1 Polynomial Interpolation

Given n distinct points, we can find a unique degree $n-1$ polynomial that passes through these points. Let the polynomial p be

$$p(x) = a_0 + a_1 x + a_2 x^2 + \cdots + a_{n-1} x^{n-1}.$$

Let the n points be

$$p(x_1) = y_1, p(x_2) = y_2, \cdots, p(x_n) = y_n,$$

where $x_1 \neq x_2 \neq \cdots \neq x_n$.

We can construct a matrix-vector equation as follows to recover the polynomial p.

$$\underbrace{\begin{bmatrix} 1 & x_1 & x_1^2 & \dots & x_1^{n-1} \\ 1 & x_2 & x_2^2 & \dots & x_2^{n-1} \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ 1 & x_n & x_n^2 & \dots & x_n^{n-1} \end{bmatrix}}_{A} \underbrace{\begin{bmatrix} a_0 \\ a_1 \\ \vdots \\ a_{n-1} \end{bmatrix}}_{\vec{a}} = \underbrace{\begin{bmatrix} y_1 \\ y_2 \\ \vdots \\ y_n \end{bmatrix}}_{\vec{y}}$$

We can solve for the a values by setting:

$$\vec{a} = A^{-1}\vec{y}$$

Note that the matrix A is known as a Vandermonde matrix whose determinant is given by

$$\det(A) = \prod_{1 \leq i < j \leq n} (x_j - x_i)$$

Since $x_1 \neq x_2 \neq \cdots \neq x_n$, the determinant is non-zero and A is always invertible.

2 Polynomial Regression

Sometimes we may want to fit our data to a polynomial with an order less than $n-1$. If we fit the data to a polynomial of order $m < n$ we get:

$$p(x) = a_0 + a_1 x + a_2 x^2 + \cdots + a_{m-1} x^{m-1}$$

Now when we construct the matrix-vector equation to recover polynomial p, we get:

$$\underbrace{\begin{bmatrix} 1 & x_1 & x_1^2 & \dots & x_1^{m-1} \\ 1 & x_2 & x_2^2 & \dots & x_2^{m-1} \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ 1 & x_m & x_m^2 & \dots & x_m^{m-1} \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ 1 & x_n & x_n^2 & \dots & x_n^{m-1} \end{bmatrix}}_{A} \underbrace{\begin{bmatrix} a_0 \\ a_1 \\ \vdots \\ a_{m-1} \end{bmatrix}}_{\vec{a}} = \underbrace{\begin{bmatrix} y_1 \\ y_2 \\ \vdots \\ y_m \\ \vdots \\ y_n \end{bmatrix}}_{\vec{y}}$$

With this matrix equation, we have n equations with m unknowns, which means our system is over-defined (since $m < n$). One way to find the best fitting a values for this polynomial is to use least-squares, where you set:

$$\vec{a} = (A^T A)^{-1} A^T \vec{y}$$

3 Lagrange Interpolation

In practice, to approximate some unknown or complex function $f(x)$, we take n evaluations/samples of the function, denoted by $\{(x_i, y_i \triangleq f(x_i));\ 0 \le i \le n-1\}$. For the rest of this question, we will consider the following three points: $\{(0,3),(1,4),(3,-6)\}$.

a) **Using the interpolation method discussed above, find the matrix A such that $A\vec{a} = \vec{y}$.**

b) **What are the coefficients a_0, a_1, a_2?**

c) Observe that this system very quickly becomes frustrating to solve—as n increases, the difficulty of calculating the inverse increases far more quickly.

This is where *Lagrange interpolation* can be useful; the idea of Lagrange interpolation is that, instead of writing the polynomial in question in terms of $\{1, x, x^2\}$, we will write it in terms of $\{L_0(x), L_1(x), L_2(x)\}$, where each

$$L_i(x_j) = \begin{cases} 1 & \text{if } j = i \\ 0 & \text{if } j \ne i \end{cases}$$

With that, the problem reduces to finding these new coefficients b_0, b_1, b_2 of the function

$$f(x) = b_0 L_0(x) + b_1 L_1(x) + b_2 L_2(x)$$

such that $f(x_i) = y_i,\ \forall i = 0, 1, 2$. **What are these coefficients b_i?**

d) **Show that if we define**

$$L_i(x) = \prod_{j=0; j\ne i}^{n-1} \frac{(x - x_j)}{(x_i - x_j)}$$

then the condition requested from part (c) is satisfied.

e) Based on the previous two parts, write down the explicit form of $f(x)$ that passes through the samples $\{(0,3),(1,4),(3,-6)\}$ in terms of x as opposed to $L_i(x)$. The resulting formula is the so called Lagrange polynomial which passes through the n sampled points. Does this agree with the previous method?

f) Now, suppose instead we wanted to use regression to fit our 3 points to a linear system $f(x) = a_0 + a_1 x$. What are the best-fit coefficients a_0 and a_1 in this situation?