Homework 1

This homework is due on Friday, January 28, 2022, at 11:59PM. Self-grades and HW Resubmissions are due on the following Friday, Feburary 4, 2022, at 11:59PM.

1. **Group Formation Survey**

   Please fill out this group formation survey if you are interested in getting matched up in a study group. We highly recommend joining a study group in order to foster a sense of community in the course and learn from others. EECS 16B is a pretty fast-paced course, and you can benefit quite a bit from your peers' perspectives on the material.

   Within a few weeks, you should get an email informing you of the group you have been matched with. It is respectful and professional behavior to follow up with your group members; completing this survey suggests you are interested in joining a group, after all – we hope you stay true to your word!

   Special shoutout to Prof. Ranade's group formation research team for making this possible!

   Just so you have an answer to put down for this question, write down whether you filled out the survey or not.

2. Reading Lecture Notes

Staying up to date with lectures is an important part of the learning process in this course. Here are links to the notes that you need to read for this week: Note j, Note 1

(a) **Have you seen the vector representation of complex numbers in Note j before?** Question 3 explores the topics in more details.

(b) **Have you solved differential equations of this form before?**

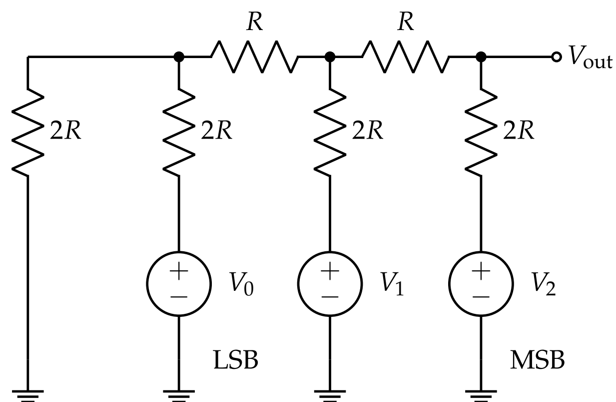
$$\frac{\mathrm{d}}{\mathrm{d}t}x(t) = \alpha x(t) \tag{1}$$

Questions 6 and 7 explore the topics of Note 1 in more details.

3. Digital-Analog Converter

A digital-analog converter (DAC) is one of the key interface components between the digital and the analog world. It is a circuit for converting a digital representation of a number (binary) into a corresponding analog voltage. In this problem, we will consider a DAC made out of resistors only (resistive DAC) called the R-$2R$ ladder. This DAC will help us generate the analog voltages from the digital representation, and later will also help us digitize the analog voltages when we will be building analog to digital interfaces in Lab 3, in part based on this ladder-DAC.

Here is the circuit for a 3-bit resistive DAC.



Let $b_0, b_1, b_2 = \{0, 1\}$ (that is, either 1 or 0), and let the voltage sources $V_0 = b_0 V_{DD}$, $V_1 = b_1 V_{DD}$, $V_2 = b_2 V_{DD}$, where V_{DD} is the supply voltage.

As you may have noticed, (b_2, b_1, b_0) represents a 3-bit binary (unsigned) number where each of b_i is a binary bit. b_0 is the least significant bit (LSB) and b_2 is the most significant bit (MSB). We will now analyze how this converter functions.

(a) **Solve for V_{out} in terms of V_{DD} and the binary bits b_2, b_1, b_0.**

(b) **If $b_2, b_1, b_0 = 0, 1, 1$, what is V_{out}?** Express your answer in terms of V_{DD}.

(c) **If $b_2, b_1, b_0 = 1, 0, 1$, what is V_{out}?** Express your answer in terms of V_{DD}.

(d) **If $b_2, b_1, b_0 = 1, 1, 0$, what is V_{out}?** Express your answer in terms of V_{DD}.

(e) **If $b_2, b_1, b_0 = 1, 1, 1$, what is V_{out}?** Express your answer in terms of V_{DD}.

(f) **Explain how your results above show that the resistive DAC converts the 3-bit binary number (b_2, b_1, b_0) to the output analog voltage V_{out}.**

4. Complex Numbers

Recall that a complex number $z \in \mathbb{C}$ is a number that can be expressed in the form

$$z = x + \mathrm{j}y \tag{2}$$

where $x, y \in \mathbb{R}$ and $j^2 = -1$. This is known as the Cartesian form of a complex number. We call x the real part of z and denote it $\mathrm{Re}\{z\} = x$. We call y the imaginary part of z and denote it $\mathrm{Im}\{z\} = y$.

Complex numbers can be visualized as vectors on the complex plane, as in Figure 1.

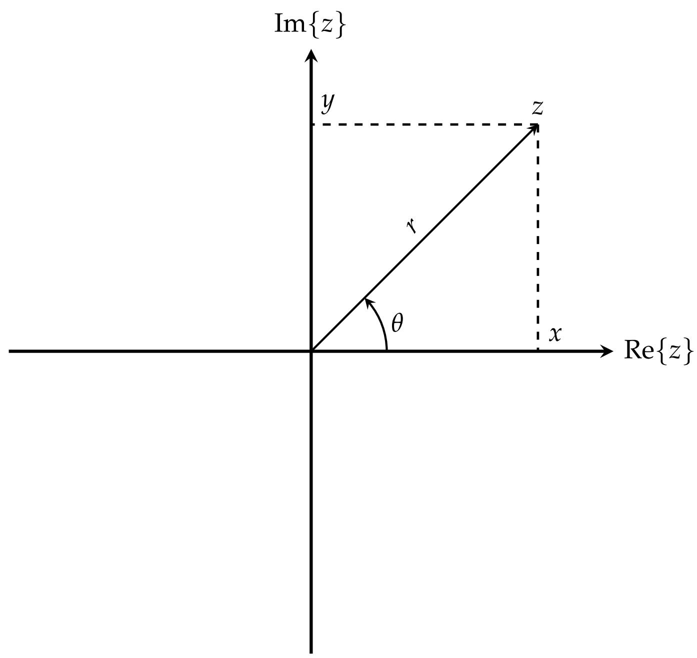


Figure 1: Complex plane

In this question, we will derive the polar form of a complex number and use this form to make some interesting conclusions.

(a) **Write an expression for the length of the vector z (as in Figure 1) in terms of x and y.** This is the magnitude of a complex number and is denoted by $|z|$ or r.
(*HINT: Use the Pythagorean theorem.*)

(b) **Write expressions for x and y in terms of r and θ.**

(c) **Substitute for x and y in Equation 2.** Use Euler's identity[1] $\mathrm{e}^{\mathrm{j}\theta} = \cos(\theta) + \mathrm{j}\sin(\theta)$ to **conclude that**

$$z = r\mathrm{e}^{\mathrm{j}\theta}. \tag{3}$$

(d) In the complex plane, **sketch the set of all the complex numbers such that $|z| = 1$. What are the z values where the sketched figure intersects the real axis and the imaginary axis?**

(e) Assume $z = r\mathrm{e}^{\mathrm{j}\theta}$. **Show that** $\overline{z} = r\mathrm{e}^{-\mathrm{j}\theta}$. Recall that the complex conjugate of a complex number $z = x + \mathrm{j}y$ is $\overline{z} = x - \mathrm{j}y$.

(f) **Show (by direct calculation) that**

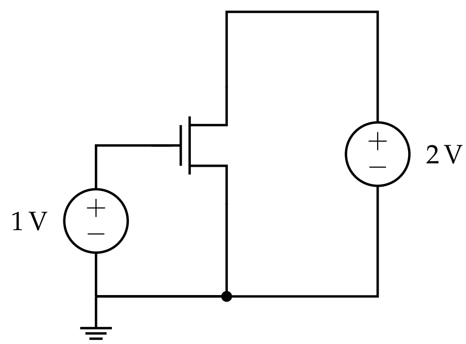
$$r^2 = z\overline{z}. \tag{4}$$

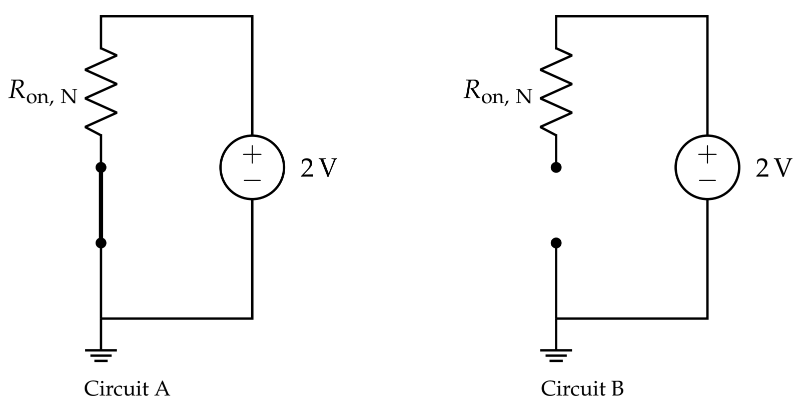
[1] also known as de Moivre's Theorem.

5. Transistor Behavior

For all NMOS devices in this problem, $V_{tn} = 0.5\,\text{V}$. For all PMOS devices in this problem, $|V_{tp}| = 0.6\,\text{V}$.

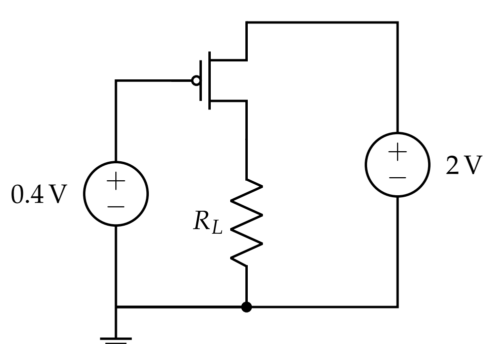
(a) Which is the equivalent circuit as seen form the voltage source on the right-hand side of the circuit? **Fill in the correct bubble.**

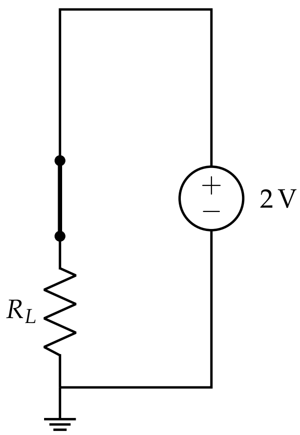




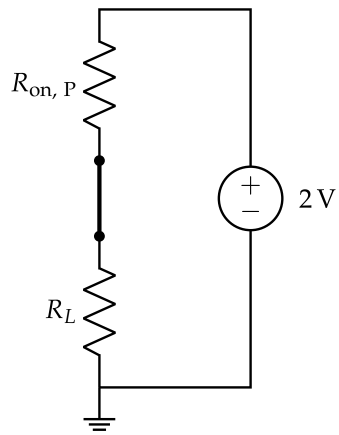
	A	B
Equivalent Circuit	◯	◯

(b) Which is the equivalent circuit as seen form the voltage source on the right-hand side of the circuit? **Fill in the correct bubble.**

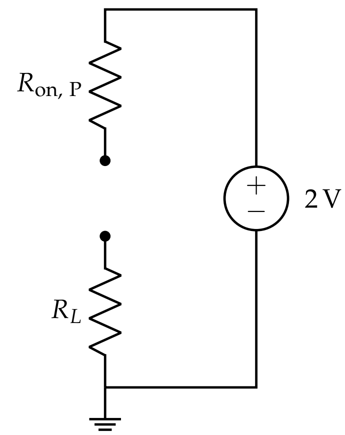




Circuit A



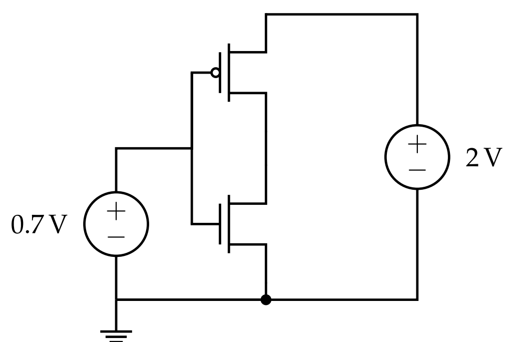
Circuit B

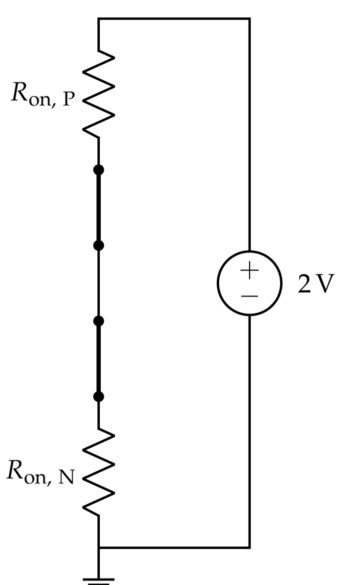


Circuit C

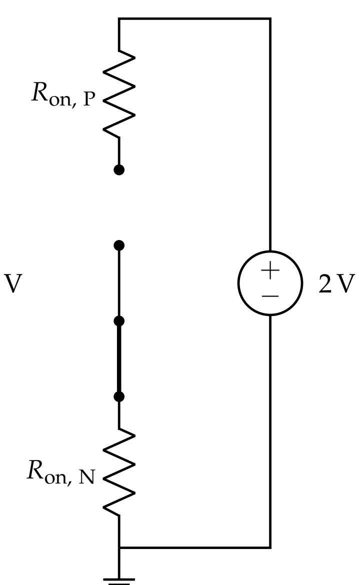
	A	B	C
Equivalent Circuit	○	○	○

(c) Which is the equivalent circuit as seen form the voltage source on the right-hand side of the circuit? **Fill in the correct bubble.**

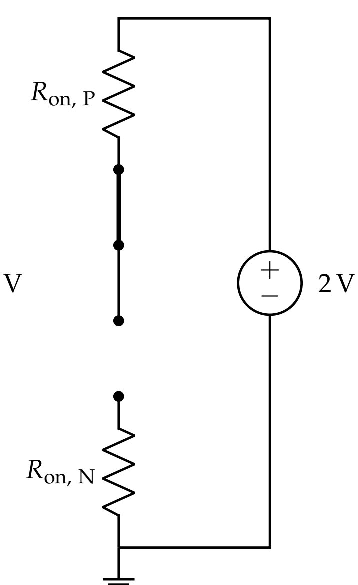




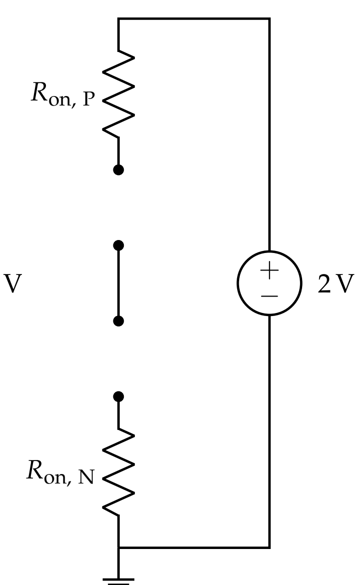
Circuit A



Circuit B



Circuit C



Circuit D

	A	B	C	D
Equivalent Circuit	◯	◯	◯	◯

6. Existence and uniqueness of solutions to differential equations

When doing circuits or systems analysis, we sometimes model our system via a differential equation, and would often like to solve it to get the system trajectory. To this end, we would like to verify that a solution to our differential equation exists and is unique, so that our model is physically meaningful. There is a general approach to doing this, which is demonstrated in this problem.

We would like to show that there is a unique function $x\colon \mathbb{R} \to \mathbb{R}$ which satisfies

$$\frac{\mathrm{d}}{\mathrm{d}t}x(t) = \alpha x(t) \tag{5}$$

$$x(0) = x_0. \tag{6}$$

In order to do this, we will first verify that a solution x_d exists. To show that x_d is the unique solution, we will take an arbitrary solution y and show that $x_d(t) = y(t)$ for every t.

(a) First, let us show that a solution to our differential equation exists. **Verify that** $x_d(t) := x_0 e^{\alpha t}$ **satisfies eq. (5) and eq. (6).**

(b) Now, let us show that our solution is unique. As mentioned before, suppose $y\colon \mathbb{R} \to \mathbb{R}$ also satisfies eq. (5) and eq. (6).

We want to show that $y(t) = x_d(t)$ for all t. Our strategy is to show that $\frac{y(t)}{x_d(t)} = 1$ for all t.

However, this particular differential equation poses a problem: if $x_0 = 0$, then $x_d(t) = 0$ for all t, so that the quotient is not well-defined. To patch this method, we would like to avoid using any function with x_0 in the denominator. One way we can do this is consider a modification of the quotient $\frac{y(t)}{x_d(t)} = \frac{y(t)}{x_0 e^{\alpha t}}$; in particular, we consider the function $z(t) := \frac{y(t)}{e^{\alpha t}}$.

Show that $z(t) = x_0$ **for all** t**, and explain why this means that** $y(t) = x_d(t)$ **for all** t**.**

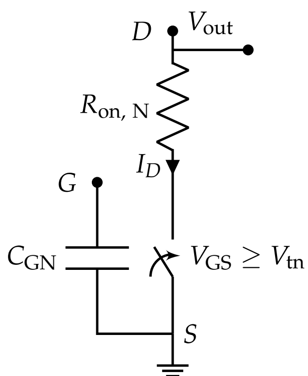
(*HINT: Show first that* $z(0) = x_0$ *and then that* $\frac{\mathrm{d}}{\mathrm{d}t}z(t) = 0$*. Argue that these two facts imply that* $z(t) = x_0$ *for all* t*. Then show that this implies* $y(t) = x_d(t)$ *for all* t*.*)

(*HINT: Remember that we said* y *is any solution to eq.* (5) *and eq.* (6)*, so we only know these properties of* y*. If you need something about* y *to be true, see if you can show it from eq.* (5) *and eq.* (6)*.*)

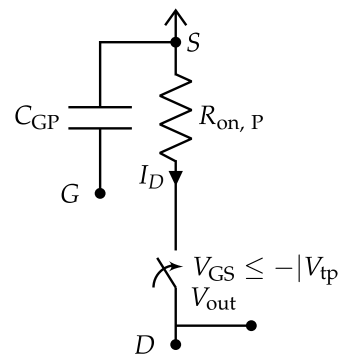
(*HINT: When taking* $\frac{\mathrm{d}}{\mathrm{d}t}z(t)$*, remember to use the quotient rule, along with what we know about* y*.*)

7. Transistor Switch Model

We can improve our resistor-switch model of the transistor by adding in a gate capacitance. In this model, the gate capacitances C_{GN} and C_{GP} represent the lumped physical capacitance present on the gate node of all transistor devices. This capacitance is important as it determines the delay of a transistor logic chain.



(a) NMOS Transistor Resistor-switch-capacitor model



(b) PMOS Transistor Resistor-switch-capacitor model. Note we have drawn this so that it aligns with the inverter.

You have two CMOS inverters made from NMOS and PMOS devices. Both NMOS and PMOS devices have an "on resistance" of $R_{on,\,N} = R_{on,\,P} = 1\,k\Omega$, and each has a gate capacitance (input capacitance) of $C_{GN} = C_{GP} = 1\,fF$ (fF = femto-Farads = 1×10^{-15} F). We assume the "off resistance" (the resistance when the transistor is off) is infinite (i.e., the transistor acts as an open circuit when off). The supply voltage V_{DD} is 1V. The two inverters are connected in series, with the output of the first inverter driving the input of the second inverter (Figure 9).

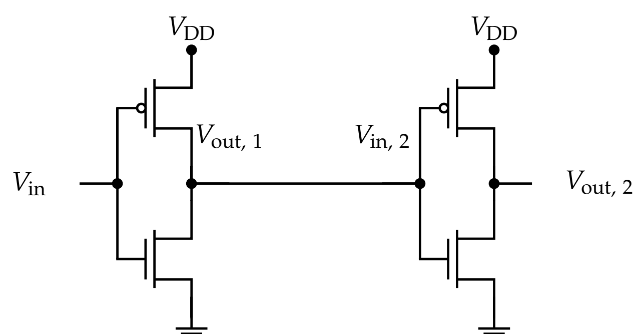


Figure 9: CMOS Inverter chain

(a) Assume the input to the first inverter has been low ($V_{in} = 0$ V) for a long time, and then switches at time $t = 0$ to high ($V_{in} = V_{DD}$).

Draw a simple RC circuit and write a differential equation describing the output voltage of the first inverter ($V_{out,\,1}$) for time $t \geq 0$.

Don't forget that the second inverter is "loading" the output of the first inverter — you need to think about both of them.

(b) Given the initial conditions in part (a), **solve for $V_{out,\,1}(t)$.**

(c) **Sketch the output voltage of the first inverter, showing clearly (1) the initial value, (2) the initial slope, (3) the asymptotic value, and (4) the time that it takes for the voltage to decay to roughly 1/3 of its initial value.**

(d) A long time later, the input to the first inverter switches low again.
Solve for $V_{\text{out, }1}(t)$**.**
Sketch the output voltage of the first inverter ($V_{\text{out, }1}$**), showing clearly (1) the initial value, (2) the initial slope, and (3) the asymptotic value.**

8. **(OPTIONAL) Make Your Own Problem.**

   Write your own problem about content covered in the course thus far, and provide a thorough solution to it.

   NOTE: This can be a totally new problem, a modification on an existing problem, or a Jupyter part for a problem that previously didn't have one. Please cite all sources for anything (including course material) that you used as inspiration.

   NOTE: High-quality problems may be used as inspiration for the problems we choose to put on future homeworks or exams.

9. **Homework Process and Study Group**

   Citing sources and collaborators are an important part of life, including being a student!

   We also want to understand what resources you find helpful and how much time homework is taking, so we can change things in the future if possible.

   (a) **What sources (if any) did you use as you worked through the homework?**

   (b) **If you worked with someone on this homework, who did you work with?**
   List names and student ID's. (In case of homework party, you can also just describe the group.)

   (c) **Roughly how many total hours did you work on this homework? Write it down here where you'll need to remember it for the self-grade form.**

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