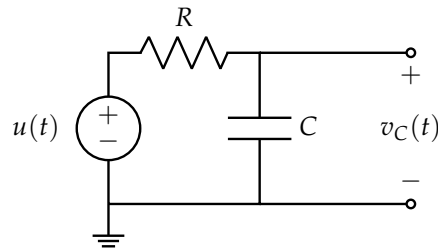


## Discussion 8B

## 1. Stability Examples and Counterexamples

- (a) Consider the circuit below with  $R = 1 \Omega$ ,  $C = 0.5 \text{ F}$ , and  $u(t)$  is some function bounded between  $-K$  and  $K$  for some constant  $K \in \mathbb{R}$  (for example  $K \cos(t)$ ). Furthermore assume that  $v_C(0) = 0 \text{ V}$  (that the capacitor is initially discharged).



This circuit can be modeled by the differential equation

$$\frac{dv_C(t)}{dt} = -2v_C(t) + 2u(t) \quad (1)$$

**Show that the differential equation is always stable (that is, as long as the input  $u(t)$  is bounded,  $v_C(t)$  also stays bounded).** Consider what this means in the physical circuit. *HINT: You may want to use the triangle inequality, i.e.  $|a + b| \leq |a| + |b|$ , and the triangle inequality for integrals, i.e.  $\left| \int_a^b f(x) dx \right| \leq \int_a^b |f(x)| dx$ . When we use  $|\cdot|$  notation here, we will take this to mean the magnitude, rather than the absolute value (since we can be dealing with complex numbers).*

- (b) **(PRACTICE)** Now, suppose that in the circuit of part 1.a we replaced the resistor with an inductor as in fig. 1.

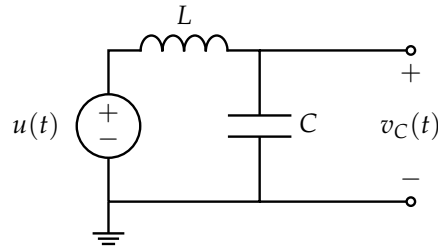


Figure 1: The original circuit with an inductor in place of the resistor.

Let  $L = 1$  mH. Repeat part 1.a for the new circuit (with an inductor). Consider the following process to arrive at the result:

- i. Derive the system of differential equations using KCL, KVL, and NVA. Show that the system is  $\frac{d}{dt} \begin{bmatrix} v_C(t) \\ i_L(t) \end{bmatrix} = \begin{bmatrix} 0 & \frac{1}{C} \\ -\frac{1}{L} & 0 \end{bmatrix} \begin{bmatrix} v_C(t) \\ i_L(t) \end{bmatrix} + \begin{bmatrix} 0 \\ \frac{1}{L} \end{bmatrix} u(t)$  with the initial condition being  $\begin{bmatrix} v_C(0) \\ i_L(0) \end{bmatrix} = \vec{0}$ .
- ii. Solve the matrix differential equation, using diagonalization if needed. Show that the diagonalized system has a solution

$$\vec{y}(t) = \begin{bmatrix} \frac{1}{2LC} e^{j\frac{1}{\sqrt{LC}}t} \int_0^t e^{-j\frac{1}{\sqrt{LC}}\theta} u(\theta) d\theta \\ \frac{1}{2LC} e^{-j\frac{1}{\sqrt{LC}}t} \int_0^t e^{j\frac{1}{\sqrt{LC}}\theta} u(\theta) d\theta \end{bmatrix} \quad (2)$$

where  $\vec{y}(t) = V^{-1} \begin{bmatrix} v_C(t) \\ i_L(t) \end{bmatrix}$  for change of basis matrix  $V$ . You may use the fact that the eigenvalue, eigenvector pairs of  $\begin{bmatrix} 0 & \frac{1}{C} \\ -\frac{1}{L} & 0 \end{bmatrix}$  are  $\left( j\frac{1}{\sqrt{LC}}, \begin{bmatrix} -j\sqrt{\frac{L}{C}} \\ 1 \end{bmatrix} \right)$  and  $\left( -j\frac{1}{\sqrt{LC}}, \begin{bmatrix} j\sqrt{\frac{L}{C}} \\ 1 \end{bmatrix} \right)$ .

- iii. Apply a similar process from part 1.a to show that, if we have a bounded input  $u(t)$ , then the system can grow unboundedly. When showing that a system is unstable, it suffices to choose a bounded  $u(t)$  that makes the system unbounded. We can choose  $u(t) = 2 \cos\left(\frac{1}{\sqrt{LC}}t\right) = e^{j\frac{1}{\sqrt{LC}}t} + e^{-j\frac{1}{\sqrt{LC}}t}$ <sup>1</sup>. HINT: You may use the fact that  $i_L(t) = y_1(t) + y_2(t)$ .

Hint: You might find it useful to revisit the process of generating the state-space equations for  $v_C(t)$  and  $i_L(t)$  as done in Note 4 for the LC Tank. The difference is that here, we have an input voltage.

<sup>1</sup>The natural frequency of this system is  $\omega_n = \frac{1}{\sqrt{LC}}$ . If we excite this system at a period equal to the natural frequency, we can make it grow unboundedly. This is similar to pushing a swing at the same rate it swings, which makes it swing farther.

- (c) Thus far, we have dealt with continuous systems so it also makes sense to consider discrete systems. Consider the discrete system

$$x[i + 1] = 2x[i] + u[i] \quad (3)$$

with  $x[0] = 0$ .

**Is the system stable or unstable? If unstable, find a bounded input sequence  $u[i]$  that causes the system to “blow up”.**

## 2. Changing behavior through feedback

In this question, we discuss how *feedback control* can be used to change the effective behavior of a system.

(a) Consider the scalar system:

$$x[i + 1] = 0.9x[i] + u[i] + w[i] \quad (4)$$

where  $u[i]$  is the control input we get to apply based on the current state and  $w[i]$  is the external disturbance, each at time  $i$ .

**Is the system stable? If  $|w[i]| \leq \epsilon$ , what can you say about  $|x[i]|$  at all times  $i$  if you further assume that  $u[i] = 0$  and the initial condition  $x[0] = 0$ ? How big can  $|x[i]|$  get?**

(b) Suppose that we decide to choose a control law  $u[i] = fx[i]$  to apply in feedback. **Given a specific  $\lambda$ , you want the system to behave like:**

$$x[i + 1] = \lambda x[i] + w[i] \quad (5)$$

**To do so, how would you pick  $f$ ?**

*NOTE:* In this case,  $w[i]$  can be thought of like another input to the system, except we can't control it.

(c) **For the previous part, which  $f$  would you choose to minimize how big  $|x[i]|$  can get?**

- (d) **What if instead of a 0.9, we had a 3 in the original eq. (4). Would system stability change? Would our ability to control  $\lambda$  change?**

- (e) Now suppose that we have a vector-valued system with a vector-valued control:

$$\vec{x}[i + 1] = A\vec{x}[i] + B\vec{u}[i] + \vec{w}[i] \quad (6)$$

where we further assume that  $B$  is an invertible square matrix. Further, suppose we decide to apply linear feedback control using a square matrix  $F$  so we choose  $\vec{u}[i] = F\vec{x}[i]$ .

**Given a specific  $A_{CL}$  we want the system to behave like:**

$$\vec{x}[i + 1] = A_{CL}\vec{x}[i] + \vec{w}[i]? \quad (7)$$

**How would you pick  $F$  given knowledge of  $A, B$  and the desired goal dynamics  $A_{CL}$ ? Will this work for any desired  $A_{CL}$ ?**

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