1 Inner Products

An inner product $\langle \cdot, \cdot \rangle$ on a vector space $V$ over $\mathbb{R}$ is a function that takes in two vectors and outputs a scalar, such that $\langle \cdot, \cdot \rangle$ is symmetric, linear, and positive-definite.

- Symmetry: $\langle \vec{u}, \vec{v} \rangle = \langle \vec{v}, \vec{u} \rangle$
- Scaling: $\langle c \vec{u}, \vec{v} \rangle = c \langle \vec{u}, \vec{v} \rangle$ and $\langle \vec{u}, c \vec{v} \rangle = c \langle \vec{u}, \vec{v} \rangle$
- Additivity: $\langle \vec{u} + \vec{v}, \vec{w} \rangle = \langle \vec{u}, \vec{w} \rangle + \langle \vec{v}, \vec{w} \rangle$ and $\langle \vec{u}, \vec{v} + \vec{w} \rangle = \langle \vec{u}, \vec{v} \rangle + \langle \vec{u}, \vec{w} \rangle$
- Positive-definite: $\langle \vec{u}, \vec{u} \rangle \geq 0$ with $\langle \vec{u}, \vec{u} \rangle = 0$ if and only if $\vec{u} = \vec{0}$

For two vectors, $\vec{u}, \vec{v} \in \mathbb{R}^n$, the standard inner product is $\langle \vec{u}, \vec{v} \rangle = \vec{u}^T \vec{v}$. We define the norm, or the magnitude, of a vector $\vec{v}$ to be $\|\vec{v}\| = \sqrt{\langle \vec{v}, \vec{v} \rangle} = \sqrt{\vec{v}^T \vec{v}}$. For any non-zero vector, we can normalize, i.e., set its magnitude to 1 while preserving its direction, by dividing the vector by its norm $\frac{\vec{v}}{\|\vec{v}\|}$.

Orthogonality and Orthonormality

The inner product lets us define the angle between two vectors through the equation

$$\langle \vec{u}, \vec{v} \rangle = \|\vec{u}\| \|\vec{v}\| \cos \theta$$

(1)

Notice that if the angle $\theta$ between two vectors is $\pm 90^\circ$, the inner product $\langle \vec{u}, \vec{v} \rangle = 0$.

Therefore, we define two vectors $\vec{u}$ and $\vec{v}$ to be orthogonal to each other if $\langle \vec{u}, \vec{v} \rangle = \vec{u}^T \vec{v} = 0$. A set of vectors is orthogonal if any two vectors in this set are orthogonal to each other.

Furthermore, we define two vectors $\vec{u}$ and $\vec{v}$ to be orthonormal to each other if they are orthogonal to each other and their norms are 1. A set of vectors is orthonormal if any two vectors in this set are orthogonal to each other and every vector has a norm of 1. In fact, for any two vectors $\vec{u}$ and $\vec{v}$ in an orthonormal set,

$$\langle \vec{u}, \vec{v} \rangle = \vec{u}^T \vec{v} = \begin{cases} 1, & \text{if } \vec{u} = \vec{v} \\ 0, & \text{otherwise} \end{cases}$$

Unitary Matrices

An orthogonal or unitary matrix is a square matrix whose columns are orthonormal with respect to the inner product. To avoid any confusion, we will often refer to these matrices as orthonormal matrices.

$$U = \begin{bmatrix} \vec{u}_1 & \cdots & \vec{u}_n \end{bmatrix}, \quad \vec{u}_i^T \vec{u}_j = \begin{cases} 1, & \text{if } i = j \\ 0, & \text{otherwise} \end{cases}$$

Note that $U^T U = UU^T = I$, so the inverse of a unitary matrix is its transpose $U^{-1} = U^T$.

Since the columns of a unitary matrix are orthonormal vectors, we can interpret these matrices as “rotation” and “reflection” matrices of the standard axes. This also implies that $\|U\vec{v}\| = ||\vec{v}||$ for any vector $\vec{v}$. 

2 Spectral Theorem

Let $A$ be an $n \times n$ symmetric matrix with real entries. Then the following statements will be true.

1. All eigenvalues of $A$ are real.

2. $A$ has $n$ linearly independent eigenvectors $\in \mathbb{R}^n$.

3. $A$ has orthogonal eigenvectors, i.e., $A = V \Lambda V^{-1} = V \Lambda V^T$, where $\Lambda$ is a diagonal matrix and $V$ is an orthonormal matrix. We say that $A$ is orthogonally diagonalizable.

Recall that a matrix $A$ is symmetric if $A = A^T$. Furthermore, if $A$ is of the form $B^T B$ for some arbitrary matrix $B$, then all of the eigenvalues of $A$ are non-negative, i.e., $\lambda \geq 0$.

a) Prove the following: All eigenvalues of a symmetric matrix $A$ are real.

Hint: Let $(\lambda, \overline{v})$ be an eigenvalue/vector pair. Then $A \overline{v} = \lambda \overline{v}$ and take the complex conjugate and transpose of both sides. Try to show that $\overline{\lambda} = \lambda$.

b) Prove the following: For any symmetric matrix $A$, any two eigenvectors corresponding to distinct eigenvalues of $A$ are orthogonal.

Hint: Let $\overline{v}_1$ and $\overline{v}_2$ be eigenvectors of $A$ with eigenvalues $\lambda_1 \neq \lambda_2$.

\[
A \overline{v}_1 = \lambda_1 \overline{v}_1 \\
A \overline{v}_2 = \lambda_2 \overline{v}_2
\]

Take the transpose of the second equation and show that $\lambda_1 \langle \overline{v}_1, \overline{v}_2 \rangle = \lambda_2 \langle \overline{v}_1, \overline{v}_2 \rangle$. 

c) Prove the following: For any matrix $A$, $A^T A$ is symmetric and only has non-negative eigenvalues. 

*Hint:* Consider the quantity $\|A \vec{v}\|^2$. Remember that norms are positive-definite.
3 Outer Products

An outer product $\otimes$ is a function that takes two vectors and outputs a matrix. We define $\vec{x} \otimes \vec{y} = \vec{x} \vec{y}^T$.

a) Let $\vec{x} = \begin{bmatrix} 1 \\ 3 \\ -2 \end{bmatrix}$ and $\vec{y} = \begin{bmatrix} 4 \\ 2 \\ -1 \end{bmatrix}$.

(i) Compute the outer-product $A = \vec{x} \vec{y}^T$.
(ii) What is the shape of the matrix $A$?
(iii) What is the rank of $A$?

b) Let $B = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{bmatrix}$.

(i) Write $B$ as an outer-product of two vectors $\vec{x}$ and $\vec{y}$.
(ii) What is the rank of $B$?
c) Let \( C = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 1 & 1 & 1 & 1 \\ 1 & 0 & 0 & 0 \end{bmatrix} \).

(i) Write \( C \) as a sum of outer-products: \( \bar{x}\bar{y}^T + \bar{u}\bar{u}^T \).

(ii) What is the rank of \( C \)?

d) Let \( D = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \).

(i) Write \( D \) as a sum of outer-products.

(ii) What is the rank of \( D \)?