels to be evenly-spaced for all 27 ($= 3^3$) possible 3-digit ternary numbers. Can you design a resistor DAC to achieve this?

(b) **If $b_2, b_1, b_0 = 0, 1, 1$, what is V_{out}?** Express your answer in terms of V_{DD}.

Solution: Plugging into the equation (8) from part (a), we get

$$V_{\text{out}} = \frac{3V_{\text{DD}}}{8}. \tag{12}$$

(c) **If $b_2, b_1, b_0 = 1, 0, 1$, what is V_{out}?** Express your answer in terms of V_{DD}.

Solution: Plugging into the equation (8) from part (a), we get

$$V_{\text{out}} = \frac{5V_{\text{DD}}}{8}. \tag{13}$$

(d) **If $b_2, b_1, b_0 = 1, 1, 0$, what is V_{out}?** Express your answer in terms of V_{DD}.

Solution: Plugging into the equation (8) from part (a), we get

$$V_{\text{out}} = \frac{3V_{\text{DD}}}{4}. \tag{14}$$

(e) **If $b_2, b_1, b_0 = 1, 1, 1$, what is V_{out}?** Express your answer in terms of V_{DD}.

Solution: Plugging into the equation (8) from part (a), we get

$$V_{\text{out}} = \frac{7V_{\text{DD}}}{8}. \tag{15}$$

(f) **Explain how your results above show that the resistive DAC converts the 3-bit binary number (b_2, b_1, b_0) to the output analog voltage V_{out}.**

Solution: Every increment of $\frac{1}{8}V_{\text{DD}}$ on V_{DD} represents an increment of 1 to the 3-bit binary number $(b_2 b_1 b_0)$.

Alternatively, you can view V_{DD} as being 1 and then these are the first binary digits after the "decimal point."

For example, if $V_{\text{out}} = \frac{5}{8}V_{\text{DD}}$, the input was 5 in binary (1 0 1)$\rightarrow$ $(b_2 = 1\ b_1 = 0\ b_0 = 1)$.

2. Existence and uniqueness of solutions to differential equations

When doing circuits or systems analysis, we sometimes model our system via a differential equation, and would often like to solve it to get the system trajectory. To this end, we would like to verify that a solution to our differential equation exists and is unique, so that our model is physically meaningful. There is a general approach to doing this, which is demonstrated in this problem.

We would like to show that there is a unique function $x\colon \mathbb{R} \to \mathbb{R}$ which satisfies

$$\frac{\mathrm{d}}{\mathrm{d}t}x(t) = \alpha x(t) \tag{16}$$

$$x(0) = x_0. \tag{17}$$

In order to do this, we will first verify that a solution x_d exists. To show that x_d is the unique solution, we will take an arbitrary solution y and show that $x_d(t) = y(t)$ for every t.

(a) First, let us show that a solution to our differential equation exists. **Verify that $x_d(t) := x_0 \mathrm{e}^{\alpha t}$ satisfies eq. (16) and eq. (17).**

Solution: We first verify eq. (16).

$$\begin{aligned} \frac{\mathrm{d}}{\mathrm{d}t}x_d(t) &= \frac{\mathrm{d}}{\mathrm{d}t}\left(x_0 \mathrm{e}^{\alpha t}\right) && (18)\\ &= x_0 \frac{\mathrm{d}}{\mathrm{d}t}\mathrm{e}^{\alpha t} && (19)\\ &= x_0 \cdot \alpha \mathrm{e}^{\alpha t} && (20)\\ &= \alpha \cdot x_0 \mathrm{e}^{\alpha t} && (21)\\ &= \alpha x_d(t). && (22) \end{aligned}$$

Now we verify eq. (17).

$$\begin{aligned} x_d(0) &= x_0 \mathrm{e}^{\alpha \cdot 0} && (23)\\ &= x_0 \mathrm{e}^{0} && (24)\\ &= x_0. && (25) \end{aligned}$$

(b) Now, let us show that our solution is unique. As mentioned before, suppose $y\colon \mathbb{R} \to \mathbb{R}$ also satisfies eq. (16) and eq. (17).

We want to show that $y(t) = x_d(t)$ for all t. Our strategy is to show that $\frac{y(t)}{x_d(t)} = 1$ for all t. However, this particular differential equation poses a problem: if $x_0 = 0$, then $x_d(t) = 0$ for all t, so that the quotient is not well-defined. To patch this method, we would like to avoid using any function with x_0 in the denominator. One way we can do this is consider a modification of the quotient $\frac{y(t)}{x_d(t)} = \frac{y(t)}{x_0 \mathrm{e}^{\alpha t}}$; in particular, we consider the function $z(t) := \frac{y(t)}{\mathrm{e}^{\alpha t}}$.

Show that $z(t) = x_0$ for all t, and explain why this means that $y(t) = x_d(t)$ for all t.

(HINT: Show first that $z(0) = x_0$ and then that $\frac{\mathrm{d}}{\mathrm{d}t}z(t) = 0$. Argue that these two facts imply that $z(t) = x_0$ for all t. Then show that this implies $y(t) = x_d(t)$ for all t.)

(HINT: Remember that we said y is any solution to eq. (16) and eq. (17), so we only know these properties of y. If you need something about y to be true, see if you can show it from eq. (16) and eq. (17).)

(HINT: When taking $\frac{\mathrm{d}}{\mathrm{d}t}z(t)$, remember to use the quotient rule, along with what we know about y.)

Solution: The solution goes in four stages, as per the hint.

Step 1. We show that $z(0) = x_0$. Indeed, using eq. (17),

$$z(0) = \frac{y(0)}{\mathrm{e}^{\alpha \cdot 0}} = \frac{x_0}{\mathrm{e}^0} = \frac{x_0}{1} = x_0. \tag{26}$$

Step 2. We show that $\frac{\mathrm{d}}{\mathrm{d}t}z(t) = 0$. Indeed, using the quotient rule from calculus and eq. (16),

$$\begin{aligned}
\frac{\mathrm{d}}{\mathrm{d}t}z(t) &= \frac{\mathrm{d}}{\mathrm{d}t}\frac{y(t)}{\mathrm{e}^{\alpha t}} && (27)\\
&= \frac{\mathrm{e}^{\alpha t}\left(\frac{\mathrm{d}}{\mathrm{d}t}y(t)\right) - y(t)\left(\frac{\mathrm{d}}{\mathrm{d}t}\mathrm{e}^{\alpha t}\right)}{\mathrm{e}^{2\alpha t}} && (28)\\
&= \frac{\mathrm{e}^{\alpha t}(\alpha y(t)) - y(t)(\alpha \mathrm{e}^{\alpha t})}{\mathrm{e}^{2\alpha t}} && (29)\\
&= \frac{\alpha \mathrm{e}^{\alpha t} y(t) - \alpha \mathrm{e}^{\alpha t} y(t)}{\mathrm{e}^{2\alpha t}} && (30)\\
&= \frac{0}{\mathrm{e}^{2\alpha t}} && (31)\\
&= 0. && (32)
\end{aligned}$$

Step 3. We show that $z(t) = x_0$ for all t. Indeed, since $\frac{\mathrm{d}}{\mathrm{d}t}z(t) = 0$, we know that $z(t)$ is a constant. Since $z(0) = x_0$, this gives that $z(t)$ is the constant value x_0, and hence $z(t) = x_0$ for all t.

Step 4. We show that $y(t) = x_d(t)$ for all t. Indeed, since $z(t) = x_0$ and $z(t) = \frac{y(t)}{\mathrm{e}^{\alpha t}}$, we have $x_0 = \frac{y(t)}{\mathrm{e}^{\alpha t}}$. We multiply both sides by $\mathrm{e}^{\alpha t}$ to get $y(t) = x_0 \mathrm{e}^{\alpha t}$. But this is just $x_d(t)$, so $y(t) = x_d(t)$ for all t.

3. Hambley P4.7

The capacitor shown in Figure 1 is charged to a voltage of 50 V prior to $t = 0$.

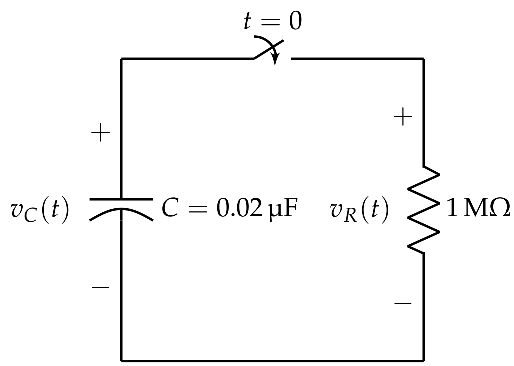


Figure 1: P4.7

(a) Find expressions for the voltage across the capacitor $v_C(t)$ and the voltage across the resistor $v_R(t)$.

Solution: We have that $RC = 0.02\,\text{s}$. Hence, using the derivation from lecture,

$$v_C(t) = \begin{cases} 50 & t < 0 \\ 50\mathrm{e}^{-\frac{t}{0.02}} = 50\mathrm{e}^{-50t} & t > 0 \end{cases} \tag{33}$$

and

$$v_R(t) = \begin{cases} 0 & t < 0 \\ 50\mathrm{e}^{-\frac{t}{0.02}} = 50\mathrm{e}^{-50t} & t > 0 \end{cases} \tag{34}$$

(b) Find an expression for the power delivered to the resistor.

Solution: We have that

$$p_R(t) = \frac{v_R(t)^2}{R} = \frac{2500}{10^6}\mathrm{e}^{-100t} = 2.5 \times 10^{-3}\mathrm{e}^{-100t}\,\text{W} \tag{35}$$

for $t > 0$, and $p_R(t) = 0\,\text{W}$ for $t < 0$

(c) Integrate the power from $t = 0$ to $t = \infty$ to find the energy delivered.

Solution: We have that

$$\begin{aligned} W &= \int_0^\infty p_R(t)\,\mathrm{d}t && (36) \\ &= 2.5 \times 10^{-3} \int_0^\infty \mathrm{e}^{-100t}\,\mathrm{d}t && (37) \\ &= -2.5 \times 10^{-5} \left(\mathrm{e}^{-100t}\big|_0^\infty \right) && (38) \\ &= 25\,\mu\text{J} && (39) \end{aligned}$$

(d) Show that the energy delivered to the resistor is equal to the energy stored in the capacitor prior to $t = 0$.

Solution: The initial energy stored in the capacitance is

$$W = \frac{1}{2}C(v_C(0))^2 \tag{40}$$
$$= \frac{1}{2} \times 0.02 \times 10^{-6} \times 50^2 \tag{41}$$
$$= 25\,\mu\text{J} \tag{42}$$

4. Hambley P4.3 and P4.4

(a) The initial voltage across the capacitor shown in Figure 2 is $v_c(0+) = 0$. Find an expression for the voltage across the capacitor as a function of time, and sketch it to scale versus time.

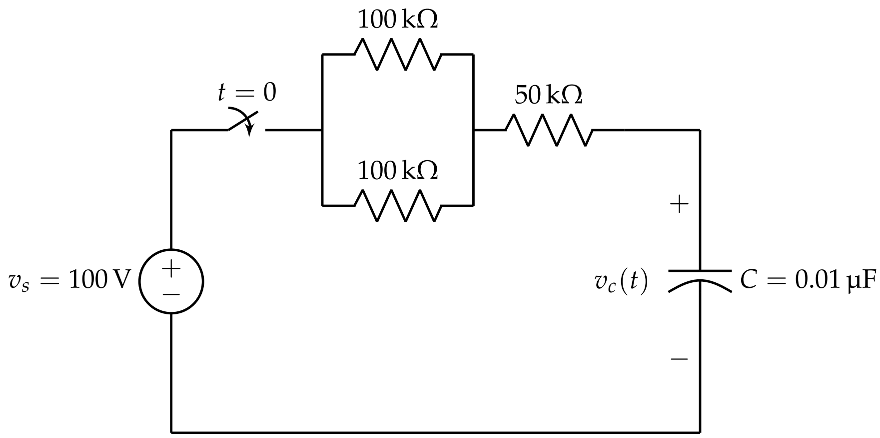
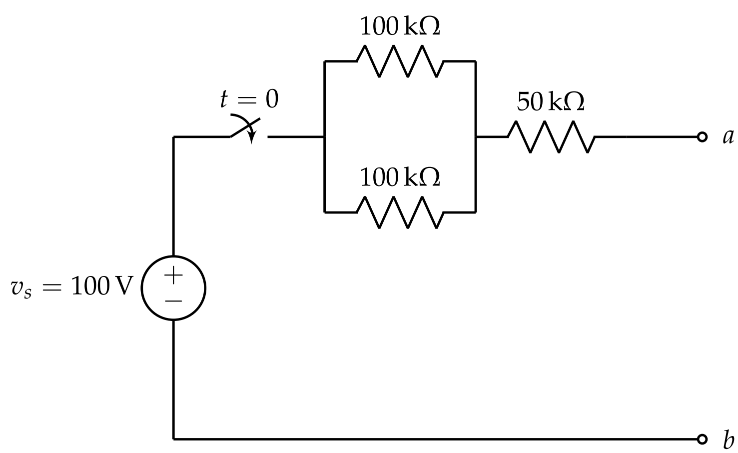


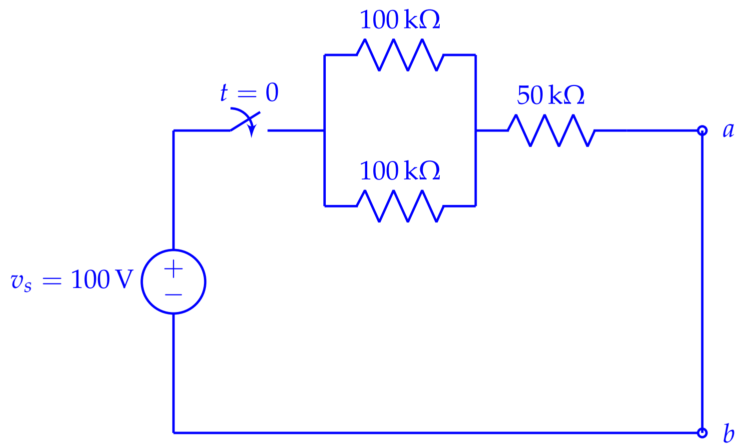
Figure 2: P4.3 Modified

(HINT: Consider simplifying the circuit using Thevenin equivalent circuits. That is, consider the following circuit, which is exactly Figure 2 without the capacitor:

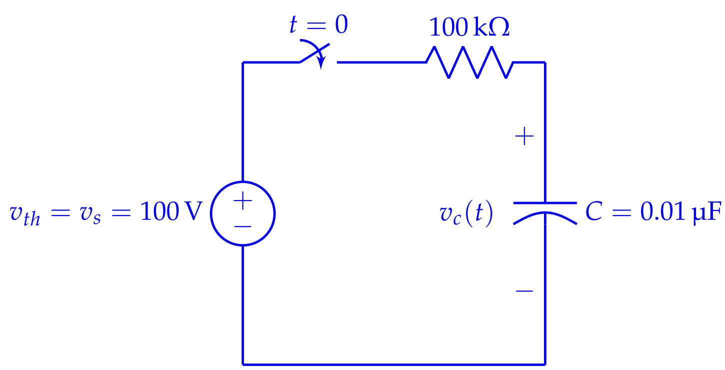


Find a Thevenin equivalent circuit and use this to simplify Figure 2.)

Solution: First, we can follow the hint to simplify the circuit. We can find V_{th} by measuring the voltage across terminals a and b. Since there is no current flowing in the circuit, $v_{th} = v_s = 100\,\text{V}$. Next, to find i_{no}, we connect terminals a and b as follows:



We can find the current between terminals a and b by first combining the resistors to find an equivalent resistance. Namely, we combine the two $100\,\mathrm{k\Omega}$ parallel resistors to obtain a single $50\,\mathrm{k\Omega}$ resistor, and we combine this resistor in series with the other $50\,\mathrm{k\Omega}$ resistor to obtain an equivalent resistance of $100\,\mathrm{k\Omega}$. Thus, $i_{no} = \frac{V_{th}}{100\,\mathrm{k\Omega}} = 1\,\mathrm{mA}$. To find R_{th}, we compute $R_{th} = \frac{v_{th}}{i_{no}} = 100\,\mathrm{k\Omega}$. Hence, an equivalent circuit to the one shown in Figure 2 is



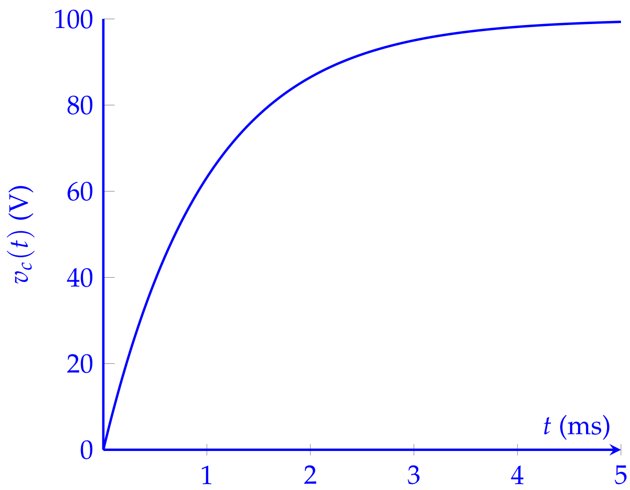
Now, we can solve for $v_c(t)$. Recall the derivation performed in lecture:

$$v_c(t) = v_s\left(1 - e^{-\frac{t}{\tau}}\right) + v_c(0+)e^{-\frac{t}{\tau}} \tag{43}$$

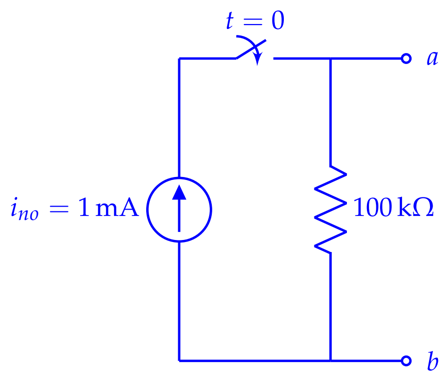
where $\tau = RC = 1\,\mathrm{ms}$. In this specific case, we have $v_c(0+) = 0$ and $v_s = 100$, so we are left with

$$v_c(t) = 100\left(1 - e^{-\frac{t}{10^{-3}}}\right) \tag{44}$$

The graph of this is plotted below:



Note: You can also simplify the circuit in the hint using a current source, which would give you



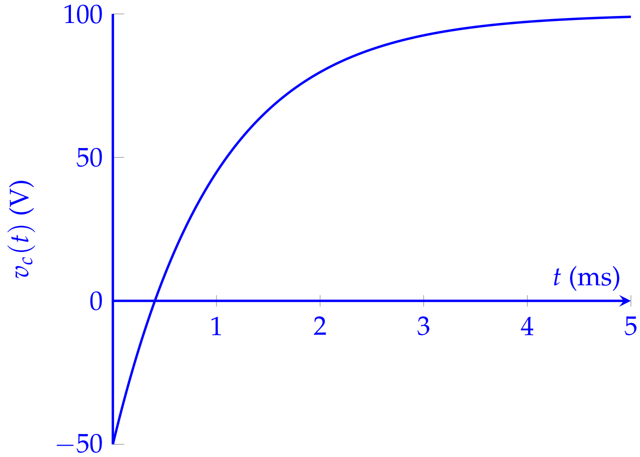
If you obtained the correct answer using this equivalent circuit, you can award yourself full credit.

(b) Repeat part (a) for an initial voltage $v_c(0+) = -50\,\text{V}$.

Solution: Using the same solution as above but setting $v_c(0+) = -50\,\text{V}$, we end up with

$$v_c(t) = 100\left(1 - e^{-\frac{t}{10^{-3}}}\right) - 50e^{-\frac{t}{10^{-3}}} \tag{45}$$

which is plotted below:


100
50
0
−50
$v_c(t)$ (V)
t (ms)
1
2
3
4
5

5. Hambley P4.18

Consider the circuit shown in Figure 3. Prior to $t = 0$, $v_1 = 100\,\text{V}$ and $v_2 = 0$.

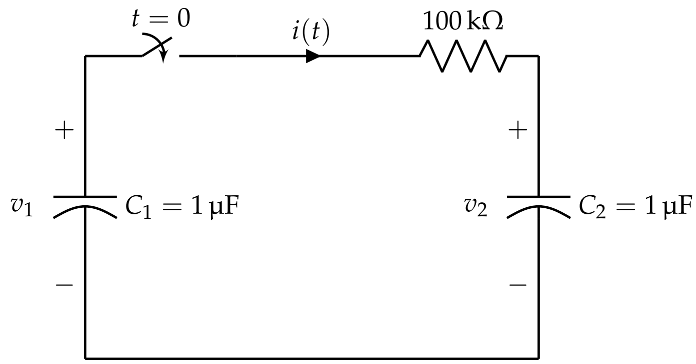


Figure 3: P4.18

(a) Immediately after the switch is closed, what is the value of the current (i.e., what is the value of $i(0+)$)?

Solution: The voltage across the capacitors cannot change instantaneously. Therefore,

$$i(0+) = \frac{v_1 - v_2}{100\,\text{k}\Omega} = \frac{100}{100 \times 10^3} = 1\,\text{mA} \tag{46}$$

(b) Write the KVL equation for the circuit in terms of the current and initial voltages. Take the derivative to obtain a differential equation.

Solution: Applying KVL, we have

$$-v_1(t) + Ri(t) + v_2(t) = 0 \tag{47}$$

$$-\left(\frac{1}{C_1}\int_0^t -i(t')\,\mathrm{d}t' - v_1(0)\right) + Ri(t) + \left(\frac{1}{C_2}\int_0^t i(t')\,\mathrm{d}t' + v_2(0)\right) = 0 \tag{48}$$

Taking a derivative with respect to time and rearranging, we obtain

$$\frac{\mathrm{d}i(t)}{\mathrm{d}t} + \frac{1}{R}\left(\frac{1}{C_1} + \frac{1}{C_2}\right)i(t) = 0 \tag{49}$$

(c) What is the value of the time constant in this circuit?

Solution: The time constant is $\tau = R\frac{C_1C_2}{C_1+C_2} = 50\,\text{ms}$.

(d) Find an expression for the current as a function of time.

Solution: From part (b),

$$i(t) = i(0)\mathrm{e}^{-\frac{t}{\tau}} \tag{50}$$

$$= \mathrm{e}^{-\frac{t}{50\times10^{-3}}}\,\text{mA} \tag{51}$$

(e) Find the value that v_2 approaches as t becomes very large.

Solution: We have

$$v_2(\infty) = \frac{1}{C_2}\int_0^\infty i(t)\,\mathrm{d}t + v_2(0) \tag{52}$$

$$= 10^6 \times 10^{-3}\int_0^\infty \mathrm{e}^{-\frac{t}{50\times10^{-3}}}\,\mathrm{d}t + 0 \tag{53}$$

$$= 10^3 \times \left(-50\times10^{-3}\right)\left(\mathrm{e}^{-\frac{t}{50\times10^{-3}}}\right)_0^\infty \tag{54}$$

$$= 50\,\mathrm{V} \tag{55}$$

6. Op-Amp Stability

In this question we will revisit the basic op-amp model that was introduced in EECS 16A and we will add a capacitance C_{out} to make the model more realistic (refer to figure 4). Now that we have the tools to do so, we will study the behavior of the op-amp in positive and negative feedback (refer to figure 5).

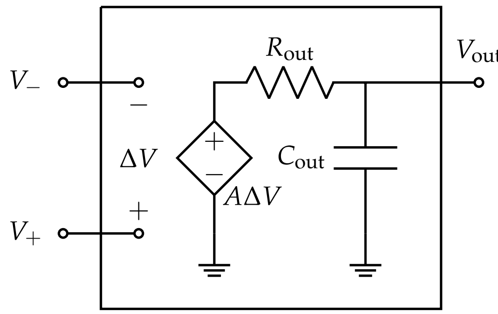
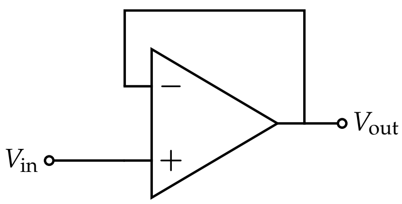
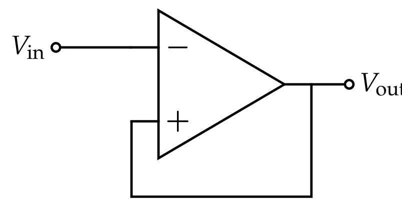


Figure 4: Op-amp model: $\Delta V = V_+ - V_-$



(a) Buffer in negative feedback



(b) "Buffer" in positive feedback that doesn't actually work as a buffer.

Figure 5: Op-amp in buffer configuration

(a) Using the op-amp model in figure 4 and the buffer in negative-feedback configuration in figure 5a, **draw a combined circuit**. Remember that $\Delta V = V_+ - V_-$, the voltage difference between the positive and negative labeled input terminals of the op-amp. *NOTE*: Here, we have used the equivalent model for the op-amp gain. In more advanced analog circuits courses, it is traditional to use a controlled current source with a resistor in parallel instead.

Solution: Please refer to Figure 6 for the completed circuit.

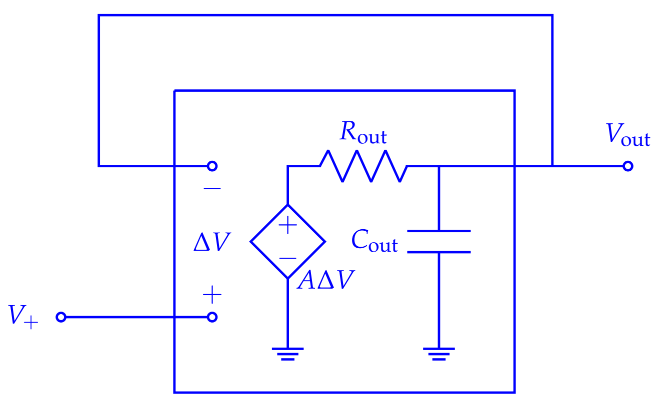


Figure 6: Negative-feedback buffer configuration using given op-amp model

(b) Let's look at the op-amp in negative feedback. From our discussions in EECS 16A, we know that the buffer in figure 5a should work with $V_{\text{out}} \approx V_{\text{in}}$ by the golden rules. **Write a differential-equation for V_{out} by replacing the op-amp with the given model and show what the solution will be as a function of time for a static V_{in}. What does it converge to as $t \to \infty$?** Note: We assume the gain $A > 1$ for all parts of the question. For this part, you can assume the initial condition $V_{\text{out}}(0) = 0$.

Solution: We have $\Delta V = V_{\text{in}} - V_{\text{out}}$. Next, we can write the following branch equations:

$$i = C_{\text{out}} \frac{\mathrm{d}}{\mathrm{d}t} V_{\text{out}} \tag{56}$$

$$A\Delta V = V_R + V_{\text{out}} \tag{57}$$

$$A(V_{\text{in}} - V_{\text{out}}) = RC \frac{\mathrm{d}}{\mathrm{d}t} V_{\text{out}} + V_{\text{out}} \tag{58}$$

Simplifying the last line, we get:

$$AV_{\text{in}} = RC \frac{\mathrm{d}}{\mathrm{d}t} V_{\text{out}} + (1+A)V_{\text{out}} \tag{59}$$

$$\frac{\mathrm{d}}{\mathrm{d}t} V_{\text{out}} + \frac{1+A}{RC} V_{\text{out}} - \frac{A}{RC} V_{\text{in}} = 0 \tag{60}$$

$$\tag{61}$$

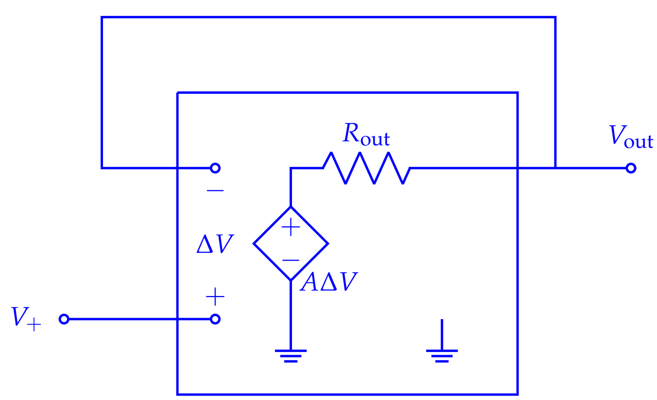
We can solve this differential equation using the homogeneous and particular solution method. The homogeneous differential equation in this case is:

$$\frac{\mathrm{d}}{\mathrm{d}t} V_{\text{out},h} + \frac{1+A}{RC} V_{\text{out},h} = 0 \tag{62}$$

This is a common first order differential equation with the solution:

$$V_{\text{out},h} = C e^{-\frac{1+A}{RC} t} \tag{63}$$

For the particular solution, we can analyze the DC steady-state of our circuit (the capacitor is replaced with an open circuit):



With the capacitor replaced with an open circuit, we can see that in DC steady-state:

$$A\Delta V = V_{\text{out}} \tag{64}$$

$$A(V_{\text{in}} - V_{\text{out}}) = V_{\text{out}} \tag{65}$$

$$V_{\text{out}} = \frac{A}{A+1} V_{\text{in}} \tag{66}$$

Thus, the particular solution for our differential equation will be:

$$V_{\text{out},p} = \frac{A}{A+1} V_{\text{in}} \tag{67}$$

Thus, the combined solution is:

$$V_{\text{out}} = V_{\text{out},h} + V_{\text{out},p} = \mathrm{C}\mathrm{e}^{-\frac{1+A}{RC}t} + \frac{A}{A+1} V_{\text{in}} \tag{68}$$

With $V_{\text{out}}(0) = 0$, we find that $C = -\frac{A}{A+1} V_{\text{in}}$ and the full solution is:

$$V_{\text{out}} = V_{\text{out},h} + V_{\text{out},p} = \frac{A}{A+1} V_{\text{in}} (1 - \mathrm{e}^{-\frac{1+A}{RC}t}) \tag{69}$$

Since $A > 1$, the exponent is negative, hence as $t \to \infty$, the solution will converge to $V_{\text{out}} \to \frac{A}{A+1} V_{\text{in}}$.

(c) Next, let's look at the op-amp in positive feedback. We know that the configuration given in figure 5b is unstable and V_{out} will just rail. **Again, using the op-amp model in figure 4, show that V_{out} does not converge and hence the output will rail. For positive DC input $V_{\text{in}} > 0$, will V_{out} rail to the positive or negative side? Explain.** For this part, you can assume the initial condition $V_{\text{out}}(0) = 0$.

Solution: This time, we have $\Delta V = V_{\text{out}} - V_{\text{in}}$. Next, we can write the following branch equations:

$$i = C_{\text{out}} \frac{\mathrm{d}}{\mathrm{d}t} V_{\text{out}} \tag{70}$$

$$A \Delta V = V_R + V_{\text{out}} \tag{71}$$

$$A(V_{\text{out}} - V_{\text{in}}) = RC \frac{\mathrm{d}}{\mathrm{d}t} V_{\text{out}} + V_{\text{out}} \tag{72}$$

Simplifying the last line, we get:

$$-AV_{\text{in}} = RC \frac{\mathrm{d}}{\mathrm{d}t} V_{\text{out}} + (1 - A) V_{\text{out}} \tag{73}$$

$$\frac{\mathrm{d}}{\mathrm{d}t} V_{\text{out}} + \frac{1-A}{RC} V_{\text{out}} + \frac{A}{RC} V_{\text{in}} = 0 \tag{74}$$

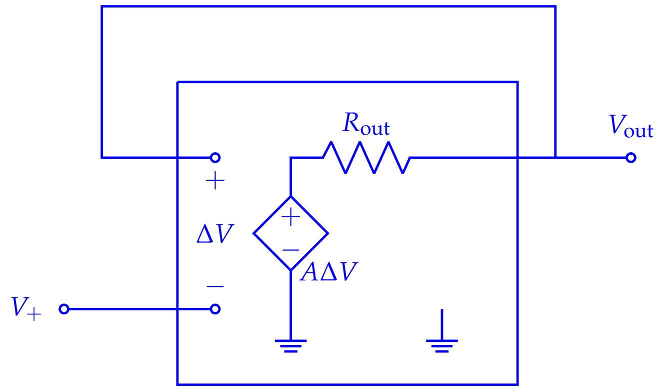
We can solve this differential equation using the homogeneous and particular solution method. The homogeneous differential equation in this case is:

$$\frac{\mathrm{d}}{\mathrm{d}t} V_{\text{out},h} - \frac{A-1}{RC} V_{\text{out},h} = 0 \tag{75}$$

This is a common first order differential equation with the solution:

$$V_{\text{out},h} = \mathrm{C}\mathrm{e}^{\frac{A-1}{RC}t} \tag{76}$$

For the particular solution, we can analyze the DC steady-state of our circuit (the capacitor is replaced with an open circuit (note that the terminals have switched in this case compared to the last part):



With the capacitor replaced with an open circuit, we can see that in DC steady-state:

$$A\Delta V = V_{out} \tag{77}$$

$$A(V_{out} - V_{in}) = V_{out} \tag{78}$$

$$V_{out} = \frac{A}{A-1} V_{in} \tag{79}$$

Thus, the particular solution for our differential equation will be:

$$V_{out,p} = \frac{A}{A-1} V_{in} \tag{80}$$

Thus, the combined solution is:

$$V_{out} = V_{out,h} + V_{out,p} = Ce^{\frac{A-1}{RC}t} + \frac{A}{A-1} V_{in} \tag{81}$$

With $V_{out}(0) = 0$, we find that $C = -\frac{A}{A-1} V_{in}$ and the full solution is:

$$V_{out} = -\frac{AV_{in}}{A-1}\left(e^{\frac{A-1}{RC}t} - 1\right) \tag{82}$$

Since $A > 1$, the exponent is positive, hence as $t \to \infty$, the solution will be unbounded, and $V_{out} \to -\infty$. Of course, it can't grow to negative infinity, and so we can conclude that V_{out} will rail to the negative side if V_{out} had started at zero. The same exact story would hold if V_{out} started anywhere below V_{in}.

The case of V_{out} starting out significantly greater than V_{in} deserves some mention, although not necessary for full credit on this question. In this case, the initial condition for $\widetilde{V}_{out}$ would be positive, and would proceed to unstably attempt to run away to positive infinity. It would be stopped at the positive rail, where it would stay.

Essentially, all that matters is the initial condition of $\widetilde{V}_{out}$ — start out positive, then we rail to a positive rail for V_{out}. Start out negative, then we rail to the negative rail for V_{out}. An op-amp in positive feedback retains its comparator-like character.

(d) For an ideal op-amp, we can assume that it has an infinite gain, i.e., $A \to \infty$. **Under these assumptions, show that the op-amp in negative feedback behaves as an ideal buffer, i.e., $V_{out} = V_{in}$.**

Solution: Taking the limit of our solution in part (a),

$$V_{out} = \lim_{A\to\infty} \frac{AV_{in}}{A+1}\left(1 - e^{-\frac{A+1}{RC}t}\right) \tag{83}$$

$$= V_{\text{in}} \tag{84}$$

The coefficient of the exponent goes to 1 whereas the exponent itself goes to $-\infty$, and hence $1 - e^{-\infty} \to 1$.

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