Discussion 9B

1. Uncontrollability

Recall that, for a n-dimensional, discrete-time linear system to be controllable, we require that the controllability matrix $\mathcal{C} = \begin{bmatrix} A^{n-1}B & A^{n-2}B & \dots & AB & B \end{bmatrix}$ to be rank n.

Consider the following discrete-time system with the given initial state:

$$\vec{x}[i+1] = \begin{bmatrix} 2 & 0 & 0 \\ -3 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix} \vec{x}[i] + \begin{bmatrix} 0 \\ 0 \\ 2 \end{bmatrix} u[i] \tag{1}$$

$$\vec{x}[0] = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} \tag{2}$$

(a) **Is the system controllable?**

Solution:

$$\mathcal{C} = \begin{bmatrix} A^2B & AB & B \end{bmatrix} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 2 & 0 \\ 2 & 0 & 2 \end{bmatrix} \tag{3}$$

Since the controllability matrix $\mathcal{C}$ only has rank 2, the system is not controllable. We would need it to be rank 3 here to span the full space $\mathbb{R}^3$.

(b) **Show that we can write the ith state as**

$$\vec{x}[i] = \begin{bmatrix} 2^i \\ -3x_1[i-1] + x_3[i-1] \\ x_2[i-1] + 2u[i-1] \end{bmatrix} \tag{4}$$

Is it possible to reach $\vec{x}[\ell] = \begin{bmatrix} -2 \\ 4 \\ 6 \end{bmatrix}$ for some ℓ? If so, for what input sequence $u[i]$ up to $i = \ell - 1$?

Solution: We can write:

$$\vec{x}[i] = \begin{bmatrix} 2 & 0 & 0 \\ -3 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix} \vec{x}[i-1] + \begin{bmatrix} 0 \\ 0 \\ 2 \end{bmatrix} u[i-1] \tag{5}$$

$$= \begin{bmatrix} 2 & 0 & 0 \\ -3 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix} \begin{bmatrix} x_1[i-1] \\ x_2[i-1] \\ x_3[i-1] \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ 2 \end{bmatrix} u[i-1] \tag{6}$$

$$= \begin{bmatrix} 2x_1[i-1] \\ -3x_1[i-1] + x_3[i-1] \\ x_2[i-1] + 2u[i-1] \end{bmatrix} \tag{7}$$

$$= \begin{bmatrix} 2^i x_1[0] \\ -3x_1[i-1] + x_3[i-1] \\ x_2[i-1] + 2u[i-1] \end{bmatrix} \quad (8)$$

$$= \begin{bmatrix} 2^i \\ -3x_1[i-1] + x_3[i-1] \\ x_2[i-1] + 2u[i-1] \end{bmatrix} \quad (9)$$

Note that in this expression for $\vec{x}[i]$, $x_1[i] = 2^i$ is decoupled from all other states and inputs. From this expression we also see that there is no choice of inputs us to get to $x_1[\ell] = -2$. Therefore, we will never be able to reach $\vec{x}[\ell] = \begin{bmatrix} -2 \\ 4 \\ 6 \end{bmatrix}$ for any ℓ.

(c) **Is it possible to reach $\vec{x}[\ell] = \begin{bmatrix} 2 \\ -3 \\ -2 \end{bmatrix}$ for some ℓ? For what input sequence $u[i]$ for $i = 0$ to $i = \ell - 1$?**

HINT: Use the result for $\vec{x}[i]$ from the previous part.

Solution: We need $\ell = 1$ since $x_1[i] = 2^i x_1[0] = 2^i$. Hence,

$$\vec{x}[1] = \begin{bmatrix} 2 \\ -3 \\ 2u[0] \end{bmatrix} \quad (10)$$

We realize that the first two entries of $\vec{x}[1]$ are exactly what we want. Thus, we have to choose $u[0]$ so the third entry is -2. If we choose $u[0] = -1$, then we have reached our desired state.

Thus we see that a system being uncontrollable does not mean we are unable to reach anything at all, but just that the range that can be reached is limited.

(d) **Find the set of all $\vec{x}[2]$, given that you are free to choose any $u[0]$ and $u[1]$.**

Solution:

$$\vec{x}[1] = \begin{bmatrix} 2 \\ -3 \\ 2u[0] \end{bmatrix} \quad (11)$$

$$\vec{x}[2] = \begin{bmatrix} 2 & 0 & 0 \\ -3 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix} \begin{bmatrix} 2 \\ -3 \\ 2u[0] \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ 2 \end{bmatrix} u[1] \quad (12)$$

$$= \begin{bmatrix} 4 \\ -6 + 2u[0] \\ -3 + 2u[1] \end{bmatrix} \quad (13)$$

Since we can set $u[0]$ and $u[1]$ arbitrarily, we can reach any state of the form $\begin{bmatrix} 4 \\ 0 \\ 0 \end{bmatrix} + \text{span}\left(\begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \right)$ after two timesteps. This means that we can reach any value for $\vec{x}[2]$; contrast this with how the first component of the state vector is fixed at 4 after two timesteps, and cannot be changed by the inputs.

Alternative Solution:

Notice that we can write

$$\vec{x}[2] = A^2\vec{x}[0] + ABu[0] + Bu[1] = \begin{bmatrix} 4 \\ 0 \\ 0 \end{bmatrix} + \begin{bmatrix} B & AB \end{bmatrix} \begin{bmatrix} u[1] \\ u[0] \end{bmatrix} \tag{14}$$

Hence, $\vec{x}[2]$ will be $\begin{bmatrix} 4 \\ 0 \\ 0 \end{bmatrix}$ plus whatever is in the column space of

$$\begin{bmatrix} B & AB \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 2 & 0 \\ 0 & 2 \end{bmatrix} \tag{15}$$

This gives the same answer as before, i.e. that

$$\vec{x}[2] = \begin{bmatrix} 4 \\ 0 \\ 0 \end{bmatrix} + \vec{p} \tag{16}$$

where $\vec{p} \in \text{span}\left(\begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}\right)$.

Any desired $x_2[i]$ and $x_3[i]$ that we can possibly reach can be obtained in only two or fewer timesteps. Hence, every reachable state can be written as

$$\vec{x}[i] = \begin{bmatrix} 2^i \\ 0 \\ 0 \end{bmatrix} + \vec{p} \tag{17}$$

with $\vec{p}$ defined as above. This will also tell us why the desired goal in part **1.**b is unreachable.

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