The following notes are useful for this discussion: Note 18.

1. **Jacobian and Linear Approximation**

Recall that for a scalar-valued function \( f(\vec{x}, \vec{y}) : \mathbb{R}^n \times \mathbb{R}^k \to \mathbb{R} \) with vector-valued arguments, we can linearize the function at \((\vec{x}_*, \vec{y}_*)\):

\[
\hat{f}(\vec{x}, \vec{y}) = f(\vec{x}_*, \vec{y}_*) + \sum_{i=1}^{n} \frac{\partial f(\vec{x}_*, \vec{y}_*)}{\partial x_i} (x_i - x_{i,*}) + \sum_{j=1}^{k} \frac{\partial f(\vec{x}_*, \vec{y}_*)}{\partial y_j} (y_j - y_{j,*}).
\]

(1)

In order to simplify this equation, we can define the following two vector quantities:

\[
J_x f = \begin{bmatrix} \frac{\partial f}{\partial x_1} & \cdots & \frac{\partial f}{\partial x_n} \end{bmatrix}
\]

(2)

\[
J_y f = \begin{bmatrix} \frac{\partial f}{\partial y_1} & \cdots & \frac{\partial f}{\partial y_k} \end{bmatrix}
\]

(3)

(a) When the function \( \hat{f}(\vec{x}, \vec{y}) : \mathbb{R}^n \times \mathbb{R}^k \to \mathbb{R}^m \) takes in vectors and outputs a vector (rather than a scalar), we can view each dimension in \( \hat{f} \) independently as a separate function \( f_i \), and linearize each of them as above:

\[
\hat{f}(\vec{x}, \vec{y}) = \begin{bmatrix} \hat{f}_1(\vec{x}, \vec{y}) \\ \hat{f}_2(\vec{x}, \vec{y}) \\ \vdots \\ \hat{f}_m(\vec{x}, \vec{y}) \end{bmatrix} = \begin{bmatrix} f_1(\vec{x}_*, \vec{y}_*) + J_x f_1 \cdot (\vec{x} - \vec{x}_*) + J_y f_1 \cdot (\vec{y} - \vec{y}_*) \\ f_2(\vec{x}_*, \vec{y}_*) + J_x f_2 \cdot (\vec{x} - \vec{x}_*) + J_y f_2 \cdot (\vec{y} - \vec{y}_*) \\ \vdots \\ f_m(\vec{x}_*, \vec{y}_*) + J_x f_m \cdot (\vec{x} - \vec{x}_*) + J_y f_m \cdot (\vec{y} - \vec{y}_*) \end{bmatrix}
\]

(4)

We can rewrite this in a clean way with the Jacobian of a vector-valued function:

\[
J_x \hat{f} = \begin{bmatrix} J_x f_1 \\ J_x f_2 \\ \vdots \\ J_x f_m \end{bmatrix} = \begin{bmatrix} \frac{\partial f_1}{\partial x_1} & \cdots & \frac{\partial f_1}{\partial x_n} \\ \vdots & \ddots & \vdots \\ \frac{\partial f_m}{\partial x_1} & \cdots & \frac{\partial f_m}{\partial x_n} \end{bmatrix},
\]

(5)

and similarly

\[
J_y \hat{f} = \begin{bmatrix} \frac{\partial f_1}{\partial y_1} & \cdots & \frac{\partial f_1}{\partial y_k} \\ \vdots & \ddots & \vdots \\ \frac{\partial f_m}{\partial y_1} & \cdots & \frac{\partial f_m}{\partial y_k} \end{bmatrix}.
\]

(6)

Then, the linearization becomes

\[
\hat{f}(\vec{x}, \vec{y}) = \hat{f}(\vec{x}_*, \vec{y}_*) + J_x \hat{f}(\vec{x}_*, \vec{y}_*) \cdot (\vec{x} - \vec{x}_*) + J_y \hat{f}(\vec{x}_*, \vec{y}_*) \cdot (\vec{y} - \vec{y}_*).
\]

(7)

Let \( \vec{x} = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \) and \( \hat{f}(\vec{x}) = \begin{bmatrix} x_1^2 x_2 \\ x_1 x_2^2 \end{bmatrix} \). Find \( J_x \hat{f} \), applying the definition above.
(b) Evaluate the approximation of $\vec{f}$ using $\vec{x}_* = \begin{bmatrix} 2 \\ 3 \end{bmatrix}$ at the point $\begin{bmatrix} 2.01 \\ 3.01 \end{bmatrix}$, and compare with $\vec{f}\left(\begin{bmatrix} 2.01 \\ 3.01 \end{bmatrix}\right)$.

Recall the definition that $\vec{f}(\vec{x}) = \begin{bmatrix} x_1^2 x_2 \\ x_1 x_2^2 \end{bmatrix}$.

(c) Let $\vec{x}$ and $\vec{y}$ be vectors with 2 rows, and let $\vec{w}$ be another vector with 2 rows. Let $\vec{f}(\vec{x}, \vec{y}) = \vec{x}\vec{y}^\top \vec{w}$.

Find $J_{\vec{x}}\vec{f}$ and $J_{\vec{y}}\vec{f}$.

(d) (PRACTICE) Continuing the above part, find the linear approximation of $\vec{f}$ near $\vec{x} = \vec{y} = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$ and with $\vec{w} = \begin{bmatrix} 2 \\ 1 \end{bmatrix}$.
2. Linearizing a Two-state System

We have a two-state nonlinear system defined by the following differential equation:

\[
\frac{d}{dt} \begin{bmatrix} \beta(t) \\ \gamma(t) \end{bmatrix} = \frac{d}{dt} \vec{x}(t) = \begin{bmatrix} -2\beta(t) + \gamma(t) \\ g(\gamma(t)) + u(t) \end{bmatrix} = \vec{f}(\vec{x}(t), u(t)) \tag{8}
\]

where \( \vec{x}(t) = \begin{bmatrix} \beta(t) \\ \gamma(t) \end{bmatrix} \) and \( g(\cdot) \) is a nonlinear function with the following graph:

![Graph of \( g(\gamma) \)]

The \( g(\cdot) \) is the only nonlinearity in this system. We want to linearize this entire system around a operating point/equilibrium. Any point \( \vec{x}_* \) is an operating point if \( \vec{f}(\vec{x}_*(t), u_*(t)) = \vec{0} \).

(a) If we have fixed \( u_*(t) = -1 \), what values of \( \gamma \) and \( \beta \) will ensure \( \frac{d}{dt} \vec{x}(t) = \vec{f}(\vec{x}(t), u(t)) = \vec{0} \)?

(b) Now that you have the three operating points, linearize the system about the operating point \( (\vec{x}_3^*, u_*) \) (that which has the largest value for \( \gamma \)). Specifically, what we want is as follows. Let \( \delta \vec{x}_i(t) = \vec{x}_i(t) - \vec{x}_i^* \) for \( i = 1, 2, 3 \), and \( \delta u(t) = u(t) - u_* \). We can in principle write the linearized system for each operating point in the following form:

\[
(\text{linearization about } (\vec{x}_i^*, u_*)) \quad \frac{d}{dt} \delta \vec{x}_i(t) = A_i \delta \vec{x}_i(t) + B_i \delta u(t) + \vec{w}_i(t) \tag{9}
\]

where \( \vec{w}_i(t) \) is a disturbance that also includes the approximation error due to linearization. For this part, find \( A_i \) and \( B_i \).

We have provided below the function \( g(\gamma) \) and its derivative \( \frac{dg}{d\gamma} \).
(c) Which of the operating points are stable? Which are unstable?

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