1 Diagonalization

Consider an $n \times n$ matrix A that has n linearly independent eigenvalue/eigenvector pairs $(\lambda_1, \vec{v}_1), \ldots, (\lambda_n, \vec{v}_n)$ that can be put into a matrices V and Λ.

$$V = \begin{bmatrix} | & & | \\ \vec{v}_1 & \ldots & \vec{v}_n \\ | & & | \end{bmatrix} \quad \Lambda = \begin{bmatrix} \lambda_1 & & \\ & \ddots & \\ & & \lambda_n \end{bmatrix}$$

a) Show that $AV = V\Lambda$.

b) Use the fact in part (a) to conclude that $A = V\Lambda V^{-1}$.

2 Systems of Differential Equations

Consider a system of differential equations (valid for $t \geq 0$)

$$\frac{d}{dt}x_1(t) = -4x_1(t) + x_2(t) \tag{1}$$

$$\frac{d}{dt}x_2(t) = 2x_1(t) - 3x_2(t) \tag{2}$$

with initial conditions $x_1(0) = 3$ and $x_2(0) = 3$.

a) Write out the system of differential equations and initial conditions in the matrix/vector form

$$\frac{d}{dt}\vec{x}(t) = A\vec{x}(t) \tag{3}$$

b) Find the eigenvalues λ_1, λ_2 and eigenspaces for the differential matrix A.

c) Let us define a new variable $\vec{z} = V^{-1}\vec{x}$. Use the diagonalization of $A = V\Lambda V^{-1}$ to rewrite the original differential equation in terms of $z_i(t)$ and a diagonal matrix Λ.

$$\frac{d}{dt}\vec{z}(t) = \Lambda\vec{z}(t) \tag{4}$$

Remember to find the new initial conditions $z_1(0), z_2(0)$.

d) Solve the differential equation for $z_i(t)$.

e) Convert your solutions $z_i(t)$ back into the original variables to find the solution $x_i(t)$.

f) We can solve this equation using a slightly shorter approach by observing that the solutions for $x_i(t)$ will all be of the form

$$x_i(t) = \sum_k c_k e^{\lambda_k t}$$

where λ_k is an eigenvalue of our differential equation relation matrix A.

Since we have observed that the solutions will include $e^{\lambda_i t}$ terms, once we have found the eigenvalues for our differential equation matrix, we can guess the forms of the $x_i(t)$ as

$$\begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix} = \begin{bmatrix} \alpha_1 e^{\lambda_1 t} + \alpha_2 e^{\lambda_2 t} \\ \beta_1 e^{\lambda_1 t} + \beta_2 e^{\lambda_2 t} \end{bmatrix}$$

where $\alpha_1,\ \alpha_2,\ \beta_1,\ \beta_2$ are all constants.

Take the derivative to write out

$$\begin{bmatrix} \frac{d}{dt} x_1(t) \\ \frac{d}{dt} x_2(t) \end{bmatrix}.$$

and connect this to the given differential equation.

Solve for $x_i(t)$ from this form of the derivative.