Last Discussion! $\ddot{\frown}$

Gram Schmidt on Complex vectors

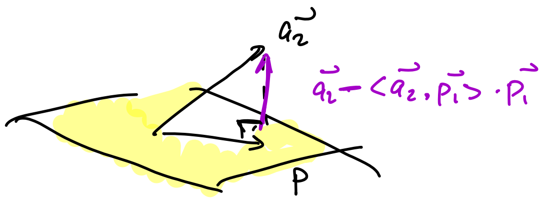
conjugate transpose

Recall: $\vec{u}, \vec{v} \in \mathbb{C}^n$, $\langle \vec{u}, \vec{v} \rangle = \vec{v}^{*} \vec{u}$

$\|\vec{u}\| = \sqrt{\langle \vec{u}, \vec{u} \rangle} \in \mathbb{R}$

Recall: Gram Schmidt

$\{\vec{a_1}, \dots \vec{a_n}\} \xrightarrow{G.S.} \{\vec{p_1}, \dots \vec{p_n}\}$ where $\vec{p_1}, \dots \vec{p_n}$ forms an orthonormal basis



① $\vec{p_1} = \dfrac{\vec{a_1}}{\|\vec{a_1}\|}$

② $\vec{z_2} = \vec{a_2} - \langle \vec{a_2}, \vec{p_1} \rangle \cdot \vec{p_1}$

$\vec{p_2} = \dfrac{\vec{z_2}}{\|\vec{z_2}\|}$

⋮

③ $\vec{z_3} = \vec{a_3} - \langle \vec{a_3}, \vec{p_1} \rangle \cdot \vec{p_1} - \langle \vec{a_3}, \vec{p_2} \rangle \cdot \vec{p_2}$

⋮

1(a) $\vec{a_1} = \begin{bmatrix} i \\ -1 \\ 0 \end{bmatrix}, \vec{a_2} = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}, \vec{a_3} = \begin{bmatrix} 0 \\ i \\ 1 \end{bmatrix}$

$\vec{p_1} = \dfrac{\vec{a_1}}{\|\vec{a_1}\|}$; $\|\vec{a_1}\| = \sqrt{\langle \vec{a_1}, \vec{a_1} \rangle} = \sqrt{[-i \ -1 \ 0]\begin{bmatrix} i \\ -1 \\ 0 \end{bmatrix}} = \sqrt{2}$

$\therefore \vec{p_1} = \begin{bmatrix} i/\sqrt{2} \\ -1/\sqrt{2} \\ 0 \end{bmatrix}$

$\vec{z_2} = \vec{a_2} - \underbrace{\langle \vec{a_2}, \vec{p_1} \rangle}_{\vec{p_1}^{*} \vec{a_2}} \cdot \vec{p_1}$

$$= \begin{bmatrix}0\\0\\1\end{bmatrix} - \cancelto{0}{\begin{bmatrix}-j/\sqrt{2} & -1/\sqrt{2} & 0\end{bmatrix}\begin{bmatrix}0\\0\\1\end{bmatrix}} \cdot \vec{p}_1$$

$$= \begin{bmatrix}0\\0\\1\end{bmatrix} \longrightarrow \vec{p}_2 = \vec{z}_2$$

$$\vec{z}_3 = \vec{a}_3 - \underbrace{\langle \vec{a}_3, \vec{p}_1 \rangle}_{\vec{p}_1^{\,*}\vec{a}_3} \cdot \vec{p}_1 - \underbrace{\langle \vec{a}_3, \vec{p}_2 \rangle}_{\vec{p}_2^{\,*}\vec{a}_3} \cdot \vec{p}_2$$

$$= \begin{bmatrix}0\\j\\1\end{bmatrix} - \left(\begin{bmatrix}-j/\sqrt{2} & -1/\sqrt{2} & 0\end{bmatrix}\begin{bmatrix}0\\j\\1\end{bmatrix}\right) \cdot \vec{p}_1 - \left(\begin{bmatrix}0 & 0 & 1\end{bmatrix}\begin{bmatrix}0\\j\\1\end{bmatrix}\right) \cdot \vec{p}_2$$

$$= \begin{bmatrix}0\\j\\1\end{bmatrix} - \left(\frac{-j}{\sqrt{2}}\right)\begin{bmatrix}j/\sqrt{2}\\-1/\sqrt{2}\\0\end{bmatrix} - \begin{bmatrix}0\\0\\1\end{bmatrix}$$

$$= \begin{bmatrix}-1/2\\j/2\\0\end{bmatrix}$$

$$\vec{p}_3 = \frac{\vec{z}_3}{\|\vec{z}_3\|} \; ; \quad \|\vec{z}_3\| = \sqrt{\langle \vec{z}_3, \vec{z}_3 \rangle} = \sqrt{\begin{bmatrix}-1/2 & -j/2 & 0\end{bmatrix}\begin{bmatrix}-1/2\\j/2\\0\end{bmatrix}}$$

$$= \sqrt{\frac{1}{4} + \frac{1}{4} + 0} = \frac{1}{\sqrt{2}}$$

$$\vec{p}_3 = \sqrt{2} \cdot \begin{bmatrix}-1/2\\j/2\\0\end{bmatrix} = \begin{bmatrix}-1/\sqrt{2}\\j/\sqrt{2}\\0\end{bmatrix}$$

$\therefore$ orthonormal set, $\{\vec{p}_1, \vec{p}_2, \vec{p}_3\}$

$$= \left\{ \begin{bmatrix}j/\sqrt{2}\\-1/\sqrt{2}\\0\end{bmatrix}, \begin{bmatrix}0\\0\\1\end{bmatrix}, \begin{bmatrix}1/\sqrt{2}\\-j/\sqrt{2}\\0\end{bmatrix} \right\}$$

$A\vec{x} = \vec{y}$, $A \in \mathbb{R}^{m \times n}$

1. A is a square, $\vec{x} = A^{-1}\vec{y}$
2. A is tall, [$m > n$], $\vec{x} = (A^T A)^{-1} A^T \vec{y}$ $\longrightarrow$ think system ID
3. A is wide, [$m < n$], $\vec{x} = (U \Sigma V^T)^{-1} \vec{y}$ pseudo inverse

   the minimum norm solution!

Controllability matrix

(b) / (c) D data matrix   measurement $\vec{y}$ $[m > n]$

$$D\vec{x} = \vec{y}$$

least squares solution (data $\in \mathbb{R}$) : $\vec{x} = (D^T D)^{-1} D^T \vec{y}$

" " " (data $\in \mathbb{C}$) : $\vec{x} = (D^* D)^{-1} D^* \vec{y}$

$\rightarrow$ if D is orthonormal, $D^T D = I_{n\times n}$ (real case)

$D^* D = I_{n\times n}$ (complex case)

$$\therefore \quad \vec{x} = \cancelto{I_{n\times n}}{(D^* D)^{-1}} D^* \vec{y}$$

$$= D^* \vec{y}$$