Discussion 9A

1. Eigenvalue Placement in Discrete Time

Recall that, for a discrete linear system to be stable, we require that all of the eigenvalues of A in $\vec{x}[i+1] = A\vec{x}[i] + B\vec{u}[i]$ must have magnitude less than 1.

Consider the following linear discrete time system

$$\vec{x}[i+1] = \begin{bmatrix} 0 & 1 \\ 2 & -1 \end{bmatrix} \vec{x}[i] + \begin{bmatrix} 1 \\ 0 \end{bmatrix} u[i] + \vec{w}[i] \tag{1}$$

(a) **Is the system given in eq. (1) stable?**

Solution: For notation's sake, let's write the system in the familiar form

$$\vec{x}[i+1] = A\vec{x}[i] + \vec{b}u[i] + \vec{w}[i] \tag{2}$$

where

$$A = \begin{bmatrix} 0 & 1 \\ 2 & -1 \end{bmatrix} \qquad \vec{b} = \begin{bmatrix} 1 \\ 0 \end{bmatrix} \tag{3}$$

We have to calculate the eigenvalues of matrix A. Doing so, we find:

$$\det(A - \lambda I) = 0 \implies \lambda_1 = 1, \lambda_2 = -2 \tag{4}$$

Since there exists a λ such that $|\lambda| \geq 1$ (in fact, both λ_1 and λ_2 satisfy this inequality), the system is unstable.

(b) We can attempt to stabilize the system by implementing closed loop feedback. That is, we choose our input $u[i]$ so that the system is stable. **If we were to use state feedback as in eq. (5), what is an equivalent representation for this system? Write your answer as $\vec{x}[i+1] = A_{\text{CL}}\vec{x}[i]$ for some matrix A_{CL}.**

$$u[i] = \begin{bmatrix} f_1 & f_2 \end{bmatrix} \vec{x}[i] \tag{5}$$

HINT: If you're having trouble parsing the expression for $u[i]$, note that $\begin{bmatrix} f_1 & f_2 \end{bmatrix}$ is a row vector, while $\vec{x}[i]$ is column vector. What happens when we multiply a row vector with a column vector like this?)

Solution: The closed loop system using state feedback has the form

$$\vec{x}[i+1] = \begin{bmatrix} 0 & 1 \\ 2 & -1 \end{bmatrix} \vec{x}[i] + \begin{bmatrix} 1 \\ 0 \end{bmatrix} u[i] \tag{6}$$

$$= \begin{bmatrix} 0 & 1 \\ 2 & -1 \end{bmatrix} \vec{x}[i] + \begin{bmatrix} 1 \\ 0 \end{bmatrix} \left(\begin{bmatrix} f_1 & f_2 \end{bmatrix} \vec{x}[i] \right) \tag{7}$$

$$= \left(\begin{bmatrix} 0 & 1 \\ 2 & -1 \end{bmatrix} + \begin{bmatrix} 1 \\ 0 \end{bmatrix} \begin{bmatrix} f_1 & f_2 \end{bmatrix} \right) \vec{x}[i] \tag{8}$$

$$= \left(\begin{bmatrix} 0 & 1 \\ 2 & -1 \end{bmatrix} + \begin{bmatrix} f_1 & f_2 \\ 0 & 0 \end{bmatrix} \right) \vec{x}[i] \tag{9}$$

$$= \underbrace{\begin{bmatrix} f_1 & 1+f_2 \\ 2 & -1 \end{bmatrix}}_{A_{\mathrm{CL}}} \vec{x}[i]. \tag{10}$$

(c) **Find the appropriate state feedback constants, f_1, f_2, that place the eigenvalues of the state space representation matrix at $\lambda_1 = -\frac{1}{2}, \lambda_2 = \frac{1}{2}$.**

Solution: From the previous part we have computed the closed loop system as

$$\vec{x}[i+1] = \underbrace{\begin{bmatrix} f_1 & 1+f_2 \\ 2 & -1 \end{bmatrix}}_{A_{\mathrm{CL}}} \vec{x}[i] \tag{11}$$

Thus, finding the eigenvalues of the above system we have

$$0 = \det(A - \lambda I) \tag{12}$$

$$= \det\left(\begin{bmatrix} f_1 - \lambda & 1+f_2 \\ 2 & -1-\lambda \end{bmatrix}\right) \tag{13}$$

$$= \lambda^2 + (1 - f_1)\lambda + (-f_1 - 2f_2 - 2) \tag{14}$$

We want to place the eigenvalues at $\lambda_1 = -\frac{1}{2}$ and $\lambda_2 = \frac{1}{2}$. This means that we should choose the constants f_1 and f_2 so that the characteristic equation is

$$0 = \left(\lambda - \frac{1}{2}\right)\left(\lambda + \frac{1}{2}\right) = \lambda^2 - \frac{1}{4} = \lambda^2 + 0\lambda - \frac{1}{4} \tag{15}$$

Thus, we can match the coefficients of λ in the polynomial above, which indicates we should choose f_1 and f_2 satisfying the following system of equations:

$$0 = 1 - f_1 \tag{16}$$

$$-\frac{1}{4} = -f_1 - 2f_2 - 2 \tag{17}$$

We can solve this two variable, two equation system and find that $f_1 = 1$, $f_2 = -\frac{11}{8}$.

Alternatively, we know what the eigenvalues are; we can plug in each λ into characteristic polynomial, and doing so will yield the same system of equations in f_1, f_2.

(d) **Is the system now stable in closed-loop, using the control feedback coefficients f_1, f_2 that we derived above?**

Solution: Yes, the closed loop system has eigenvalues $\lambda_1 = -\frac{1}{2}, \lambda_2 = \frac{1}{2}$, which means that A_{CL} satisfies our condition that all of its eigenvalues have magnitude less than 1.

(e) Suppose that instead of $\begin{bmatrix} 1 \\ 0 \end{bmatrix} u[i]$ in eq. (1), we had $\begin{bmatrix} 1 \\ 1 \end{bmatrix} u[i]$ as the way that the discrete-time control acted on the system. In other words, the system is as given in eq. (18).

$$\vec{x}[i+1] = \begin{bmatrix} 0 & 1 \\ 2 & -1 \end{bmatrix} \vec{x}[i] + \begin{bmatrix} 1 \\ 1 \end{bmatrix} u[i] \tag{18}$$

Determine whether the system is controllable or not.

Solution:

$$\mathcal{C} = \begin{bmatrix} A\vec{b} & \vec{b} \end{bmatrix} \tag{19}$$

$$A\vec{b} = \begin{bmatrix} 0 & 1 \\ 2 & -1 \end{bmatrix} \begin{bmatrix} 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \end{bmatrix} \tag{20}$$

$$\mathcal{C} = \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix} \tag{21}$$

As we can see, the controllability matrix is not full rank and therefore the system is not controllable.

(f) Let's say we still try and apply closed loop feedback to our system. Let's use $u[i] = \begin{bmatrix} f_1 & f_2 \end{bmatrix} \vec{x}[i]$ to try and control the system. **Show that the resulting closed-loop state space matrix is**

$$A_{\text{CL}} = \begin{bmatrix} f_1 & f_2+1 \\ f_1+2 & f_2-1 \end{bmatrix} \tag{22}$$

Is it possible to stabilize this system?

Solution:

$$\begin{aligned} \vec{x}[i+1] &= \left(\begin{bmatrix} 0 & 1 \\ 2 & -1 \end{bmatrix} + \begin{bmatrix} 1 \\ 1 \end{bmatrix} \begin{bmatrix} f_1 & f_2 \end{bmatrix} \right) \vec{x}[i] && (23) \\ &= \left(\begin{bmatrix} 0 & 1 \\ 2 & -1 \end{bmatrix} + \begin{bmatrix} f_1 & f_2 \\ f_1 & f_2 \end{bmatrix} \right) \vec{x}[i] && (24) \\ &= \underbrace{\begin{bmatrix} f_1 & f_2+1 \\ f_1+2 & f_2-1 \end{bmatrix}}_{A_{\text{CL}}} \vec{x}[i] && (25) \end{aligned}$$

Finding the eigenvalues λ:

$$\begin{aligned} 0 &= \det\left(\begin{bmatrix} f_1-\lambda & f_2+1 \\ f_1+2 & f_2-1-\lambda \end{bmatrix} \right) && (26) \\ &= (f_1-\lambda)(f_2-1-\lambda) - (f_1+2)(f_2+1) && (27) \\ &= f_1(f_2-1) - f_1\lambda - \lambda(f_2-1) + \lambda^2 - (f_1f_2 + f_1 + 2f_2 + 2) && (28) \\ &= f_1f_2 - f_1 - f_1\lambda - \lambda f_2 + \lambda + \lambda^2 - f_1f_2 - f_1 - 2f_2 - 2 && (29) \\ &= \lambda^2 + (1-f_1-f_2)\lambda - 2(1+f_1+f_2) && (30) \\ &= (\lambda+2)(\lambda-(1+f_1+f_2)) && (31) \end{aligned}$$

We can see that the eigenvalue at $\lambda = -2$ cannot be moved, so we cannot arbitrarily change our eigenvalues with this control input. Since there will always be an eigenvalue with $|\lambda| \geq 1$, then we cannot stabilize the system.

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