1. Eigenvalue Placement in Discrete Time

Recall that, for a discrete linear system to be stable, we require that all of the eigenvalues of $A$ in $\vec{x}[i + 1] = A\vec{x}[i] + B\vec{u}[i]$ must have magnitude less than 1.

Consider the following linear discrete time system

$$\vec{x}[i + 1] = \begin{bmatrix} 0 & 1 \\ 2 & -1 \end{bmatrix} \vec{x}[i] + \begin{bmatrix} 1 \\ 0 \end{bmatrix} \vec{u}[i] + \vec{w}[i]$$

(a) Is the system given in eq. (1) stable?

(b) We can attempt to stabilize the system by implementing closed loop feedback. That is, we choose our input $\vec{u}[i]$ so that the system is stable. If we were to use state feedback as in eq. (2), what is an equivalent representation for this system? Write your answer as $\vec{x}[i + 1] = A_{CL}\vec{x}[i]$ for some matrix $A_{CL}$.

$$\vec{u}[i] = \begin{bmatrix} f_1 & f_2 \end{bmatrix} \vec{x}[i]$$

**HINT:** If you’re having trouble parsing the expression for $\vec{u}[i]$, note that $\begin{bmatrix} f_1 & f_2 \end{bmatrix}$ is a row vector, while $\vec{x}[i]$ is column vector. What happens when we multiply a row vector with a column vector like this?
(c) Find the appropriate state feedback constants, $f_1, f_2$, that place the eigenvalues of the state space representation matrix at $\lambda_1 = -\frac{1}{2}, \lambda_2 = \frac{1}{2}$.

(d) Is the system now stable in closed-loop, using the control feedback coefficients $f_1, f_2$ that we derived above?

(e) Suppose that instead of $1 0 \# u[i] \[i\]$ in eq. (1), we had $1 1 \# u[i] \[i\]$ as the way that the discrete-time control acted on the system. In other words, the system is as given in eq. (3).

$$\vec{x}[i + 1] = \begin{bmatrix} 0 & 1 \\ 2 & -1 \end{bmatrix} \vec{x}[i] + \begin{bmatrix} 1 \\ 1 \end{bmatrix} u[i]$$

Determine whether the system is controllable or not.
(f) Let’s say we still try and apply closed loop feedback to our system. Let’s use $u[i] = \begin{bmatrix} f_1 & f_2 \end{bmatrix} \vec{x}[i]$ to try and control the system. **Show that the resulting closed-loop state space matrix is**

$$A_{CL} = \begin{bmatrix} f_1 & f_2 + 1 \\ f_1 + 2 & f_2 - 1 \end{bmatrix}$$

(4)

**Is it possible to stabilize this system?**

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