

Discussion 12B

1. Computing the SVD: A “Tall” Matrix Example

Define the matrix

$$A = \begin{bmatrix} 1 & -1 \\ -2 & 2 \\ 2 & -2 \end{bmatrix}. \quad (1)$$

Here, we expect $U \in \mathbb{R}^{3 \times 3}$, $\Sigma \in \mathbb{R}^{3 \times 2}$, and $V \in \mathbb{R}^{2 \times 2}$ (recall that U and V must be square since they are orthonormal matrices).

In this problem, we will walk through the SVD algorithm, prove some important theorems about the SVD matrices and column/null spaces, and consider an alternate way to approach the SVD.

- (a) Let’s start by trying to write A as an outer product in the form of $\sigma \vec{u} \vec{v}^\top$ where both \vec{u} and \vec{v}^\top have unit norm. (HINT: Are the columns of A linearly independent or dependent? What does that tell us about how we can represent them?)

- (b) In this part, we will walk through Algorithm 7 in [Note 16](#). This algorithm applies for a general matrix $A \in \mathbb{R}^{m \times n}$.

- i. **Find $r := \text{rank}(A)$. Compute $A^\top A$ and diagonalize it using the spectral theorem (i.e. find V and Λ).**
- ii. **Unpack $V := \begin{bmatrix} V_r & V_{n-r} \end{bmatrix}$ and unpack $\Lambda := \begin{bmatrix} \Lambda_r & 0_{r \times (n-r)} \\ 0_{(n-r) \times r} & 0_{(n-r) \times (n-r)} \end{bmatrix}$.**
- iii. **Find $\Sigma_r := \Lambda_r^{1/2}$ and then find $\Sigma := \begin{bmatrix} \Sigma_r & 0_{r \times (n-r)} \\ 0_{(m-r) \times r} & 0_{(m-r) \times (n-r)} \end{bmatrix}$.**
- iv. **Find $U_r := AV_r \Sigma_r^{-1}$, where $U_r \in \mathbb{R}^{3 \times 1}$ and then extend the basis defined by columns of U_r to find $U \in \mathbb{R}^{3 \times 3}$.**

(HINT: How can we extend a basis, and why is that needed here?)

- v. Use the previous parts to write the full SVD of A .
- vi. If we were to calculate the SVD of our matrix using a calculator, are we *guaranteed* to always get the same SVD? Why or why not?

(c) We now want to create the SVD of A^T . **What are the relationships between the matrices composing A and the matrices composing A^T ?**

Contributors:

- Anish Muthali.
- Neelesh Ramachandran.
- Druv Pai.
- John Maidens.
- Nikhil Jain.
- Chancharik Mitra.