1 RC Circuits

In this problem, we will be using differential equations to find the voltage across a capacitor over time in an RC circuit. We set up our problem by first defining three functions over time: \( I(t) \) is the current at time \( t \), \( V(t) \) is the voltage across the circuit at time \( t \), and \( V_C(t) \) is the voltage across the capacitor at time \( t \).

Recall from 16A that the voltage across a resistor is defined as \( V_R = R I_R \) where \( I_R \) is the current across the resistor. Also, recall that the voltage across a capacitor is defined as \( V_C = \frac{Q}{C} \) where \( Q \) is the charge across the capacitor.

![Figure 1: Example Circuit](image)

a) First, find an equation that relates the current through the capacitor \( I_C(t) \) with the voltage across the capacitor \( V_C(t) \).

b) Using nodal analysis, write a differential equation for the capacitor voltage \( V_C(t) \). Note that this is also the voltage for the node \( n_2 \).
c) Let’s suppose that at \( t = 0 \), the capacitor is charged to a voltage \( V_{DD} \) (\( V_C(0) = V_{DD} \)). Let’s also assume that \( V(t) = 0 \) for all \( t \geq 0 \).

![Figure 2: Circuit for part (d)](image)

Solve the differential equation for \( V_C(t) \) for \( t \geq 0 \).
d) Now, let’s suppose that we start with an uncharged capacitor \( V_C(0) = 0 \). We apply some constant voltage \( V(t) = V_{DD} \) across the circuit. Solve the differential equation for \( V_C(t) \) for \( t \geq 0 \).

![Circuit for part (e)](image)
2 Graphing RC Responses

Consider the following RC Circuit with a single resistor $R$, capacitor $C$, and voltage source $V(t)$.

![Example Circuit](image)

Figure 4: Example Circuit

a) Let’s suppose that at $t = 0$, the capacitor is charged to a voltage $V_{DD}$ ($V_c(0) = V_{DD}$) and that $V(t) = 0$ for all $t \geq 0$. Plot the response $V_c(t)$.

b) Now let’s suppose that at $t = 0$, the capacitor is uncharged ($V_c(0) = 0$) and that $V(t) = V_{DD}$ for all $t \geq 0$. Plot the response $V_c(t)$. 

To better understand our responses, we now define a **time constant** which is a measure of how long it takes for the capacitor to charge or discharge. Mathematically, we define $\tau$ as the time at which $V_C(\tau)$ is $\frac{1}{e} = 36.8\%$ away from its steady state value.

![Graph showing different values of capacitor voltage at different times, relative to $\tau$.](image)

**Figure 5**: Different values of capacitor voltage at different times, relative to $\tau$.

c) Suppose that $V_{DD} = 5\text{ V}$, $R = 100\text{ }\Omega$, and $C = 10\text{ }\mu\text{F}$. What is the time constant $\tau$ for this circuit?

d) Going back to part (b), on what order of magnitude of time (nanoseconds, milliseconds, 10’s of seconds, etc.) does this circuit settle ($V_c$ is $> 95\%$ of its value as $t \to \infty$)?
e) Give 2 ways to reduce the settling time of the circuit if we are allowed to change one component in the circuit.

f) Suppose we have a source $V(t)$ that alternates between 0 and $V_{DD} = 1$ V. Given $RC = 0.1$ s, plot the response $V_c$ if $V_c(0) = 0$.

![Time (s) vs Voltage (V) graph]

- Voltage (V): 0, 0.5, 1
- Time (s): 0, 1, 2, 3, 4

- 0 s: 0 V
- 1 s: 1 V
- 2 s: 0 V
- 3 s: 1 V
- 4 s: 0 V

g) Now suppose we have the same source $V(t)$ but $RC = 1$ s, plot the response $V_c$ if $V_c(0) = 0$.