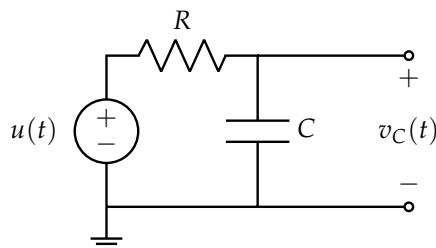


1. Stability Examples and Counterexamples

- (a) Consider the circuit below with $R = 1 \Omega$, $C = 0.5 F$, and $u(t)$ is some function bounded between $-K$ and K for some constant $K \in \mathbb{R}$ (for example $K \cos(t)$). Furthermore assume that $v_C(0) = 0 V$ (that the capacitor is initially discharged).



This circuit can be modeled by the differential equation

$$\frac{dv_C(t)}{dt} = -2v_C(t) + 2u(t) \tag{1}$$

Show that the differential equation is always stable (that is, as long as the input $u(t)$ is bounded, $v_C(t)$ also stays bounded). Consider what this means in the physical circuit. *HINT: You may want to use the triangle inequality, i.e. $|a + b| \leq |a| + |b|$, and the triangle inequality for integrals, i.e. $\left| \int_a^b f(x) dx \right| \leq \int_a^b |f(x)| dx$. When we use $|\cdot|$ notation here, we will take this to mean the magnitude, rather than the absolute value (since we can be dealing with complex numbers).*

Solution: We can apply the integral solution for a nonhomogeneous differential equation to demonstrate boundedness of the solution. The general solution to $\frac{dx(t)}{dt} = \lambda x(t) + bu(t)$ is $x(t) = x_0 e^{\lambda t} + \int_0^t e^{\lambda(t-\theta)} bu(\theta) d\theta$. Here, we can say that:

$$v_C(t) = v_C(0)e^{-2t} + \int_0^t e^{-2(t-\theta)} 2u(\theta) d\theta \tag{2}$$

$$= v_C(0)e^{-2t} + 2 \int_0^t e^{-2(t-\theta)} u(\theta) d\theta \tag{3}$$

We wish to show $|v_C(t)| \leq M$ for all $t \geq 0$, where $M \in \mathbb{R}$ is some constant (this is another way to say that something is “bounded”). We can take the absolute value around eq. (3) as follows:

$$|v_C(t)| = \left| v_C(0)e^{-2t} + 2 \int_0^t e^{-2(t-\theta)} u(\theta) d\theta \right| \tag{4}$$

$$\leq \left| v_C(0)e^{-2t} \right| + \left| 2 \int_0^t e^{-2(t-\theta)} u(\theta) d\theta \right| \tag{5}$$

$$\leq \left| v_C(0)e^{-2t} \right| + 2 \int_0^t \left| e^{-2(t-\theta)} u(\theta) \right| d\theta \tag{6}$$

$$= |v_C(0)|e^{-2t} + 2 \int_0^t e^{-2(t-\theta)} |u(\theta)| d\theta \tag{7}$$

where we use the traditional triangle inequality to obtain eq. (5) and the integral triangle inequality to obtain eq. (6). We know $v_C(0) = 0$, so the first term is 0. Even if it is nonzero, we

may assume that it is some finite constant. Furthermore, $0 \leq e^{-2t} \leq 1$ for $t \geq 0$ (it is a decaying exponential). Hence, the $|v_C(0)|e^{-2t}$ term is bounded. Next, we are allowed to assume that $|u(t)| \leq K$ from the statement of the problem. This will let us obtain

$$|v_C(t)| \leq 2 \int_0^t e^{-2(t-\theta)} \underbrace{|u(\theta)|}_{\leq K} d\theta \quad (8)$$

$$\leq 2K \int_0^t e^{-2(t-\theta)} d\theta \quad (9)$$

$$= K(1 - e^{-2t}) \quad (10)$$

Because $e^{-2t} \geq 0$, $1 - e^{-2t} \leq 1$. Hence, $|v_C(t)| \leq K$ so $v_C(t)$ is bounded.

- (b) **(PRACTICE)** Now, suppose that in the circuit of part 1.a we replaced the resistor with an inductor as in fig. 1.

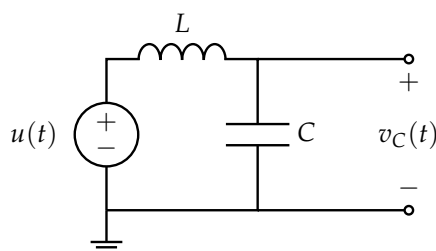


Figure 1: The original circuit with an inductor in place of the resistor.

Let $L = 1$ mH. Repeat part 1.a for the new circuit (with an inductor). Consider the following process to arrive at the result:

- i. Derive the system of differential equations using KCL, KVL, and NVA. Show that the system is $\frac{d}{dt} \begin{bmatrix} v_C(t) \\ i_L(t) \end{bmatrix} = \begin{bmatrix} 0 & \frac{1}{C} \\ -\frac{1}{L} & 0 \end{bmatrix} \begin{bmatrix} v_C(t) \\ i_L(t) \end{bmatrix} + \begin{bmatrix} 0 \\ \frac{1}{L} \end{bmatrix} u(t)$ with the initial condition being $\begin{bmatrix} v_C(0) \\ i_L(0) \end{bmatrix} = \vec{0}$.
- ii. Solve the matrix differential equation, using diagonalization if needed. Show that the diagonalized system has a solution

$$\vec{y}(t) = \begin{bmatrix} \frac{1}{2LC} e^{j\frac{1}{\sqrt{LC}}t} \int_0^t e^{-j\frac{1}{\sqrt{LC}}\theta} u(\theta) d\theta \\ \frac{1}{2LC} e^{-j\frac{1}{\sqrt{LC}}t} \int_0^t e^{j\frac{1}{\sqrt{LC}}\theta} u(\theta) d\theta \end{bmatrix} \quad (11)$$

where $\vec{y}(t) = V^{-1} \begin{bmatrix} v_C(t) \\ i_L(t) \end{bmatrix}$ for change of basis matrix V . You may use the fact that the

eigenvalue, eigenvector pairs of $\begin{bmatrix} 0 & \frac{1}{C} \\ -\frac{1}{L} & 0 \end{bmatrix}$ are $\left(j\frac{1}{\sqrt{LC}}, \begin{bmatrix} -j\sqrt{\frac{L}{C}} \\ 1 \end{bmatrix} \right)$ and $\left(-j\frac{1}{\sqrt{LC}}, \begin{bmatrix} j\sqrt{\frac{L}{C}} \\ 1 \end{bmatrix} \right)$.

- iii. Apply a similar process from part 1.a to show that, if we have a bounded input $u(t)$, then the system can grow unboundedly. When showing that a system is unstable, it suffices to choose a bounded $u(t)$ that makes the system unbounded. We can choose $u(t) =$

$$2 \cos\left(\frac{1}{\sqrt{LC}}t\right) = e^{j\frac{1}{\sqrt{LC}}t} + e^{-j\frac{1}{\sqrt{LC}}t} \text{ }^1. \text{ HINT: You may use the fact that } i_L(t) = y_1(t) + y_2(t).$$

Hint: You might find it useful to revisit the process of generating the state-space equations for $v_C(t)$ and $i_L(t)$ as done in Note 4 for the LC Tank. The difference is that here, we have an input voltage.

Solution: 1.(b)i:

First, we begin forming the vector state-space equation, which involves relating $v_C(t)$ and $i_L(t)$ to their derivatives and the input voltage.

$$C \frac{dv_C(t)}{dt} = i_C(t) = i_L(t) \quad (12)$$

$$\Rightarrow \frac{dv_C(t)}{dt} = \frac{1}{C} i_L(t) \quad (13)$$

$$L \frac{di_L(t)}{dt} = v_L(t) = u(t) - v_C(t) \quad (14)$$

$$\Rightarrow \frac{di_L(t)}{dt} = \frac{1}{L} v_L(t) = -\frac{1}{L} v_C(t) + \frac{1}{L} u(t) \quad (15)$$

Combining this info, we find:

$$\frac{d}{dt} \begin{bmatrix} v_C(t) \\ i_L(t) \end{bmatrix} = \underbrace{\begin{bmatrix} 0 & \frac{1}{C} \\ -\frac{1}{L} & 0 \end{bmatrix}}_A \underbrace{\begin{bmatrix} v_C(t) \\ i_L(t) \end{bmatrix}}_{\vec{x}(t)} + \underbrace{\begin{bmatrix} 0 \\ \frac{1}{L} \end{bmatrix}}_{\vec{b}} u(t) \quad (16)$$

1.(b)ii:

This is not a diagonal system, so we have to diagonalize it first. We start by solving for the eigenvalues and eigenvectors of A :

$$\lambda_1 = j\frac{1}{\sqrt{LC}} \quad \vec{v}_1 = \begin{bmatrix} -j\sqrt{\frac{L}{C}} \\ 1 \end{bmatrix} \quad (17)$$

$$\lambda_2 = -j\frac{1}{\sqrt{LC}} \quad \vec{v}_2 = \begin{bmatrix} j\sqrt{\frac{L}{C}} \\ 1 \end{bmatrix} \quad (18)$$

Note that these eigenvalues are purely imaginary. This will be helpful later. Our change of basis matrix is $V = \begin{bmatrix} -j\sqrt{\frac{L}{C}} & j\sqrt{\frac{L}{C}} \\ 1 & 1 \end{bmatrix}$, so we can define our change of basis as $\vec{y}(t) = V^{-1}\vec{x}(t)$. Note that the new diagonal system will be

$$\frac{d}{dt} \vec{y}(t) = \begin{bmatrix} j\frac{1}{\sqrt{LC}} & 0 \\ 0 & -j\frac{1}{\sqrt{LC}} \end{bmatrix} \vec{y}(t) + V^{-1}\vec{b}u(t) \quad (19)$$

$$= \begin{bmatrix} j\frac{1}{\sqrt{LC}} & 0 \\ 0 & -j\frac{1}{\sqrt{LC}} \end{bmatrix} \vec{y}(t) + \left(\begin{bmatrix} -j\sqrt{\frac{L}{C}} & j\sqrt{\frac{L}{C}} \\ 1 & 1 \end{bmatrix}^{-1} \begin{bmatrix} 0 \\ \frac{1}{L} \end{bmatrix} \right) u(t) \quad (20)$$

$$= \begin{bmatrix} j\frac{1}{\sqrt{LC}} & 0 \\ 0 & -j\frac{1}{\sqrt{LC}} \end{bmatrix} \vec{y}(t) + \begin{bmatrix} \frac{1}{2LC} \\ \frac{1}{2LC} \end{bmatrix} u(t) \quad (21)$$

¹The natural frequency of this system is $\omega_n = \frac{1}{\sqrt{LC}}$. If we excite this system at a period equal to the natural frequency, we can make it grow unboundedly. This is similar to pushing a swing at the same rate it swings, which makes it swing farther.

so our system of equations is

$$\frac{d}{dt}y_1(t) = j\frac{1}{\sqrt{LC}}y_1(t) + \frac{1}{2LC}u(t) \quad (22)$$

$$\frac{d}{dt}y_2(t) = -j\frac{1}{\sqrt{LC}}y_2(t) + \frac{1}{2LC}u(t) \quad (23)$$

$$(24)$$

Recall that $\vec{x}(0) = \vec{0}$, so $\vec{y}(t) = \vec{0}$ (where $\vec{0}$ is a vector of all zeros). Solving this differential equation now, we get

$$y_1(t) = \underbrace{y_1(0)}_0 e^{j\frac{1}{\sqrt{LC}}t} + \int_0^t e^{j\frac{1}{\sqrt{LC}}(t-\theta)} \left(\frac{1}{2LC}u(\theta) \right) d\theta \quad (25)$$

$$y_2(t) = \underbrace{y_2(0)}_0 e^{-j\frac{1}{\sqrt{LC}}t} + \int_0^t e^{-j\frac{1}{\sqrt{LC}}(t-\theta)} \left(\frac{1}{2LC}u(\theta) \right) d\theta \quad (26)$$

Simplifying and stacking the solutions in vector form,

$$\begin{bmatrix} v_C(t) \\ i_L(t) \end{bmatrix} = \vec{x}(t) = V \begin{bmatrix} \frac{1}{2LC}e^{j\frac{1}{\sqrt{LC}}t} \int_0^t e^{-j\frac{1}{\sqrt{LC}}\theta} u(\theta) d\theta \\ \frac{1}{2LC}e^{-j\frac{1}{\sqrt{LC}}t} \int_0^t e^{j\frac{1}{\sqrt{LC}}\theta} u(\theta) d\theta \end{bmatrix} \quad (27)$$

1.(b)iii:

We wish to show $\vec{x}(t)$ is unbounded, given some bounded input $u(t)$. When showing a vector is bounded, we can show that all of its individual, scalar entries are bounded. Alternatively, when showing a vector is unbounded, it is enough to show that one of its entries will be unbounded. Note that $i_L(t) = y_1(t) + y_2(t)$ (which we see by computing $\vec{x}(t) = V\vec{y}(t)$). We can show that this quantity is unbounded. Recall that

$$y_1(t) = \frac{e^{j\frac{1}{\sqrt{LC}}t}}{2LC} \int_0^t e^{-j\frac{1}{\sqrt{LC}}\theta} u(\theta) d\theta \quad (28)$$

$$y_2(t) = \frac{e^{-j\frac{1}{\sqrt{LC}}t}}{2LC} \int_0^t e^{j\frac{1}{\sqrt{LC}}\theta} u(\theta) d\theta \quad (29)$$

$$\implies i_L(t) = \frac{e^{j\frac{1}{\sqrt{LC}}t}}{2LC} \int_0^t e^{-j\frac{1}{\sqrt{LC}}\theta} u(\theta) d\theta + \frac{e^{-j\frac{1}{\sqrt{LC}}t}}{2LC} \int_0^t e^{j\frac{1}{\sqrt{LC}}\theta} u(\theta) d\theta \quad (30)$$

Now, we have to make some choice of a bounded input $u(t)$ so the entire term is unbounded as $t \rightarrow \infty$. We can choose $u(t) = e^{-j\frac{1}{\sqrt{LC}}t} + e^{j\frac{1}{\sqrt{LC}}t} = 2\cos\left(\frac{1}{\sqrt{LC}}t\right)$ which is a bounded sinusoidal function. We can first compute $i_L(t)$ with this input:

$$i_L(t) = \frac{e^{j\frac{1}{\sqrt{LC}}t}}{2LC} \int_0^t e^{-j\frac{1}{\sqrt{LC}}\theta} \left(e^{-j\frac{1}{\sqrt{LC}}\theta} + e^{j\frac{1}{\sqrt{LC}}\theta} \right) d\theta + \frac{e^{-j\frac{1}{\sqrt{LC}}t}}{2LC} \int_0^t e^{j\frac{1}{\sqrt{LC}}\theta} \left(e^{-j\frac{1}{\sqrt{LC}}\theta} + e^{j\frac{1}{\sqrt{LC}}\theta} \right) d\theta \quad (31)$$

$$= \frac{e^{j\frac{1}{\sqrt{LC}}t}}{2LC} \int_0^t 1 + e^{-j\frac{2}{\sqrt{LC}}\theta} d\theta + \frac{e^{-j\frac{1}{\sqrt{LC}}t}}{2LC} \int_0^t 1 + e^{j\frac{2}{\sqrt{LC}}\theta} d\theta \quad (32)$$

$$= \frac{e^{j\frac{1}{\sqrt{LC}}t}}{2LC} \left(t + \frac{1 - e^{-j\frac{2}{\sqrt{LC}}t}}{j\frac{2}{\sqrt{LC}}} \right) + \frac{e^{-j\frac{1}{\sqrt{LC}}t}}{2LC} \left(t + \frac{e^{j\frac{2}{\sqrt{LC}}t} - 1}{j\frac{2}{\sqrt{LC}}} \right) \quad (33)$$

$$= \frac{t}{LC} \left(\frac{e^{j\frac{1}{\sqrt{LC}}t} + e^{-j\frac{1}{\sqrt{LC}}t}}{2} \right) + \frac{1}{\sqrt{LC}} \left(\frac{e^{j\frac{1}{\sqrt{LC}}t} - e^{-j\frac{1}{\sqrt{LC}}t}}{2j} \right) \quad (34)$$

$$= \frac{t}{LC} \cos\left(\frac{t}{\sqrt{LC}}\right) + \frac{1}{\sqrt{LC}} \sin\left(\frac{t}{\sqrt{LC}}\right) \quad (35)$$

Notice that the cos and sin terms are bounded, but the cos term is multiplied by a t , so as $t \rightarrow \infty$, $i_L(t) \rightarrow \infty$. Hence, the system is unstable. Generally, we say a system with eigenvalues having negative real part implies stability. Here, the real part of the eigenvalues is 0, so the system is unstable.

- (c) Thus far, we have dealt with continuous systems so it also makes sense to consider discrete systems. Consider the discrete system

$$x[i+1] = 2x[i] + u[i] \quad (36)$$

with $x[0] = 0$.

Is the system stable or unstable? If unstable, find a bounded input sequence $u[i]$ that causes the system to “blow up”.

Solution: Notice that, if we had the system

$$x[i+1] = 2x[i] \quad (37)$$

then we can write $x[i+1] = 2^i x[1]$. So, if we can somehow make $x[1]$ nonzero using a bounded input (e.g. equal to 1, for simplicity), then as $i \rightarrow \infty$, $x[i+1] \rightarrow \infty$. We know that $x[0] = 0$, and that $x[1] = 2x[0] + u[0] = u[0]$. Hence, we can set $u[0] = 1$ and then $x[1] = 1$. We have achieved what we wanted, i.e. to make $x[1]$ a nonzero value using the bounded input $u[0] = 1$. Now, for the other timesteps $i > 0$, we can set $u[i] = 0$ since that would leave us with the system in eq. (37). Written explicitly, our bounded input is

$$u[i] = \begin{cases} 1 & i = 0 \\ 0 & i > 0 \end{cases} \quad (38)$$

2. Changing behavior through feedback

In this question, we discuss how *feedback control* can be used to change the effective behavior of a system.

(a) Consider the scalar system:

$$x[i + 1] = 0.9x[i] + u[i] + w[i] \quad (39)$$

where $u[i]$ is the control input we get to apply based on the current state and $w[i]$ is the external disturbance, each at time i .

Is the system stable? If $|w[i]| \leq \epsilon$, what can you say about $|x[i]|$ at all times i if you further assume that $u[i] = 0$ and the initial condition $x[0] = 0$? How big can $|x[i]|$ get?

Solution: The system is stable, as $\lambda = 0.9 \implies |\lambda| < 1$. We can say that $|x[i]|$ is bounded at all time if the disturbance is bounded. Unrolling the system's recursion and extrapolating the general form,

$$x[0] = 0 \quad (40)$$

$$x[1] = w[0] \quad (41)$$

$$x[2] = 0.9w[0] + w[1] \quad (42)$$

$$x[3] = 0.9^2w[0] + 0.9w[1] + w[2] \quad (43)$$

$$\vdots \quad (44)$$

$$x[i] = \sum_{k=0}^{i-1} 0.9^k w[i - k - 1]. \quad (45)$$

We can check that this form works by plugging it into our recursion:

$$x[i + 1] = 0.9x[i] + w[i] = 0.9 \left(\sum_{k=0}^{i-1} 0.9^k w[i - k - 1] \right) + w[i] \quad (46)$$

$$= \sum_{k=0}^{i-1} 0.9^{k+1} w[i - k - 1] + w[i] = \sum_{k=0}^i 0.9^k w[i - k] \quad (47)$$

which is exactly what our formula predicts. So,

$$|x[i]| = \left| \sum_{k=0}^{i-1} 0.9^k w[i - k - 1] \right| \leq \sum_{k=0}^{i-1} \left| 0.9^k w[i - k - 1] \right| = \sum_{k=0}^{i-1} 0.9^k \epsilon. \quad (48)$$

In the limit as $i \rightarrow \infty$, by the geometric series formula,

$$|x[i]| \leq \frac{\epsilon}{1 - 0.9} = 10\epsilon \quad (49)$$

(b) Suppose that we decide to choose a control law $u[i] = f x[i]$ to apply in feedback. **Given a specific λ , you want the system to behave like:**

$$x[i + 1] = \lambda x[i] + w[i]? \quad (50)$$

To do so, how would you pick f ?

NOTE: In this case, $w[i]$ can be thought of like another input to the system, except we can't control it.

Solution: We can control the system to have any value of λ , as long as we're not limited on the values of f .

$$x[i + 1] = 0.9x[i] + fx[i] + w[i] = \lambda x[i] + w[i]. \quad (51)$$

Fitting terms, $f = \lambda - 0.9$. Note we can get a $\lambda > 1$ if we so desire; there is nothing stopping us from putting arbitrarily big/small λ by the choice of f .

- (c) **For the previous part, which f would you choose to minimize how big $|x[i]|$ can get?**

Solution: From eq. (50), in order to have the minimum bound on $|x[i]|$, $\lambda = 0$. To get this λ , $f = -0.9$. In the limit as $i \rightarrow \infty$ in this case,

$$|x[i]| \leq \frac{\epsilon}{1 - 0} = \epsilon \quad (52)$$

The minimum bound on $|x(i)| = \epsilon$ is the same bound as on the disturbance.

- (d) **What if instead of a 0.9, we had a 3 in the original eq. (39). Would system stability change? Would our ability to control λ change?**

Solution: If our system were now,

$$x[i + 1] = 3x[i] + u[i] + w[i], \quad (53)$$

the system would no longer be stable. However, we can still choose any λ using closed loop feedback. In this case, $f = \lambda - 3$.

- (e) Now suppose that we have a vector-valued system with a vector-valued control:

$$\vec{x}[i + 1] = A\vec{x}[i] + B\vec{u}[i] + \vec{w}[i] \quad (54)$$

where we further assume that B is an invertible square matrix. Further, suppose we decide to apply linear feedback control using a square matrix F so we choose $\vec{u}[i] = F\vec{x}[i]$.

Given a specific A_{CL} we want the system to behave like:

$$\vec{x}[i + 1] = A_{CL}\vec{x}[i] + \vec{w}[i]? \quad (55)$$

How would you pick F given knowledge of A, B and the desired goal dynamics A_{CL} ? Will this work for any desired A_{CL} ?

Solution: Since in this case our input is the same rank as our output, we can arbitrarily choose the matrix A_{CL} . As long as B is invertible (as given), we can define:

$$\vec{x}[i + 1] = A\vec{x}[i] + B\vec{u}[i] + \vec{w}[i] \quad (56)$$

$$= A\vec{x}[i] + BF\vec{x}[i] + \vec{w}[i] \quad (57)$$

$$= (A + BF)\vec{x}[i] + \vec{w}[i] \quad (58)$$

$$= A_{CL}\vec{x}[i] + \vec{w}[i] \quad (59)$$

Therefore, matching terms,

$$A + BF = A_{\text{CL}} \implies F = B^{-1}(A_{\text{CL}} - A). \quad (60)$$

Contributors:

- Anish Muthali.
- Neelesh Ramachandran.
- Sidney Buchbinder.
- Anant Sahai.
- Kareem Ahmad.
- Regina Eckert.
- Kumar Krishna Agrawal.