Rational Transfer Functions

When we write the transfer function of an arbitrary circuit, it always takes the following form. This is called a "rational transfer function." We also like to factor the numerator and denominator, so that they become easier to work with and plot:

$$H(\omega) = K\frac{(j\omega)^{N_{z0}}\left(1 + j\frac{\omega}{\omega_{z1}}\right)\left(1 + j\frac{\omega}{\omega_{z2}}\right)\cdots\left(1 + j\frac{\omega}{\omega_{zn}}\right)}{(j\omega)^{N_{p0}}\left(1 + j\frac{\omega}{\omega_{p1}}\right)\left(1 + j\frac{\omega}{\omega_{p2}}\right)\cdots\left(1 + j\frac{\omega}{\omega_{pm}}\right)} \tag{1}$$

Here, we define the constants ω_z as "zeros" and ω_p as "poles."

Bode Plots

Bode plots provide us with a simple and easy tool to plot these transfer functions by hand. Always remember that Bode plots are an approximation; if you want the precisely correct plots, you need to use numerical methods (like solving using MATLAB or iPython).

When we make Bode plots, we plot the frequency and magnitude on a logarithmic scale, and the angle in either degrees or radians. We use the logarithmic scale because it allows us to break up complex transfer functions into its constituent components.

For two transfer functions $H_1(\omega)$ and $H_2(\omega)$, if $H(\omega) = H_1(\omega) \cdot H_2(\omega)$,

$$\log\left|H(\omega)\right| = \log\left|H_1(\omega) \cdot H_2(\omega)\right| = \log\left|H_1(\omega)\right| + \log\left|H_2(\omega)\right| \tag{2}$$

$$\angle H(\omega) = \angle(H_1(\omega) \cdot H_2(\omega)) = \angle H_1(\omega) + \angle H_2(\omega) \tag{3}$$

Decibels (Old notation you will find scattered over internet)

We define the decibel as the following:

$$20\log_{10}(|H(\omega)|) = |H(\omega)|\ [\text{dB}]$$

This means that $20\,\text{dB}$ per decade is equivalent to one order of magnitude. A decade is defined as a change in ω by a factor of 10. For example ω going from 1 to 10 is a decade.

NOTE: We won't be using dB when plotting, but understanding the conversion to dB will help when reading the Bode plot sheet on the next page.

Algorithm

Given a frequency response $H(\omega)$, you can follow the following steps to sketch the magnitude plot:

a) Break $H(\omega)$ into a product of poles and zeros and put it in "rational transfer function" form as described in Equation 1.

b) Sketch the Bode plot for the poles and zeros in an increasing order. Start with the smallest pole/zero (this could be a pole or zero at origin). If you encounter a zero, the slope of you magnitude plot increases by the order of the zero. For example, if you encounter a zero of order 2 at $\omega = 100$, the slope increases by 2 from that frequency. On the other hand, if you encounter a pole, the slope decreases by the order of the pole.

c) Add the resulting plots to get the final Bode plot.

We will not focus on plotting phase plots. However, the algorithm is similar to the one described for magnitude plots here. An order 1 zero, for example, results in a slope increase of $\frac{\pi}{4}$ rad/decade in the phase of $H(\omega)$. These are also illustrated in Figure 1

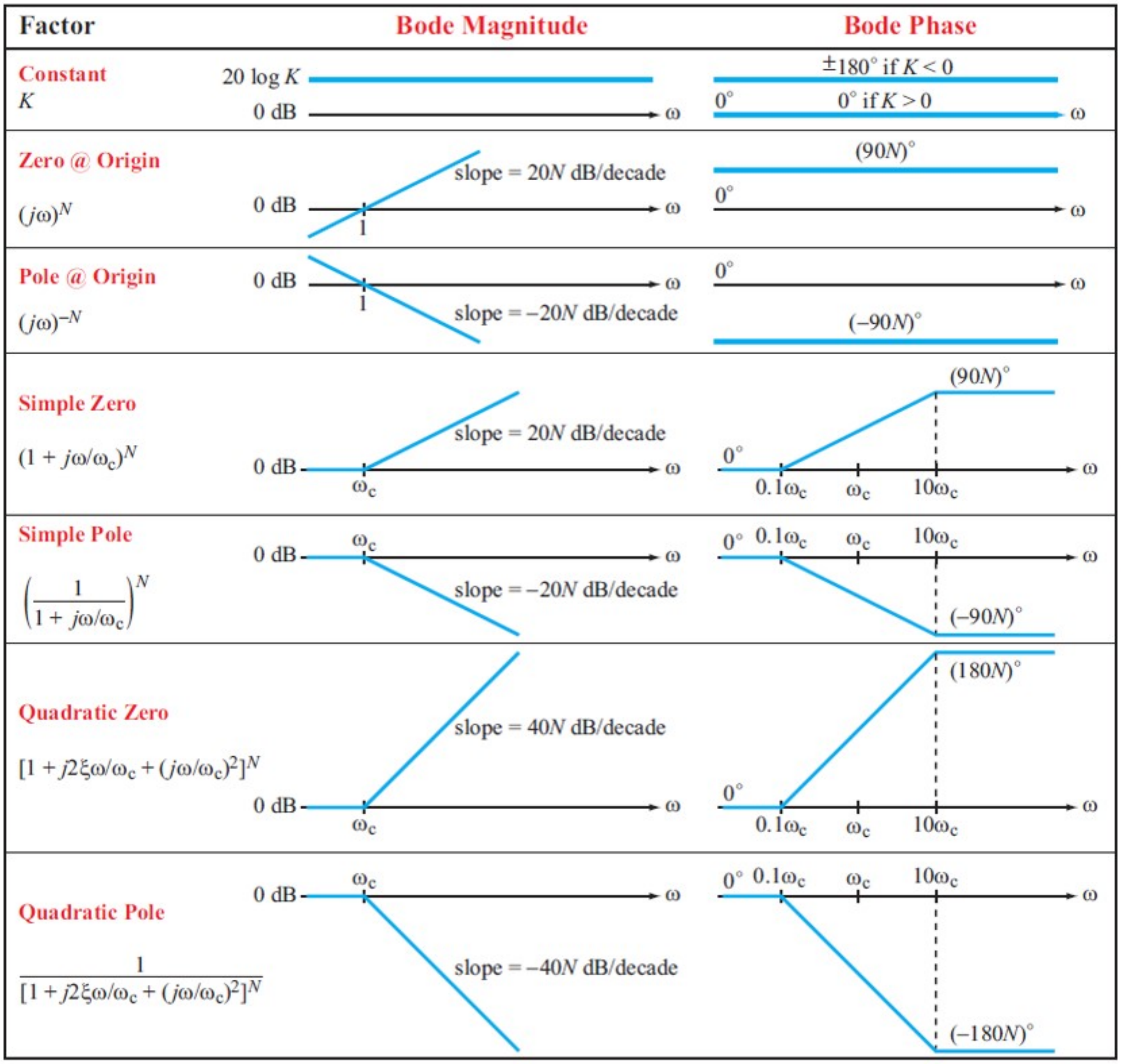


Figure 1: Reference for sketching Bode plots for transfer functions.

1 Log-Log plots

In this problem we will explore some of the ideas behind $\log - \log$ plots. Consider functions

$$f(x) = 100x^2\,,$$
$$g(x) = 10(1 + x^{10})$$

We will combine these functions in different ways to create a new function that we will call h(x). We will then analyze this newly created function. In particular, we are interested in the relationship between $\log_{10}(x)$ and $\log_{10} h(x)$ Let us assign a new name for $\log_{10}(x)$. From here, on we will call it $z \equiv \log_{10}(x)$.

Note that $\log_{10}(x)$ is well defined for $x > 0$ and we will restrict ourselves to x in that domain. The functions that we are looking at, are also strictly positive for $x > 0$.

a) $h(x) = f(x)$

Find a relationship between z and $H_1(z) \equiv \log_{10} h(x)$. What does this relationship resemble?

b) $h(x) = g(x)$

Again, find a relationship between z and $H_2(z) \equiv \log_{10} h(x)$.

Hint: You might find that is not easy to directly evaluate $\log_{10} h(x)$*. Try looking at different ranges of values of* x*. How can you approximate h(x) when* $x \ll 1$ *(say around* 10^{-5}*)? What will the approximation be when* $x \gg 1$ *(say around* 10^5*)?*

c) $h(x) = f(x) \cdot g(x)$

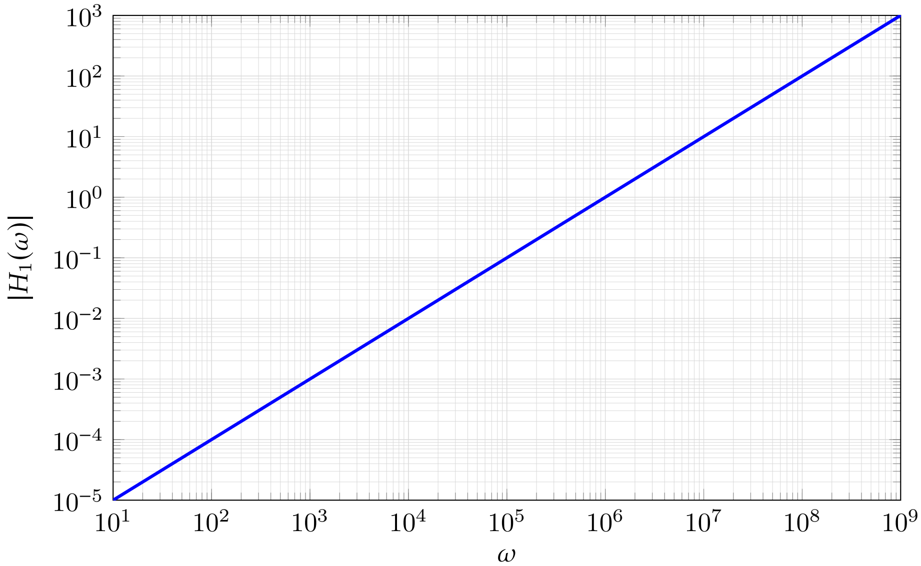
First, write out the expression for $H_3(z) \equiv \log_{10} h(x)$ in terms of $H_1(z)$ and $H_2(z)$ that we have defined previously. Now use the different domains we had used in the previous problem to write out $H_3(z)$ for $x \ll 1$ and $x \gg 1$.

d) $h(x) = \frac{f(x)}{g(x)}$

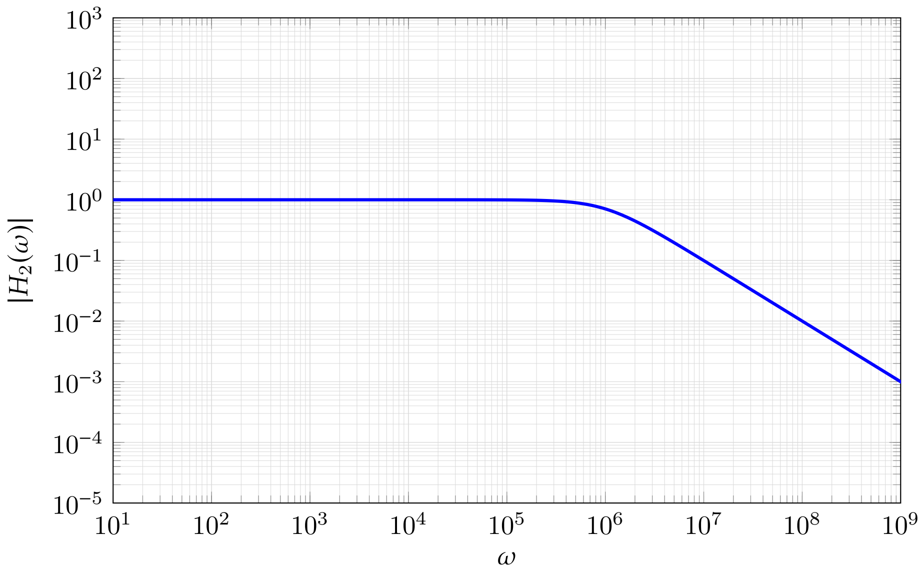
Write out the expression for $H_4(x) \equiv \log_{10} h(x)$ in terms of $H_1(z)$ and $H_2(z)$ that we have defined previously. How can we simplify these for the two domains $x \ll 1$ and $x \gg 1$.

2 Bode Plot Practice

a) **Identify the locations of the poles and zeroes in the following magnitude Bode plot. What transfer function $H_1(\omega)$ would result in this plot?**

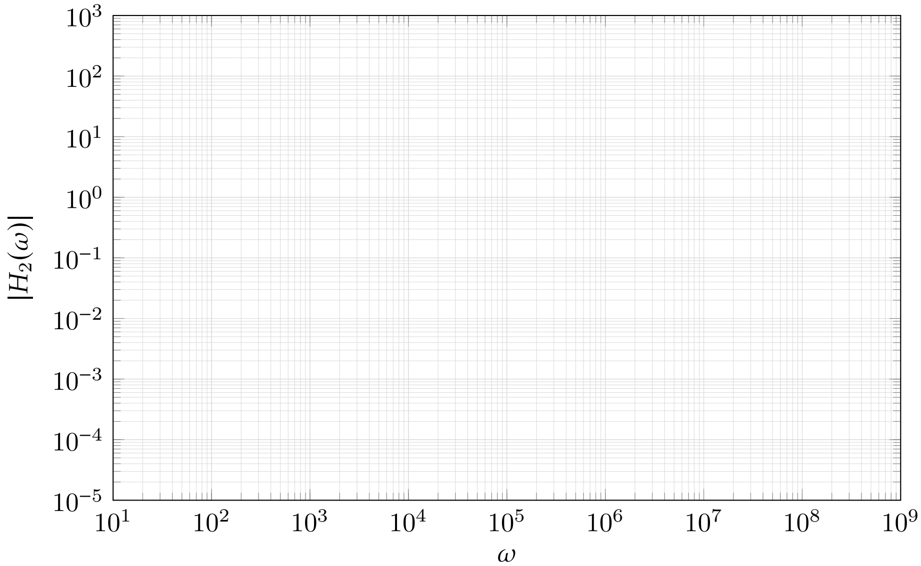


b) **Identify the locations of the poles and zeroes in the following magnitude Bode plot. What transfer function $H_2(\omega)$ would result in this plot?**



c) **Identify the locations of the poles and zeroes in the following transfer function. Then sketch the magnitude Bode plot.**

$$H_3(\omega) = \frac{\frac{j\omega}{10^6}}{1 + \frac{j\omega}{10^6}}$$



d) **Identify the locations of the poles and zeroes in the following magnitude Bode plot. Then sketch the magnitude Bode plot.**

$$H_4(\omega) = 100 \frac{(1 + \frac{j\omega}{10^7})^2}{(1 + \frac{j\omega}{10^4})(1 + \frac{j\omega}{5\times 10^4})}$$